Morphisms in categories of sets with similarity relations

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Abstract— *Morphisms of some categories of sets with similarity relations (Ω-sets) are investigated, where Ω is a complete residuated lattice. Namely a category $\mathbf{SetF}(\Omega)$ with morphisms $(A,\delta) \to (B,\gamma)$ defined as special maps $A \to B$ and a category $\mathbf{SetR}(\Omega)$ with morphisms defined as a special relations $A \times B \to \Omega$. It is proved that arbitrary maps $A \to \Omega$ and $A \times B \to \Omega$ can be extended onto morphisms $(A,\delta) \to (\Omega,\leftrightarrow)$ in $\mathbf{SetF}(\Omega)$ and morphisms $(A,\delta) \to (B,\gamma)$ in $\mathbf{SetR}(\Omega)$, respectively and that these extension processes are from categorical point of view special reflections. Moreover, if Ω is a complete Heyting algebra we also investigate morphisms in another category $\mathbf{Set}(\Omega)$ which consists of classical fuzzy sets $f : A \to \Omega$.*

Keywords— residuated lattice, Heyting algebra, similarity relations, fuzzy sets.

1 Introduction

In fuzzy set theory the category $\mathbf{SetF}(\Omega)$ of sets with similarity relations defined over a complete residuated lattice $\Omega = (\Omega, \to, \otimes, \vee, \wedge)$ is of principal importance ([5]-[10]). This category consists of objects (A,δ) (called Ω-sets), where A is a set and δ is a similarity relation, i.e. a map $\delta : A \times A \to \Omega$ such that

(a) $\delta(x,x) = 1$,

(b) $\delta(x,y) = \delta(y,x)$,

(c) $\delta(x,y) \otimes (\delta(y,y) \to \delta(y,z)) \le \delta(x,z)$.

A morphism $f : (A,\delta) \to (B,\gamma)$ in $\mathbf{SetF}(\Omega)$ is a map $f : A \to B$ such that $\gamma(f(x),f(y)) \ge \delta(x,y)$ for all $x,y \in A$.

From historical point of view there is another category consisting of sets with similarity relations defined. This category $\mathbf{SetR}(\Omega)$ is an analogy of a category of sets with relations between sets as morphisms. Objects of this category $\mathbf{SetR}(\Omega)$ are the same as in the category $\mathbf{SetF}(\Omega)$ and morphism $f : (A,\delta) \to (B,\gamma)$ are maps $f : A \times B \to \Omega$ such that

(a) $(\forall x,z \in A)(\forall y \in B) \quad \delta(z,x) \otimes f(x,y) \le f(z,y)$,

(b) $(\forall x \in A)(\forall y,z \in B) \quad f(x,y) \otimes \gamma(y,z) \le f(x,z)$,

If $f : (A,\delta) \to (B,\gamma)$ and $g : (B,\gamma) \to (C,\omega)$ are two morphisms then their composition is a relation $g \circ f : A \times C \to \Omega$ such that

$$g \circ f(x,z) = \bigvee_{y \in B} (f(x,y) \otimes g(y,z)).$$

Finally we can consider a classical category $\mathbf{Set}(\Omega)$ of fuzzy sets over Ω with objects couples (A,f), where A is a set and f is a map $A \to \Omega$ and with morphisms $u : (A,f) \to (B,g)$ such that $u : A \to B$ is map and $f(a) \le g \circ u(a)$ for all $a \in A$.

With categories $\mathbf{SetF}(\Omega)$ and $\mathbf{SetR}(\Omega)$ we can consider new sets of objects:

(a) the set $\mathbf{Mor}(\mathbf{SetR}(\Omega))$ of all morphisms in $\mathbf{SetR}(\Omega)$,

(b) the set $\mathbf{Rel}(\mathbf{SetR}(\Omega))$ of all *weak morphisms* in $\mathbf{SetR}(\Omega)$, where $f : (A,\delta) \to (B,\gamma)$ is a weak morphism in $\mathbf{SetR}(\Omega)$ if $f : A \times B \to \Omega$ is an Ω-valued relation,

(c) the set $\mathbf{Map}(\mathbf{SetF}(\Omega))$ of all *weak fuzzy sets* in $\mathbf{SetF}(\Omega)$, where $f : (A,\delta) \to (\Omega,\leftrightarrow)$ is a weak fuzzy set in $\mathbf{SetF}(\Omega)$ if $f : A \to \Omega$ is a map and where $\leftrightarrow$ is a biresiduation in a lattice Ω.

(d) the set $\mathbf{Fuz}(\mathbf{SetF}(\Omega))$ of all *fuzzy sets* in $\mathbf{SetF}(\Omega)$, i.e. morphisms $(A,\delta) \to (\Omega,\leftrightarrow)$ in the category $\mathbf{SetF}(\Omega)$, where (A,δ) is an Ω-set.

It is clear that we have

$$\mathbf{Mor}(\mathbf{SetR}(\Omega)) \hookrightarrow \mathbf{Rel}(\mathbf{SetR}(\Omega)),$$
$$\mathbf{Fuz}(\mathbf{SetF}(\Omega)) \hookrightarrow \mathbf{Map}(\mathbf{SetF}(\Omega)).$$

In this paper we want to consider some relationships between these sets. We show that new categories (denoted by the same symbols as the above sets) can be defined with the above sets as object sets. Then we will receive the following functor commutative diagram:

$$\begin{array}{ccc}
\mathbf{Fuz}(\mathbf{SetF}(\Omega)) & \xrightarrow{\hookrightarrow} & \mathbf{Map}(\mathbf{SetF}(\Omega)) \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\
\mathbf{Mor}(\mathbf{SetR}(\Omega)) & \xrightarrow{\hookrightarrow} & \mathbf{Rel}(\mathbf{SetR}(\Omega))
\end{array}$$

Finally, we show that

(a) the category $\mathbf{Mor}(\mathbf{SetR}(\Omega))$ is a full reflective subcategory in the category $\mathbf{Rel}(\mathbf{SetR}(\Omega))$,

(b) the category $\mathbf{Map}(\mathbf{SetF}(\Omega))$ is a full reflective subcategory in the category $\mathbf{Fuz}(\mathbf{SetF}(\Omega))$.

The reflections $f \to \tilde{f}$ and $g \to \widehat{g}$ of a weak morphism $f : (A,\delta) \to (B,\gamma)$ and a weak fuzzy set $g : (A,\delta) \to (\Omega,\leftrightarrow)$ can be then used for construction of a fuzzy logic formulas interpretation in corresponding categories $\mathbf{SetF}(\Omega)$ and $\mathbf{SetR}(\Omega)$.

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2 Categories of maps and relations

Let us firstly define two categories $\mathbf{Rel}(\mathbf{SetR}(\Omega))$ and $\mathbf{Mor}(\mathbf{SetR}(\Omega))$. Objects of the category $\mathbf{Rel}(\mathbf{SetR}(\Omega))$ are all weak morphisms from $\mathbf{SetR}(\Omega)$. A morphism from a weak morphism $f:(A,\delta)\to(B,\gamma)$ to a weak morphism $g:(C,\rho)\to(D,\omega)$ is a couple (u,v) of morphisms from the category $\mathbf{SetR}(\Omega)$ such that

(i) $u:(A,\delta)\to(C,\rho)$ is a morphism in $\mathbf{SetR}(\Omega)$,

(ii) $v:(B,\gamma)\to(D,\omega)$ is a morphism in $\mathbf{SetR}(\Omega)$,

(iii) $(v\circ f)(a,d)\le(g\circ u)(a,d)$ for all $a\in A, d\in D$, where the composition of a morphism and a weak morphism is formally the same as for morphisms in the category $\mathbf{SetR}(\Omega)$.

In that case we say that the diagram

$$\begin{array}{ccc}(A,\delta) & \xrightarrow{f} & (B,\gamma)\\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v}\\ (C,\rho) & \xrightarrow{g} & (D,\omega)\end{array}$$

fuzzy commutes. If $(u,v):f\to g$ and $(u_1,v_1):g\to h$ are two morphisms in $\mathbf{Rel}(\mathbf{SetR}(\Omega))$ such that $f:(A,\delta)\to(B,\gamma)$, $g:(C,\rho)\to(D,\omega)$, $h:(E,\sigma)\to(F,\tau)$ are weak morphisms then a composition $(u_1,v_1)\circ(u,v)$ is defined as $(u_1\circ u, v_1\circ v)$. Using a composition of weak morphisms we can easily prove that the following diagram fuzzy commutes

$$\begin{array}{ccc}(A,\delta) & \xrightarrow{f} & (B,\gamma)\\ {\scriptstyle u_1\circ u}\downarrow & & \downarrow{\scriptstyle v_1\circ v}\\ (E,\sigma) & \xrightarrow{h} & (F,\tau),\end{array}$$

i.e. the definition is correct.

Objects of the category $\mathbf{Mor}(\mathbf{SetR}(\Omega))$ are all morphisms $f:(A,\delta)\to(B,\gamma)$ in the category $\mathbf{SetR}(\Omega)$ and morphisms between such objects are defined formally in the same way as for the category $\mathbf{Rel}(\mathbf{SetR}(\Omega))$. It is clear that $\mathbf{Mor}(\mathbf{SetR}(\Omega))$ is a full subcategory of the category $\mathbf{Rel}(\mathbf{SetR}(\Omega))$.

Now, objects of the category $\mathbf{Map}(\mathbf{SetF}(\Omega))$ are all weak fuzzy sets $(A,\delta)\to(\Omega,\leftrightarrow)$ in the category $\mathbf{SetF}(\Omega)$, where (A,δ) are Ω-sets. If $f:(A,\delta)\to(\Omega,\leftrightarrow)$ and $g:(B,\gamma)\to(\Omega,\leftrightarrow)$ are objects in $\mathbf{Map}(\mathbf{SetF}(\Omega))$ then $u:f\to g$ is a morphism in $\mathbf{Map}(\mathbf{SetF}(\Omega))$ if $u:(A,\delta)\to(B,\gamma)$ is a morphism in $\mathbf{SetF}(\Omega)$ and $g\circ u(a)\ge f(a)$ for all $a\in A$. A composition of such morphisms is defined as a composition of corresponding morphisms in the category $\mathbf{SetF}(\Omega)$.

Finally objects of the category $\mathbf{Fuz}(\mathbf{SetF}(\Omega))$ are all fuzzy sets in the category $\mathbf{SetF}(\Omega)$, i.e. morphisms $(A,\delta)\to(\Omega,\leftrightarrow)$ in the category $\mathbf{SetF}(\Omega)$. Morphisms in the category $\mathbf{Fuz}(\mathbf{SetF}(\Omega))$ are defines similarly as in the category $\mathbf{Map}(\mathbf{SetF}(\Omega))$.

It is well known that there exists a functor $F:\mathbf{SetF}(\Omega)\to\mathbf{SetR}(\Omega)$ such that F is an identity on objects and if $f:(A,\delta)\to(B,\gamma)$ is a morphism in $\mathbf{SetF}(\Omega)$ then $F(f):A\times B\to\Omega$ is such that $F(f)(a,b)=\gamma(f(a),b)$.

Lemma 1
Let $g:(A,\delta)\to(B,\gamma)$ be a weak morphism in $\mathbf{SetR}(\Omega)$. Let $\tilde g: A\times B\to\Omega$ be defined by the formula

$$\tilde g(a,b)=\bigvee_{x\in A}\bigvee_{y\in B} g(x,y)\otimes\delta(a,x)\otimes\gamma(b,y).$$

Then

(a) $\tilde g$ is a morphism in $\mathbf{SetR}(\Omega)$,

(b) $\tilde g=\bigwedge\{h : h$ is a morphism $(A,\delta)\to(B,\gamma)$ in $\mathbf{SetR}(\Omega), h\ge g\}$,

(c) If g is a morphism in $\mathbf{SetR}(\Omega)$, then $\tilde g=g$.

Lemma 2
Let $s:(A,\delta)\to(\Omega,\leftrightarrow)$ be a weak fuzzy set in $\mathbf{SetF}(\Omega)$. Let a map $\widehat s: A\to\Omega$ be defined such that $\widehat s(a)=\bigvee_{x\in A}\delta(a,x)\otimes s(x)$ for all $a\in A$. Then

(a) $\widehat s:(A,\delta)\to(\Omega,\leftrightarrow)$ is a morphism in $\mathbf{SetF}(\Omega)$,

(b) $\widehat s=\bigwedge\{t : t$ is a morphism $(A,\delta)\to(\Omega,\leftrightarrow)$ in $\mathbf{SetF}(\Omega), t\ge s\}$.

(c) If $s:(A,\delta)\to(\Omega,\leftrightarrow)$ is a fuzzy set in $\mathbf{SetF}(\Omega)$ then $\widehat s=s$,

Now we define functors $G:\mathbf{Rel}(\mathbf{SetR}(\Omega))\to\mathbf{Mor}(\mathbf{SetR}(\Omega))$ and $H:\mathbf{Map}(\mathbf{SetF}(\Omega))\to\mathbf{Fuz}(\mathbf{SetF}(\Omega))$ which will be reflections.

Proposition 1

(a) Let a weak morphism f be an object from $\mathbf{Rel}(\mathbf{SetR}(\Omega))$ and let $(u,v):f\to g$ be a morphism in $\mathbf{Rel}(\mathbf{SetR}(\Omega))$. Let $G(f)=\tilde f$ and $G(u,v)=(u,v)$. Then

$$G:\mathbf{Rel}(\mathbf{SetR}(\Omega))\to\mathbf{Mor}(\mathbf{SetR}(\Omega))$$

is a functor.

(b) Let a weak fuzzy set f be an object in $\mathbf{Map}(\mathbf{SetF}(\Omega))$ and let $u:f\to g$ be a morphism in $\mathbf{Map}(\mathbf{SetF}(\Omega))$. Let $H(f)=\widehat f$ and $H(u)=u$. Then

$$H:\mathbf{Map}(\mathbf{SetF}(\Omega))\to\mathbf{Fuz}(\mathbf{SetF}(\Omega))$$

is a functor.

Proof. (a) Let $f:(A,\delta)\to(B,\gamma)$ and $g:(C,\rho)\to(D,\tau)$ be weak morphisms. We show that (u,v) is a morphism $\tilde f\to\tilde g$. Since $(u,v):f\to g$ is a morphism in $\mathbf{Rel}(\mathbf{SetR}(\Omega))$, we have $v\circ f\le g\circ u$. Then since v and u are morphisms in $\mathbf{SetR}(\Omega)$, we have

$$\begin{aligned} v\circ\tilde f(a,d) &= \\ \bigvee_{b,b',a'} f(a',b')\otimes\delta(a,a')\otimes\gamma(b,b')\otimes v(b,d) &\le \\ \bigvee_{b',a'} f(a',b')\otimes\delta(a,a')\otimes v(b',d) &= \\ \bigvee_{a'} v\circ f(a',d)\otimes\delta(a,a') &\le \\ \bigvee_{a'} g\circ u(a',d)\otimes\delta(a,a')\le\bigvee_{c} u(a,c)\otimes g(c,d) &\le \\ \tilde g\circ u(a,d). \end{aligned}$$

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(b) Let $f : (A, \delta) \to (\Omega, \leftrightarrow)$ and $g : (B, \gamma) \to (\Omega, \leftrightarrow)$ be weak fuzzy sets. We need to show that $\widehat{g} \circ u \geq \widehat{f}$. Since u is a morphism in $\mathbf{SetF}(\Omega)$, we have

$$\widehat{g} \circ u(a) = \bigvee_{b \in B} g(b) \otimes \gamma(b, u(a)) \geq$$
$$\bigvee_{x \in A} g(u(x)) \otimes \gamma(u(x), u(a)) \geq \bigvee_{x \in A} g \circ u(x) \otimes \delta(x, a) \geq$$
$$\bigvee_{x \in A} f(x) \otimes \delta(x, a) = \widehat{f}(a).$$

We now introduce a notion of a fuzzy reflective subcategory.

Theorem 1

(a) $\mathbf{Mor}(\mathbf{SetR}(\Omega))$ *is a full reflective subcategory in the category* $\mathbf{Rel}(\mathbf{SetR}(\Omega))$ *and* G *is a reflection.*

(b) $\mathbf{Map}(\mathbf{SetF}(\Omega))$ *is a full reflective subcategory in the category* $\mathbf{Fuz}(\mathbf{SetF}(\Omega))$ *and* H *is a reflection.*

Proof. (a) Let $f : (A, \delta) \to (B, \gamma)$ be an object in $\mathbf{Rel}(\mathbf{SetR}(\Omega))$. It is clear that $(id_A, id_B) : f \to \tilde{f}$ is a morphism in $\mathbf{Rel}(\mathbf{SetR}(\Omega))$, where for an object (X, σ) from $\mathbf{SetR}(\Omega)$ the identity morphism $id_X : (X, \sigma) \to (X, \sigma)$ is defined such that $id_X : X \times X \to \Omega$ is such that $id_X(x, y) = 1_\Omega$ if $x = y$ and 0_Ω, otherwise. Let $g : (C, \rho) \to (D, \tau)$ be an object in $\mathbf{Mor}(\mathbf{SetR}(\Omega))$ and let $(u, v) : f \to g$ be a morphism in $\mathbf{Rel}(\mathbf{SetR}(\Omega))$. Then we show that (u, v) is the unique morphism such that the diagram commutes:

$$\begin{array}{ccc} f & \xrightarrow{(id_A, id_B)} & \tilde{f} \\ \| & & \downarrow{\scriptstyle (u,v)} \\ f & \xrightarrow{(u,v)} & g \end{array}$$

In fact, it suffices to prove that $(u, v) : \widetilde{f} \to g$ is a morphism in $\mathbf{Rel}(\mathbf{SetR}(\Omega))$, i.e. that the diagram

$$\begin{array}{ccc} (A, \delta) & \xrightarrow{\widetilde{f}} & (B, \gamma) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ (C, \rho) & \xrightarrow{g} & (D, \tau) \end{array}$$

fuzzy commutes. We have $v \circ f(a, d) \leq g \circ u(a, d)$ for all $a \in A, d \in D$. Since u and v are morphisms in a category $\mathbf{SetR}(\Omega)$, we have

$$v \circ \tilde{f}(a, d) =$$
$$\bigvee_{b, b' \in B, a' \in A} f(a', b') \otimes \delta(a, a') \otimes \gamma(b, b') \otimes v(b, d) \leq$$
$$\bigvee_{b, b' \in B a' \in A} f(a', b') \otimes \delta(a, a') \otimes v(b', d) =$$
$$\bigvee_{a' \in A} \delta(a, a') \otimes v \circ f(a', d) \leq$$
$$\bigvee_{a' \in A} \delta(a, a') \otimes g \circ u(a', d) \leq g \circ u(a, d).$$

It is clear that (u, v) is the unique morphism with such property and that the diagram commutes.

(b) Let $f : (A, \delta) \to (\Omega, \leftrightarrow)$ be an object of $\mathbf{Map}(\mathbf{SetF}(\Omega))$. It is clear that $id_A : f \to \widehat{f}$ is a morphism in $\mathbf{Map}(\mathbf{SetF}(\Omega))$. Let $g : (B, \gamma) \to (\Omega, \leftrightarrow)$ be an object in $\mathbf{Fuz}(\mathbf{SetF}(\Omega))$ and let $u : f \to g$ be a morphism in $\mathbf{Map}(\mathbf{SetF}(\Omega))$. Then u is the unique morphism in $\mathbf{Fuz}(\mathbf{SetF}(\Omega))$ such that the diagram commutes:

$$\begin{array}{ccc} f & \xrightarrow{id_A} & \widehat{f} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ g & = & g. \end{array}$$

In fact, since $g \circ u(a) \geq f(a)$ and $g \circ u$ is an object in $\mathbf{Fuz}(\mathbf{SetF}(\Omega))$, we have

$$\widehat{f}(a) = \bigvee_{x \in A} \delta(a, x) \otimes f(x) \leq \bigvee_{x \in A} \delta(a, x) \otimes g \circ u(x) \leq$$
$$g \circ u(a).$$

3 Reflections and fuzzy logic models

The reflections G, H can be used for a definition of fuzzy logic formulas interpretation in categories $\mathbf{SetR}(\Omega)$ and $\mathbf{SetF}(\Omega)$, respectively (see [10]). Let J be a first order language of a fuzzy logic which consists (as classically) of a set of predicate symbols $P \in \mathscr{P}$, a set of functional symbols $f \in \mathscr{F}$ and a set of classical logical connectives $\{\wedge, \vee, \Rightarrow, \neg, \otimes\}$. Moreover J contains also a set Ω of logical constants.

Definition 1

Let $\mathbf{K}$ *be a category with products and with* Ω*-sets as objects. Then a model of a language* J *in a category* $\mathbf{K}$ *is*

$$\mathscr{D} = ((A, \delta), \{P_\mathscr{D} : P \in \mathscr{P}\}, \{f_\mathscr{D} : f \in \mathscr{F}\}),$$

where

(a) (A, δ) *is an* Ω*-set from a category* $\mathbf{K}$*,*

(b) $P_\mathscr{D} : (A, \delta) \times \cdots \times (A, \delta) \to (\Omega, \leftrightarrow)$ *is a morphism in* $\mathbf{K}$*,*

(c) $f_\mathscr{D} : (A, \delta) \times \cdots \times (A, \delta) \to (A, \delta)$ *is a morphism in a category* $\mathbf{K}$*.*

Further, let ψ (t, respectively) be a formula (term, respectively) with free variables contained in a set X of variables. Then an interpretation $\|\psi\|_{\mathscr{D},X}$ ($\|t\|_{\mathscr{D},X}$, respectively) of ψ (t, respectively) in a model $\mathscr{D}$ in a category $\mathbf{K}$ should be defined such that

(a) $\|\psi\|_{\mathscr{D},X} : (A, \delta)^X \to (\Omega, \leftrightarrow)$ is a morphism in $\mathbf{K}$,

(b) $\|t\|_{\mathscr{D},X} : (A, \delta)^X \to (A, \delta)$ is a morphism in $\mathbf{K}$,

where $(A, \delta)^X$ is a product $(A, \delta)^{|X|} = (A^{|X|}, \delta_X)$ in a category $\mathbf{K}$. We show shortly how by using reflections H and G the definition of a formula interpretation can be done in models $\mathscr{D}$ in categories $\mathbf{K} = \mathbf{SetF}(\Omega), \mathbf{SetR}(\Omega)$.

Let $\mathscr{D} = ((A, \delta), \{P_\mathscr{D} : P \in \mathscr{P}\}, \{f_\mathscr{D} : f \in \mathscr{F}\})$ be a model of a language J in a category $\mathbf{K}$, where $\mathbf{K} = \mathbf{SetF}(\Omega)$ or $\mathbf{K} = \mathbf{SetR}(\Omega)$, i.e.

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(i) (A,δ) is a Ω-set from $\mathbf{K}$,

(ii) $P_{\mathscr{D}} : (A,\delta)\times\cdots\times(A,\delta) \to (\Omega,\leftrightarrow)$ is a morphism in a category $\mathbf{K}$,

(iii) $f_{\mathscr{D}} : (A,\delta)\times\cdots\times(A,\delta) \to (A,\delta)$ is a morphism in a category $\mathbf{K}$.

Let t be a term with a set of variables contained in a set X. Then $\|t\|_{\mathscr{D},X} = \|t\|_X : (A,\delta)^X \to (A,\delta)$ is a morphism in $\mathbf{K}$ defined as follows.

(i) Let $t = x$, where $x \in X$. Then $\|t\|_X := pr_x : (A,\delta)^X \to (A,\delta)$ is a projection morphism in a category $\mathbf{K}$.

(ii) Let $t = f(t_1,\dots,t_n)$. Then $\|t\|_X$ is a composition (in $\mathbf{K}$) of morphisms

$$(A,\delta)^X \xrightarrow{\prod_i \|t_i\|_X} (A,\delta)^n \xrightarrow{f_{\mathscr{D}}} (A,\delta)\cdot$$

Hence, for $\mathbf{K} = \mathbf{SetR}(\Omega)$ we have $\|t\|_X(\mathbf{a},b) = \bigvee_{\mathbf{x}\in A^n}(\prod_i \|t_i\|_X(\mathbf{a},\mathbf{x}) \otimes f_{\mathscr{D}}(\mathbf{x},b)$, where $\mathbf{a} \in A^X$.

Now let ψ be a formula with free variables contained in a set X. In a category $\mathbf{K}$ we will define firstly a weak morphisms (i.e. *weak fuzzy set* for $\mathbf{K} = \mathbf{SetF}(\Omega)$ or a *weak morphism* for $\mathbf{K} = \mathbf{SetR}(\Omega)$) $|\psi|_{\mathscr{D},X} = |\psi|_X : (A,\delta)^X \to (\Omega,\leftrightarrow)$. A definition will be done by induction principle on a structure of ψ.

(a) Let $\psi \equiv P(t_1,\dots,t_n)$. Then (according to induction assumption) $\|t_i\|_X, P_{\mathscr{D}}$ are defined and we define $|\psi|_X$ as a composition of the following morphisms in $\mathbf{K}$:

$$(A,\delta)^X \xrightarrow{\prod_i \|t_i\|_X} (A,\delta)^n \xrightarrow{P_{\mathscr{D}}} (\Omega,\leftrightarrow).$$

(b) Let $\psi \equiv t_1 = t_2$. Then $|\psi|_X$ is a composition of the following weak morphisms:

$$(A,\delta)^X \xrightarrow{\|t_1\|_X\times\|t_2\|_X} (A,\delta)^2 \xrightarrow{\Delta_{\mathbf{K},A}} (\Omega,\leftrightarrow).$$

(c) Let $\psi \equiv \psi_1 \wedge \psi_2$. Then $|\psi|_X$ is a composition of the following weak morphisms:

$$(A,\delta)^X \xrightarrow{\|\psi\|_X\times\|\psi_2\|_X} (\Omega,\leftrightarrow)^2 \xrightarrow{\sqcap_{\mathbf{K}}} (\Omega,\leftrightarrow).$$

(d) Let $\psi \equiv \psi_1 \vee \psi_2$. Then $|\psi|_X$ is a composition of the following weak morphisms:

$$(A,\delta)^X \xrightarrow{\|\psi\|_X\times\|\psi_2\|_X} (\Omega,\leftrightarrow)^2 \xrightarrow{\sqcup_{\mathbf{K}}} (\Omega,\leftrightarrow).$$

(e) Let $\psi \equiv \sigma \Rrightarrow \tau$. Then $|\psi|_X$ is a composition of the following weak morphisms:

$$(A,\delta)^X \xrightarrow{\|\sigma\|_X\times\|\tau\|_X} (\Omega,\leftrightarrow)^2 \xrightarrow{\Rrightarrow_{\mathbf{K}}} (\Omega,\leftrightarrow).$$

(f) Let $\psi \equiv \neg\sigma$. Then $|\psi|_X$ is a composition of the following weak morphisms:

$$(A,\delta)^X \xrightarrow{\|\sigma\|_X} (\Omega,\leftrightarrow) \xrightarrow{\neg_{\mathbf{K}}} (\Omega,\leftrightarrow).$$

(g) Let $\psi \equiv (\exists x)\sigma$. Then $\|\sigma\|_{X\cup\{x\}}$ is already defined as a morphism $(A,\delta)^{X\cup\{x\}} = (A,\delta)^X \times (A,\delta) \to (\Omega,\leftrightarrow)$ in $\mathbf{K}$. Then we set

$$|\psi|_X(\mathbf{a},\alpha) = \bigvee_{x\in A} \|\sigma\|_{X\cup\{x\}}((\mathbf{a},x),\alpha), \quad \text{if } \mathbf{K} = \mathbf{SetR}(\Omega),$$
$$|\psi|_X(\mathbf{a}) = \bigvee_{x\in A} \|\sigma\|_{X\cup\{x\}}(\mathbf{a},x), \quad \text{if } \mathbf{K} = \mathbf{SetR}(\Omega).$$

The maps $\Rrightarrow_{\mathbf{K}}, \neg_{\mathbf{K}}, \sqcup_{\mathbf{K}}, \sqcap_{\mathbf{K}}, \Delta_{\mathbf{K},A}$ are defined as follows: For $\mathbf{K} = \mathbf{SetR}(\Omega)$, we have

(i) $\Delta_{\mathbf{SetR}(\Omega),A}((a,b),\alpha) = \alpha \leftrightarrow \delta(a,b)$ for all $a,b \in A, \alpha \in \Omega$.

(ii) $\sqcap_{\mathbf{SetR}(\Omega)}((\beta,\gamma),\alpha) = \alpha \leftrightarrow (\beta\wedge\gamma)$.

(iii) $\sqcup_{\mathbf{SetR}(\Omega)}((\beta,\gamma),\alpha) = \alpha \leftrightarrow (\beta\vee\gamma)$.

(iv) $\Rrightarrow_{\mathbf{SetR}(\Omega)}((\beta,\gamma),\alpha) = (\beta\otimes\alpha) \to \gamma$.

(v) $\neg_{\mathbf{SetR}(\Omega)}(\alpha,\beta) = \beta \leftrightarrow (\alpha \to 0)$.

For $\mathbf{K} = \mathbf{SetF}(\Omega)$, we have

(a) $\Delta_{\mathbf{SetF}(\Omega),A}(a,b) = \delta(a,b)$ for all $a,b \in A$.

(b) $\sqcap_{\mathbf{SetF}(\Omega)}(\beta,\gamma) = \beta\wedge\gamma$.

(c) $\sqcup_{\mathbf{SetF}(\Omega)}(\beta,\gamma) = \beta\vee\gamma$.

(d) $\Rrightarrow_{\mathbf{SetF}(\Omega)}(\beta,\gamma) = \beta\to\gamma$.

(e) $\neg_{\mathbf{SetF}(\Omega)}(\beta) = \neg\beta$.

Definition 2
Let ψ be a formula in J and let X be a set of variables containing all free variables of ψ. Let $\mathscr{D}$ be a model of J in a category $\mathbf{K} = \mathbf{SetF}(\Omega), \mathbf{SetR}(\Omega)$. Then we set

$$\|\psi\|_{\mathscr{D},X} = G(|\psi|_{\mathscr{D},X}) \quad \textit{if} \quad \mathbf{K} = \mathbf{SetR}(\Omega),$$
$$\|\psi\|_{\mathscr{D},X} = H(|\psi|_{\mathscr{D},X}) \quad \textit{if} \quad \mathbf{K} = \mathbf{SetF}(\Omega).$$

Theorem 2
For any formula ψ, $\|\psi\|_{\mathscr{D},X} : (A,\delta)^X \to (\Omega,\leftrightarrow)$ is a morphism in a category $\mathbf{K} = \mathbf{SetR}(\Omega), \mathbf{SetF}(\Omega)$.

4 Category $\mathrm{Set}(\Omega)$

To investigate a relationship between classical fuzzy sets (A,f), where $f : A \to \Omega$ is a map and Ω-sets (A,δ) it will be necessary to modify a definition of a similarity relation δ. There are several natural ways how to transform a classical fuzzy set into a similarity relation. For example, for a fuzzy set f we can define a similarity relation σ_f such that $\sigma_f(x,y) = f(x) \leftrightarrow f(y)$. We can also introduce another relation, namely $\Delta_f(x,y) = f(x)$ iff $x = y$ and $\Delta_f(x,y) = 0$, otherwise. It is clear that in that case Δ_f does not satisfies the condition $\Delta_f(x,x) = 1$ from the definition of similarity relations. Nevertheless to investigate a relationship between fuzzy sets and such similarity relations, we will introduce in this section a generalization of Ω-sets, namely we will define a new category $\mathbf{SetR}(\Omega)_*$ with objects (A,δ), where A is a set and δ satisfies only the conditions (b) and (c) from definition of a

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similarity relation. Moreover, we will require that morphisms of $\mathbf{SetR}(\Omega)_*$ are those of morphisms $f : (A, \delta) \to (B, \gamma)$ from $\mathbf{SetR}(\Omega)$ that satisfy the following additional conditions:

(a) $\gamma(x, x) = \bigvee_{y \in B} f(x, y)$, for all $x \in A$,

(b) $f(x, y) \otimes f(x, z) \leq \gamma(y, z)$.

In this section we show a deeper relationship between morphisms in $\mathbf{Set}(\Omega)$ and $\mathbf{SetR}(\Omega)_*$ but only for a very special Ω, namely for a totally ordered Heyting algebra Ω.

Proposition 2
Let Ω be a totally ordered complete Heyting algebra and let $(A, \alpha), (B, \beta)$ be objects from a category $\mathbf{Set}(\Omega)$. Let $f : A \times B \to \Omega$ be a map. Then the following statement are equivalent.

(a) $f : (A, \Delta_\alpha) \to (B, \Delta_\beta)$ *is a morphism in the category* $\mathbf{SetR}(\Omega)_*$.

(b) There exist a morphism $g : (A, \alpha) \to (B, \beta)$ *in a category* $\mathbf{Set}(\Omega)$ *such that*

(i) $(\forall a \in A, b \in B) \quad f(a, b) \leq \Delta_\beta(g(a), b)$,

(ii) $(\forall a \in A, b \in B) \quad f(a, g(a)) = \alpha(a)$.

(c) There exists a morphism $g : (A, \alpha) \to (B, \beta)$ *in a category* $\mathbf{Set}(\Omega)$ *such that*

$$(\forall a \in A, b \in B) \quad f(a, b) = \alpha(a) \wedge \Delta_\beta(g(a), b).$$

Proof. Recall that a *singleton* in $(X, \delta) \in \mathbf{SetR}(\Omega)_*$ is a map $s : X \to \Omega$ such that

$$(\forall x, y \in X) \quad s(x) \wedge \delta(x, y) \leq s(y),$$
$$(\forall x, y \in X) \quad s(x) \wedge s(y) \leq \delta(x, y).$$

The set singl (X, δ) of all singletons of (X, δ) with a function $\tau_{(X,\delta)}$ is then an object of $\mathbf{SetR}(\Omega)_*$ if we set

$$(\forall s, t \in \text{ singl }) \quad \tau_{(X,\delta)}(s, t) = \bigvee_{a \in X} s(a) \wedge t(a).$$

(a)⇒ (b). We construct a map $h : A \to$ singl (B, Δ_β) such that

$$(\forall a \in A)(\forall y \in B) \quad h(a)(y) = \bigvee_{x \in A} (f(x, y) \wedge \Delta_\alpha(a, x)).$$

We have to show that $h(a) \in$ singl (B, Δ_β) for any $a \in A$. In fact, for $y \neq y' \in B$ we have

$$h(a)(y) \wedge h(a)(y') \leq f(a, y) \wedge f(a, y') \leq \Delta_\beta(y, y') = 0,$$
$$h(a)(y) \leq f(a, y) \leq \Delta_\alpha(a, a) = \alpha(a).$$

Moreover, $h : (A, \Delta_\alpha) \to ($ singl $(B, \Delta_\beta), \tau)$ is a morphism in a category $\mathbf{SetR}(\Omega)_*$. In fact, for any $a, a' \in A$ we have

$$\tau(h(a), h(a')) =$$
$$\bigvee_{y \in B} \bigvee_{x, x' \in A} (f(x, y) \wedge \Delta_\alpha(a, x) \wedge f(x', y) \wedge \Delta_\alpha(a', x')) \geq$$
$$\geq \bigvee_{y \in B} (f(a, y) \wedge \Delta_\alpha(a, a) \wedge \Delta_\alpha(a', a)) = \Delta_\alpha(a', a),$$
$$\tau(h(a), h(a)) =$$
$$\bigvee_{y \in B} \bigvee_{x \in A} f(x, y) \wedge \Delta_\alpha(a, x) \leq \bigvee_{y \in B} f(a, y) = \Delta_\alpha(a, a).$$

Since $h(a)(y) \wedge h(a)(y') = 0$ for all $y \neq y' \in B$ and Ω is totally ordered, for any $a \in A$ there exists at most one element $g(a) \in B$ such that $h(a)(g(a)) > 0$. If $h(a)(y) = 0$ for all $y \in B$, let $g(a) \in B$ be an arbitrary element. Then $g : (A, \alpha) \to (B, \beta)$ is a morphism in $\mathbf{Set}(\Omega)$. In fact, if $h(a)(g(a)) > 0$, then we have

$$\alpha(a) = \Delta_\alpha(a, a) = \tau(h(a), h(a)) = \bigvee_{y \in B} h(a)(y)$$
$$= h(a)(g(a)) = \bigvee_{x \in A} f(x, g(a)) \wedge \Delta_\alpha(a, x) \leq$$
$$f(a, g(a)) \leq \Delta_\beta(g(a), g(a)) = \beta(g(a)).$$

If $h(a)(y) = 0$ for all $y \in B$, then $\alpha(a) = \Delta_\alpha(a, a) = 0 \leq \Delta_\beta(g(a), g(a))$. We show that the function g satisfies the conditions (i) and (ii). Let $h(a)(g(a)) > 0$. Then we have

$$0 < h(a)(g(a)) = \bigvee_{x \in A} f(x, g(a)) \wedge \Delta_\alpha(a, x) \leq f(a, g(a)).$$

Moreover, since f is a morphism in $\mathbf{SetR}(\Omega)_*$, we have

$$(\forall a, b) \quad f(a, b) \wedge f(a, g(a)) \leq \Delta_\beta(b, g(a)),$$

and it follows that $f(a, b) = 0$ for all $b \in B$ such that $b \neq g(a)$. Hence, the condition (i) holds. Since $\alpha(a) = \Delta_\alpha(a, a) = \bigvee_{y \in B} f(a, y) = f(a, g(a))$, the condition (ii) holds. Now, if $h(a)(y) = 0$ for all $y \in B$, it follows that $f(a, y) = 0$ for all $y \in B$ and the conditions (i),(ii) hold as well.

(b)⇒(c). This is a trivial computation only.

(c)⇒(a). Let $g : (A, \alpha) \to (B, \beta)$ be a morphism in $\mathbf{Set}(\Omega)$ and let $f : A \times B \to \Omega$ be a function from the statement (c). Then for any $a \in A$ we have

$$\bigvee_{b \in B} f(a, b) = \bigvee_{b \in B} (\alpha(a) \wedge \Delta_\beta(g(a), b)) =$$
$$= \alpha(a) \wedge \Delta_\beta(g(a), g(a)) = \alpha(a) \wedge \beta(g(a)) = \alpha(a).$$

and $f : (A, \Delta_\alpha) \to (B, \Delta_\beta)$ is a morphism in $\mathbf{SetR}(\Omega)_*$. ■

It is clear that Proposition 2 introduces a map

$$\varphi : \mathbf{Hom}_{\mathbf{Set}(\Omega)}((A, \alpha), (B, \beta)) \to \mathbf{Hom}_{\mathbf{SetR}(\Omega)_*}((A, \Delta_\alpha), (B, \Delta_\beta))$$

such that for any morphism $f : (A, \alpha) \to (B, \beta)$ in $\mathbf{Set}(\Omega)$, $\varphi(f) : (A, \Delta_\alpha) \to (B, \Delta_\beta)$ is a morphism in $\mathbf{SetR}(\Omega)_*$ such

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that $\varphi(f)(a,b) = \alpha(a) \wedge \Delta_\beta(f(a),b)$ for all $a \in A, b \in B$. On the other hand, there exists a relation

$$\psi \subseteq \mathbf{Hom}_{\mathbf{SetR}(\Omega)_*}((A,\Delta_\alpha),(B,\Delta_\beta)) \times \mathbf{Hom}_{\mathbf{Set}(\Omega)}((A,\alpha),(B,\beta))$$

such that $(f,g) \in \psi$ if g is a morphism from (c) (Proposition 2) for a morphism f. It should be observed that (on the contrary to φ) ψ is not a map, in general. In fact, let (A,α) be an object in $\mathbf{Set}(\Omega)$ such that $\alpha(a) = 0$ for some $a \in A$. Let $f : (A,\Delta_\alpha) \to (B,\Delta_\beta)$ be a morphism in $\mathbf{SetR}(\Omega)_*$ and let $g : (A,\alpha) \to (B,\beta)$ be a morphism in $\mathbf{Set}(\Omega)$ satisfying the condition (c) from Proposition 2. Let us define a map $g' : A \to B$ such that $g'(x) = g(x)$ for all $x \in A, x \neq a$ and $g'(a) \neq g(a)$. Then $g' : (A,\alpha) \to (B,\beta)$ is a morphism, $(f,g),(f,g') \in \psi$ and $g' \neq g$. Hence, ψ is not a map.

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