A Predator-Prey Model with Fuzzy Initial Populations

Muhammad Zaini Ahmad   Bernard De Baets

Department of Applied Mathematics, Biometrics and Process Control, Ghent University
Coupure links 653, B-9000 Ghent, Belgium
Email: mzaini_ahmad@yahoo.com, Bernard.DeBaets@ugent.be

Abstract— *The aim of this paper is to study a predator-prey population model which takes into account the uncertainty that arises when determining the initial populations of predator and prey . This model is solved numerically by means of a 4^{th}-order Runge-Kutta method. Simulations are made and a graphical representation is also provided to show the evolution of both populations over time. In addition to that, a new phase-plane in this fuzzy setting, referred to as fuzzy phase-plane, is introduced. The stability of the equilibrium points is also described. Finally, this paper points out a new research direction on fuzzy dynamical systems, especially in non-linear cases.*

Keywords— Equilibrium points, Fuzzy predator-prey population model, Fuzzy phase-plane, Fuzzy stability, Uncertainty.

1 Introduction

Nowadays, differential equations have proved a successful modelling paradigm. In a wide range of disciplines, the behaviour of an idealised version of a problem under study has been adequately described by one or more ordinary differential equations. Real-world problems, however, are pervaded with uncertainty. According to Diniz *et al.* [1], the uncertainty can arise in the experimental part, the data collection, the measurement process, as well as when determining the initial conditions. These are patently obvious when dealing with "living" material, such as soil, water, microbial populations, etc. In order to handle these problems, the use of fuzzy sets may be seen as an effective tool for a better understanding of the studied phenomena. It is therefore not surprising that there is a vast literature dealing with fuzzy differential equations (FDEs) for modelling this kind of perception. In the study of differential equations in a fuzzy environment, the term fuzzy differential equations is used for referring to differential equations with fuzzy coefficients, differential equations with fuzzy initial values or fuzzy boundary values, or even differential equations dealing with functions on the space of fuzzy intervals (see [2, 3, 4, 5, 6, 7, 8]).

In population dynamics, Bassanezi *et al.* [9] have started using FDEs to study the stability of fuzzy dynamical systems. The variables and/or the parameters in the systems under consideration were supposed to be uncertain and modelled by fuzzy sets. The authors have suggested a new definition of fixed point: if f is a continuous function, then u is a fixed point if $f([u]^\alpha) = [u]^\alpha$ for $\alpha \in [0,1]$, where $[u]^\alpha = [u_1^\alpha, u_2^\alpha] = [\min_{x\in[u_1^\alpha,u_2^\alpha]} f(x), \max_{x\in[u_1^\alpha,u_2^\alpha]} f(x)]$ (see [9] for more details). According to Ortega *et al.* [10], the proposed idea is not easy to implement in practice because several details should be treated carefully. Guo *et al.* [11] have adopted Hüllermeier's approach [7] to establish fuzzy impulsive functional differential equations and some results were applied in the logistic model and the Gompertz model. In [12], the author has established existence and uniqueness results for fuzzy functional differential equations. These results were applied in population models whose solutions define fuzzy intervals. Recently, Peixoto *et al.* [13] have studied the fuzzy predator-prey population model. The authors have elaborated the classical deterministic model by means of a fuzzy rule-based system. The fuzzy rule-based system considered in their work consists of two inputs and two outputs. The fuzzy rule base is given by 30 rules of the type: "If the number of preys is large AND the potential of predation is very small, THEN the variation of preys increase a little AND the variation of the potential of predation increase a lot". The words such as "large", "very small", "increase a little" and "increase a lot" are modelled by fuzzy sets. This might be a possible way to treat the model in a fuzzy setting. In this paper, we propose a new way of studying the fuzzy predator-prey population model. For this purpose, we only consider the initial populations of predator and prey to be fuzzy intervals.

2 Preliminary Concepts

In this section, let us first introduce the notations and some theoretical background we will be using throughout this paper. Denote $\mathcal{F}(\mathbb{R}) = \{U \mid U : \mathbb{R} \to [0,1]\}$, where U satisfies the following conditions:

1. U is normal: there exists $x_0 \in \mathbb{R}$ such that $U(x_0) = 1$;
2. U is convex: for $x, y \in \mathbb{R}$ and $0 \le \lambda \le 1$, one has
$$U(\lambda x + (1-\lambda)y) \ge \min\left(U(x), U(y)\right);$$
3. U is upper semi-continuous: for any $x_0 \in \mathbb{R}$, one has
$$U(x_0) \ge \lim_{x\to x_0^\pm} U(x);$$
4. $[U]^0 = \overline{\{x \in \mathbb{R} \mid U(x) > 0\}}$ is a compact subset of $\mathbb{R}$.

If U satisfies (1)–(4), then it is called a fuzzy interval. We can define its α-cuts as follows:

$$[U]^\alpha = \{x \in \mathbb{R} \mid U(x) \ge \alpha\}, \;\; 0 < \alpha \le 1$$

For a fuzzy interval U, its α-cuts are closed intervals in $\mathbb{R}$. Let us denote them by $[U]^\alpha = [u_1^\alpha, u_2^\alpha]$.

In what follows, we introduce some results we adopt from [14] concerning the notion of interactivity of fuzzy intervals.

ISBN: 978-989-95079-6-8

Definition 1 *Two fuzzy intervals A and B are said to be interactive fuzzy intervals if there exist $q, r \in \mathbb{R}$, $q \neq 0$, such that their joint possibility distribution is given by*

$$\begin{aligned} C(x_1, x_2) &= A(x_1) \cdot \chi_{\{qx_1+r=x_2\}}(x_1, x_2) \\ &= B(x_2) \cdot \chi_{\{qx_1+r=x_2\}}(x_1, x_2) \end{aligned} \quad (1)$$

where $\chi_{\{qx_1+r=x_2\}}(x_1, x_2)$ stands for the characteristic function of the line

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \mid qx_1 + r = x_2\right\}.$$

In this case we have

$$[C]^\alpha = \left\{(x, qx + r) \in \mathbb{R}^2 \mid x = (1 - t)a_1^\alpha + ta_2^\alpha, t \in [0, 1]\right\}$$

where $[A]^\alpha = [a_1^\alpha, a_2^\alpha]$ and $[B]^\alpha = q[A]^\alpha + r$ for any $\alpha \in [0, 1]$, and, finally

$$B(x) = A\left(\frac{x - r}{q}\right)$$

for all $x \in \mathbb{R}$.

Definition 2 *Two fuzzy intervals A and B are said to be completely positively interactive if q is positive in (1).*

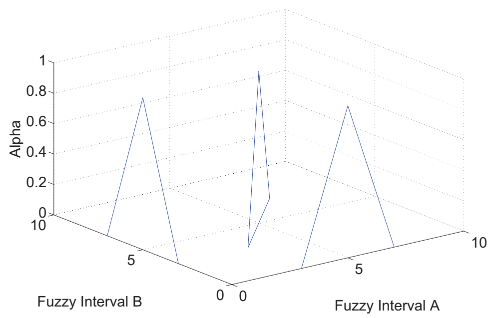

Figure 1: Completely positively interactive fuzzy intervals.

Definition 3 *Let C be a joint possibility distribution with marginal possibility distributions A and B, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. If A and B are interactive fuzzy intervals, then the extension of f via C is defined as*

$$f(A, B)(y) = \sup_{y=f(x_1,x_2)} C(x_1, x_2), \quad \forall y \in \mathbb{R}. \quad (2)$$

Remark 1 *If A and B are non-interactive fuzzy intervals, that is, their joint possibility distribution is defined by*

$$C(x_1, x_2) = \min\left(A(x_1), B(x_2)\right),$$

then (2) turns into the extension principle introduced by Zadeh [15].

Definition 4 *If A and B are two fuzzy intervals, then the distance D between A and B is defined as*

$$D(A, B) = \sup_{\alpha\in[0,1]} d_H([A]^\alpha, [B]^\alpha)$$

where

$$d_H([A]^\alpha, [B]^\alpha) =$$

$$\max\left\{\sup_{a\in[A]^\alpha} \inf_{b\in[B]^\alpha} d(a, b), \sup_{b\in[B]^\alpha} \inf_{a\in[A]^\alpha} d(a, b)\right\}$$

3 Two-Dimensional Autonomous Systems

In this section, we first consider a general two-dimensional autonomous system with crisp initial values:

$$\begin{cases} x'(t) = f(x, y), & x(t_0) = x_0 \\ y'(t) = g(x, y), & y(t_0) = y_0 \end{cases} \quad (3)$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and $x_0, y_0 \in \mathbb{R}$. This model is called an autonomous system because the functions appearing on the right-hand side of (3) do not depend on the independent variable t. Suppose that the initial values in (3) are uncertain and modelled by fuzzy intervals, then we have the following fuzzy autonomous system [16]:

$$\begin{cases} x'(t) = f(x, y), & x(t_0) = X_0 \\ y'(t) = g(x, y), & y(t_0) = Y_0 \end{cases} \quad (4)$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions and $X_0, Y_0 \in \mathcal{F}(\mathbb{R})$ with $[X_0]^\alpha = [x_{01}^\alpha, x_{02}^\alpha]$ and $[Y_0]^\alpha = [qx_{01}^\alpha + r, qx_{02}^\alpha + r]$ for $\alpha \in [0, 1]$, $q \in \mathbb{R}^+$ and $r \in \mathbb{R}$. Here we assume that X_0 and Y_0 are completely positively interactive fuzzy intervals.

A solution to (4) can be approximated by any numerical method existing in the literature. For any fixed value of t, the solution is represented as a joint possibility distribution in the xy-plane. As t varies, the solution traces out the joint possibility distributions in the plane. By plotting a collection of such joint possibility distributions, we gain graphical insight into the behaviour of solutions. In this paper, the set of joint possibility distributions and the xy-plane are called fuzzy trajectories and the fuzzy phase-plane, respectively.

4 Numerical Methods

In many cases, model (4) presented in Section 3 cannot be solved analytically and therefore we need a numerical scheme to approximate the exact solution. We consider the following 4^{th}-order Runge-Kutta method:

$$x_{i+1} = x_i + \frac{h}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] \quad (5)$$

$$y_{i+1} = y_i + \frac{h}{6}\left[L_1 + 2L_2 + 2L_3 + L_4\right] \quad (6)$$

where

$$\begin{aligned} K_1 &= f(x_i, y_i) \\ L_1 &= g(x_i, y_i) \\ K_2 &= f\left(x_i + \frac{h}{2}K_1, y_i + \frac{h}{2}L_1\right) \\ L_2 &= g\left(x_i + \frac{h}{2}K_1, y_i + \frac{h}{2}L_1\right) \\ K_3 &= f\left(x_i + \frac{h}{2}K_2, y_i + \frac{h}{2}L_2\right) \\ L_3 &= g\left(x_i + \frac{h}{2}K_2, y_i + \frac{h}{2}L_2\right) \\ K_4 &= f(x_i + hK_3, y_i + hL_3) \\ L_4 &= g(x_i + hK_3, y_i + hL_3) \end{aligned}$$

ISBN: 978-989-95079-6-8

Since the arguments x_i and y_i on the right-hand side of (5) and (6) are interactive, we can define new functions as follows:

$$F_h(x_i, y_i) = x_i + \frac{h}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] \quad (7)$$

$$G_h(x_i, y_i) = y_i + \frac{h}{6}\left[L_1 + 2L_2 + 2L_3 + L_4\right] \quad (8)$$

By using the concept of interactivity of fuzzy intervals proposed by Majlender [14], we have the following generalised 4^{th}-order Runge-Kutta method for solving (4), where X_0 and Y_0 are interactive fuzzy intervals.

$$X_{i+1} = F_h(X_i, Y_i) \quad (9)$$

$$Y_{i+1} = G_h(X_i, Y_i). \quad (10)$$

The idea is that if $X_i(u) \geq \alpha$ for some $u \in \mathbb{R}$, then there exists a unique $v \in \mathbb{R}$ that Y_i can take. Furthermore, if u is moved to the left (right), then the corresponding value (that Y_i can take) will also move to the left (right). Vice versa, if u is moved to the left (right), then the corresponding value (that Y_i can take) will move to the right (left) [14]. From this explanation, we can compute the numerical approximations as follows:

$$x^\alpha_{1,i+1} = F_h(x^\alpha_{1,i}, y^\alpha_{1,i}) \quad (11)$$

$$x^\alpha_{2,i+1} = F_h(x^\alpha_{2,i}, y^\alpha_{2,i}) \quad (12)$$

$$y^\alpha_{1,i+1} = G_h(x^\alpha_{1,i}, y^\alpha_{1,i}) \quad (13)$$

$$y^\alpha_{2,i+1} = G_h(x^\alpha_{2,i}, y^\alpha_{2,i}) \quad (14)$$

Our purpose here is to generate accurate approximations at each α-cut. We begin by making a partition of the form $t_0 < t_1 < t_2 < ... < t_{N-1} < t_N = T$ on the interval $[t_0, T]$. This partition is uniformly spaced, that is the partition points are $t_i = t_0 + ih$, $i = 0, 1, 2, ..., N$ and the partition spacing $h = \frac{T-t_0}{N} > 0$ is sufficiently small and we called it the step-size or step-length.

5 Stability

In this section, we consider the following fuzzy autonomous system in vector form [16]:

$$\mathbf{z}'(t) = \mathbf{f}(\mathbf{z}), \quad \mathbf{z}(t_0) = \mathbf{Z}_0 \quad (15)$$

where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{Z}_0 \in \mathcal{F}(\mathbb{R}^2)$ is a joint possibility distribution of which the marginal possibility distributions are interactive fuzzy intervals.

Next, we give several definitions and sufficient conditions for stability of the model studied.

Definition 5 *A point $\mathbf{z}_e$ is called an equilibrium point of (15) at time t_0, if for all $t \geq t_0$, it holds that $\mathbf{f}(\mathbf{z}_e) = 0$.*

This definition asserts that an equilibrium point $\mathbf{z}_e$ is a constant solution of (15), since $\mathbf{f}(\mathbf{z}_e) = 0$ and thus $\mathbf{z}'(t) = 0$ at such a point.

Lemma 1 *If $\mathbf{Z}$ is joint possibility distribution and $\mathbf{z}_e$ is an equilibrium point, then*

$$D(\mathbf{Z}, \mathbf{z}_e) = \sup_{\mathbf{s} \in [\mathbf{Z}]^0} d(\mathbf{s}, \mathbf{z}_e)$$

Definition 6 *An equilibrium point $\mathbf{z}_e$ of (15) is defined to be fuzzy stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $D(\mathbf{Z}(t_0), \mathbf{z}_e) < \delta$ then $D(\mathbf{Z}(t), \mathbf{z}_e) < \epsilon$ for all $t \geq t_0$.*

This definition states that all solutions of (15) that start "sufficiently close" to an equilibrium point $\mathbf{z}_e$ stay "close" to $\mathbf{z}_e$. Note that this definition of fuzzy stability does not require the solution to approach the equilibrium point.

Proposition 1 *If an equilibrium point $\mathbf{z}_e$ is stable, then it is also fuzzy stable.*

Definition 7 *An equilibrium point $\mathbf{z}_e$ of (15) is defined to be asymptotically fuzzy stable if it is fuzzy stable and there exists $\delta > 0$ such that if $D(\mathbf{Z}(t_0), \mathbf{z}_e) < \delta$ then $\lim_{t\to\infty} D(\mathbf{Z}(t), \mathbf{z}_e) = 0$ for all $t \geq t_0$.*

This definition asserts that solutions of (15) that start "sufficiently close" to an equilibrium point $\mathbf{z}_e$ not only stay "close" to $\mathbf{z}_e$, but must eventually approach the equilibrium point $\mathbf{z}_e$ as t approaches infinity. Note that this is a stronger requirement than fuzzy stability, since the equilibrium point $\mathbf{z}_e$ must be stable before one can talk about whether or not it is asymptotically fuzzy stable.

Proposition 2 *If an equilibrium point $\mathbf{z}_e$ is asymptotically stable, then it is also asymptotically fuzzy stable.*

Definition 8 *An equilibrium point $\mathbf{z}_e$ of (15) is defined to be fuzzy unstable if it is not fuzzy stable. Hence, there exists an $\epsilon > 0$ such that for any choice of $\delta > 0$, there exists $\mathbf{Z}(t)$ satisfying $D(\mathbf{Z}(t_0), \mathbf{z}_e) < \delta$ such that $D(\mathbf{Z}(t), \mathbf{z}_e) \geq \epsilon$ for some t.*

This definition states that all solutions of (15) that start "sufficiently close" to an equilibrium point $\mathbf{z}_e$ move away from $\mathbf{z}_e$ when t increases.

Proposition 3 *If an equilibrium point $\mathbf{z}_e$ is unstable, then it is also fuzzy unstable.*

6 Fuzzy Predator-Prey Population Model

Consider the following fuzzy predator-prey population model

$$\begin{cases} x'(t) = x - 0.1xy \\ y'(t) = -0.5y + 0.02xy \end{cases}$$

where the initial populations of prey and predator are interactive fuzzy intervals with $q = 1$ and $r = -5$, and they are defined as follows, respectively:

$$X_0 = \begin{cases} 0, & \text{if } x < 14, \\ x - 14, & \text{if } 14 \leq x < 15, \\ -x + 16, & \text{if } 15 \leq x < 16, \\ 0, & \text{if } x > 16 \end{cases}$$

and

$$Y_0 = \begin{cases} 0, & \text{if } y < 9, \\ y - 9, & \text{if } 9 \leq y < 10, \\ -y + 11, & \text{if } 10 \leq y < 11, \\ 0, & \text{if } y > 11 \end{cases}$$

ISBN: 978-989-95079-6-8

First we find the equilibrium points of the model by setting the derivatives equal to zero. One can note that the model has two equilibrium points: $(0, 0)$ and $(25, 10)$. The equilibrium point $(0, 0)$ is uninteresting because there are no populations to observe in the model. However, the second equilibrium point is of interest. From the numerical approximation, we can see that the point $(25, 10)$ is fuzzy stable since the supports of the joint possibility distributions move around the equilibrium point when t increases (see Fig. 4). This is true since the prey population increases when the predator population is minimum, and the prey population decreases when the predator population is maximal (see Figs. 2 and 3).

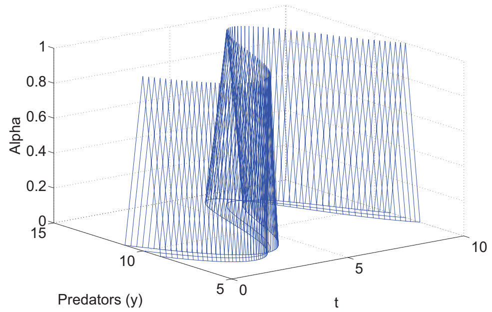

Figure 2: The evolution of the predator population over time.

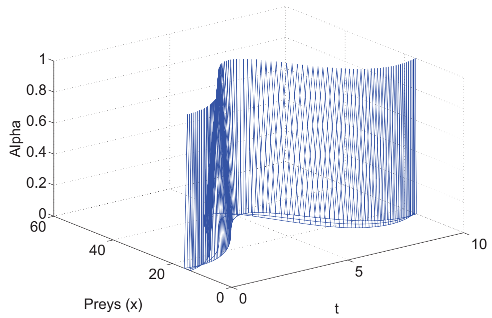

Figure 3: The evolution of the prey population over time.

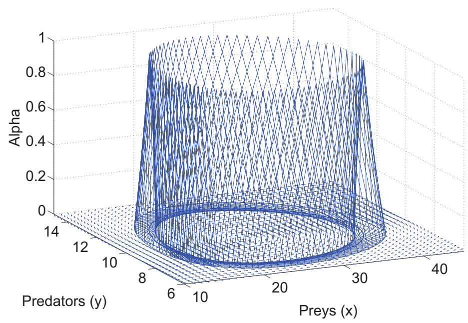

Figure 4: Fuzzy phase-plane with direction field ($N = 100$, $t = 10$, $h = 0.1$).

7 Conclusions

In this paper, we have proposed a new way of studying predator-prey population models, which takes into account the uncertainty that arises when determining the initial populations of predator and prey. By using the concept of interactivity of fuzzy intervals proposed by Majlender [14], we have developed a generalised numerical method, based on a 4^{th}-order Runge-Kutta method, to approximate the number of predators and preys over time.

Acknowledgment

This research was co-funded by the Ministry of Higher Education of Malaysia (MOHE) and partially supported by Universiti Malaysia Perlis (UniMAP) under the program "Skim Latihan Akademik IPTA (SLAI)".

References

[1] G.L. Diniz, J.F.R. Fernandes, J.F.C.A. Meyer, and L.C. Barros. A fuzzy Cauchy problem modelling the decay of the biochemical oxygen demand in water. *Proc. Joint 9th IFSA World Congress and 20th NAFIPS International Conferences*, 1:512–516, 25-28 July 2001.

[2] S. Abbasbandy and T. Allahviranloo. Numerical solution of fuzzy differential equation by Runge-Kutta method. *Nonlinear Studies*, 11(1):117–129, 2004.

[3] M. Ma, M. Friedman, and A. Kandel. Numerical solution of fuzzy differential equations. *Fuzzy Sets and Systems*, 105:133–138, 1999.

[4] B. Bede and S.G. Gal. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets and Systems*, 151:581–599, 2005.

[5] J.J. Buckley and T. Feuring. Fuzzy differential equations. *Fuzzy Sets and Systems*, 110:43–54, 2000.

[6] Y. Chalco-Cano and H. Román-Flores. On new solutions of fuzzy differential equations. *Chaos, Solitons and Fractals*, 38 (1):112–119, 2008.

[7] E. Hüllermeier. An approach to modelling and simulation of uncertain dynamical systems. *Int. J. Uncertainty Fuzziness Knowledge-Based Systems*, 5:117–137, 1997.

[8] M.T. Mizukoshi, L.C Barros, Y. Chalco-Cano, H. Román-Flores, and R.C. Bassanezi. Fuzzy differential equations and the extension principle. *Information Sciences*, 177:3627–3635, 2007.

[9] R. C Bassanezi, L.C. Barros, and P.A. Toneli. Attractor and asymptotic stability for dynamical systems. *Fuzzy Sets and Systems*, 113:473–483, 2000.

[10] N. R. S. Ortega, P. C. Sallum, and E. Massad. Fuzzy dynamical systems in epidemic modelling. *Kybernetes*, 29 (2):201–218, 2000.

[11] M. Guo, X. Xu, and R. Li. Impulsive functional differential inclusions and fuzzy population models. *Fuzzy Sets and Systems*, 138:601–615, 2003.

[12] V. Lupulescu. On a class of fuzzy functional differential equations. *Fuzzy Sets and Systems*, 160:1547–1562, 2009.

[13] M.S. Peixoto, L.C. Barros, and R.C. Bassanezi. Predator-prey fuzzy model. *Ecological Modelling*, 214:39–44, 2008.

[14] C. Carlsson, R. Fullér, and P. Majlender. Additions of completely correlated fuzzy numbers. *Proceeding of IEEE International Conference on Fuzzy Systems*, 1:353–539, 25-29 July 2004.

[15] L.A. Zadeh. Fuzzy sets. *Information and Control*, 8:338–353, 1965.

[16] S. Seikkala. On the fuzzy initial value problem. *Fuzzy Sets and Systems*, 24:319–330, 1987.

ISBN: 978-989-95079-6-8