Exchanging iterated expectations of random upper semicontinuous functions: an application to decision theory

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Abstract— *In this paper we present a procedure to deal with a kind of single-stage decision problems with imprecise utilities. In this type of problems the product measurability of the utility function is not required. So that, the involved expectations are calculated by means of iterated integrals instead of integrals over product spaces.*

Keywords— Bayesian decision analysis, Iterated expectation, Kudo-Aumann's integral, Random upper semicontinuous function, Uncertainty modeling.

1 Introduction

There are situations in which product measurability of certain mappings is not satisfied, but iterated integrals are well-defined (see [10]). This type of situations appear in single-stage decision problems where we need conditions which allows us to exchange iterated expectations in order to perform a Bayesian analysis. This paper studies the case of Bayesian analysis of single-stage decision problems with imprecise utilities and non product-measurable utility funtion.

Several studies have been developed before to evaluate imprecise utilities, see for instance Watson *et al.* [27], Tong and Bonissone [26], Dubois aand Prade [7, 8, 9], Gil and Jain [11], Billot [1], Chen and Klein [3], Gil *et al.* [13], Krätschmer [16], Bordley [2], Rébillé [22] or Rodríguez-Muñiz and López-Díaz [25].

Here we model the imprecise utilities by means of fuzzy-valued utility functions (based on the concept of random upper semicontinuous functions or fuzzy random variable). However, only in [25] the non-product measurable case has been analyzed for this type of furrzy utilities, based on the theoretical results on iterated integrals of random upper semicontinuous functions given in [24].

In this paper we gather theoretical and applied results about how to deal with the type of problems referred above.

The paper is organized as follows: Preliminaries and notation constitutes Section 2, Section 3 contains theoretical results and Section 4 gather the statistical decision analysis results.

2 Preliminaries

Let us consider $\mathcal{K}_c$ the class of nonempty compact convex subsets of $\mathbb{R}$, endowed with a semilinear structure by means of the Minkowski addition and the product by a scalar. Also, consider the *Hausdorff metric* on $\mathcal{K}_c$ (see [6]). On a measurable space $(\Omega, \mathcal{A})$ we can define $S : \Omega \to \mathcal{K}_c$ a randon set as a $\mathcal{A}|\mathcal{B}_{d_H}$-measurable mapping ([14]).

A random set S is said to be *integrably bounded with respect to measure* μ, if $\|S\| \in L^1(\Omega, \mathcal{A}, \mu)$, where $\|S\|(\omega) = \sup_{x\in S(\omega)} \|x\|$. The *integral*, or *expected value* in case of μ being a probability, of S, is given by the *Kudo-Aumann integral* and it will be denoted either by $\int_\Omega S(\omega)\, d\mu(\omega)$ or $E(S|\mu)$ ([14]).

Let $\mathcal{F}_c$ denote the class of upper semicontinuous functions or *fuzzy sets* $U : \mathbb{R} \to [0,1]$ such that $U_\alpha \in \mathcal{K}_c$ for all $\alpha \in [0,1]$, where $U_\alpha = \{x \in \mathbb{R} : U(x) \geq \alpha\}$ for $\alpha \in (0,1]$, and $U_0 = \text{cl}\{x \in \mathbb{R} : U(x) > 0\}$, cl denoting the topological closure.

The class $\mathcal{F}_c$ can be endowed with a semilinear structure, where addition and product by a scalar can be defined by means of Zadeh's extension principle ([28, 19]). On $\mathcal{F}_c$ we consider the d_∞ metric ([19]). The *magnitude* of $U \in \mathcal{F}_c$ is given by $\|U\| = d_\infty(U, \mathbf{1}_{\{0\}}) = d_H(U_0, \{0\})$.

Given a measurable space $(\Omega, \mathcal{A})$, a mapping $X : \Omega \to \mathcal{F}_c$ is said to be a *random upper semicontinuous function* (r.u.s.f. for short) if $X_\alpha : \Omega \to \mathcal{K}_c$ with $X_\alpha(\omega) = (X(\omega))_\alpha$ for all $\omega \in \Omega$, is a random set for all $\alpha \in [0,1]$ ([21, 4]).

A r.u.s.f. X is said to be *integrably bounded* with respect to a measure $\mu : \mathcal{A} \to \mathbb{R}$, if the mapping $\|X\| \in L^1(\Omega, \mathcal{A}, \mu)$, where $\|X\| : \Omega \to \mathbb{R}$ is given by $\|X\|(\omega) = \|X(\omega)\|$ for all $\omega \in \Omega$.

For an integrably bounded r.u.s.f., in [21] is defined its *integral*, denoted by $\int_\Omega X(\omega)\, d\mu(\omega)$ or $E(X|\mu)$, as the unique set in $\mathcal{F}_c$ such that $E(X|\mu)_\alpha = E\big(X_\alpha|\mu\big)$ for every $\alpha \in [0,1]$. When $\Omega = [a,b]$, we will use also the notation $\int_a^b X(\omega)\, d\mu(\omega)$.

If μ is a probability measure, an r.u.s.f. is also referred to as a *fuzzy random variable* and its integral as *the fuzzy expected value of* X.

It is possible to extend to upper semicontinuous functions the concept of *Hukuhara difference* or *Minkowski difference* for subsets ([15, 20]), so given $U, V \in \mathcal{F}_c$, its *Hukuhara difference*, denoted by $U -_h V$, is the set $W \in \mathcal{F}_c$ (if it exists) such that $U = V + W$.

Let T be a nonempty open subset of $\mathbb{R}$. A mapping $G : T \to \mathcal{F}_c$ is said to be *Hukuhara differentiable* at $t_0 \in T$ if there exists $G'(t_0) \in \mathcal{F}_c$, which is called the *Hukuhara differential of* G *at* t_0, such that

$$\lim_{\Delta t \to 0^+} d_\infty\left(\frac{G(t_0 + \Delta t) -_h G(t_0)}{\Delta t}, G'(t_0)\right)$$

$$= \lim_{\Delta t \to 0^+} d_\infty\left(\frac{G(t_0) -_h G(t_0 - \Delta t)}{\Delta t}, G'(t_0)\right) = 0.$$

The above definition ([20, 23]) is an extension of the Hukuhara's differentiability criterion for set-valued mappings ([15]).

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If a mapping G depends on more than one argument, we will make use of the usual symbol of partial derivative to indicate with respect to which argument the Hukuhara differential is considered.

Throughout the paper, for any set $\Omega \subset \mathbb{R}^k$ with $k \in \mathbb{N}$, $\mathcal{B}_\Omega$ will denote the Borel σ-field on Ω. Given $(\Omega, \mathcal{B}_\Omega)$ and $m_1, m_2 : \mathcal{B}_\Omega \to [0, \infty]$ two σ-finite measures, $m_1 \ll m_2$ will indicate that m_1 is absolutely continuous with respect to m_2, and $\frac{dm_1}{dm_2}$ will denote a Radon-Nikodym derivative of m_1 with respect to m_2. If it is supposed that there exists a continuous Radon-Nikodym derivative, then $\frac{dm_1}{dm_2}$ will denote this particular function.

As we will model the imprecise utilities by means of fuzzy-valued functions it should be necessary to rank fuzzy sets. For this purpose the ranking criterion introduced by De Campos and González [5] will be considered. Therefore, we will say that $U \in \mathcal{F}_c$ is *greater than or equal to* $W \in \mathcal{F}_c$ *in the* λ, μ*-average sense*, we will denote it by $U \geq_{\lambda,\mu} W$, if $V_\mu^\lambda(U) \geq V_\mu^\lambda(W)$, where $\lambda \in [0, 1]$ represents a kind of degree of optimism/pessimism and μ is a measure on $[0, 1]$ (see [18] for more details).

3 Theoretical results

We include in this section those theoretical results regarding the exchange of iterated integrals that are necessary to prove the applied results in Section 4.

The first result is about differentiating under the integral sign, by using Hukuhara derivative.

Proposition 3.1 *Let $(\Omega, \mathcal{A}, P)$ be a probability space with $\Omega \subset \mathbb{R}^k$, and let m denote the Borel measure on the interval $[a, b]$. For every $\omega \in \Omega$, let P_ω be a probability measure on $([a, b], \mathcal{B}_{[a,b]})$ with $P_\omega \ll m$, such that there exists a continuous Radon-Nikodym derivative.*

If $X : \Omega \times [a, b] \to \mathcal{F}_c$ satisfies the following conditions:

i) for every $\omega \in \Omega$, the mapping $X_\omega : [a, b] \to \mathcal{F}_c$, with $X_\omega(t) = X(\omega, t)$, is an integrably bounded r.u.s.f. with respect to P_ω, and X_ω is continuous $a.s.\,[P]$,

ii) there exists $h \in L^1(\Omega, \mathcal{A}, P)$, such that $\left\|X(\omega, t)\frac{dP_\omega}{dm}(t)\right\| \leq h(\omega)\, a.s.\,[P]$ for every $t \in [a, b]$, and the mapping $\omega \mapsto X(\omega, t)\frac{dP_\omega}{dm}(t)$ is continuous $a.e.\,[m]$,

iii) there exists $g \in L^1([a, b], \mathcal{B}_{[a,b]}, m)$ with $\left\|X(\omega, t)\frac{dP_\omega}{dm}(t)\right\| \leq g(t)\, a.e.\,[m]$ for every $\omega \in \Omega$,

then, the mapping

$$t \in [a, b] \mapsto \int_\Omega \Big(\int_a^t X(\omega, s)\, dP_\omega(s) \Big)\, dP(\omega)$$

is Hukuhara differentiable on (a, b), and for every $t \in (a, b)$ it holds that

$$\frac{\partial}{\partial t} \int_\Omega \left(\int_a^t X(\omega, s)\, dP_\omega(s) \right)\, dP(\omega)$$

$$= \int_\Omega X(\omega, t)\frac{dP_\omega}{dm}(t)\, dP(\omega).$$

Starting from last result, we can go one step further to prove a exchange of iterated integrals:

Theorem 3.2 *Let $(\Omega, \mathcal{B}_\Omega, P)$ be a probability space with $\Omega \subset \mathbb{R}^k$ and let m denote the Borel measure on the interval $T = [a, b]$. For every $t \in T$, let P_t be a probability measure on $(\Omega, \mathcal{B}_\Omega)$ such that $P_t \ll P$ and there exists a continuous Radon-Nikodym derivative. For every $\omega \in \Omega$, let P_ω be a probability on $(T, \mathcal{B}_T)$ such that $P_\omega \ll m$ and there exists a continuous Radon-Nikodym derivative.*

Let $X : \Omega \times T \to \mathcal{F}_c$ be a mapping satisfying that:

i) for every $t \in T$, X_t is an integrably bounded r.u.s.f. with respect to P_t,

ii) for every $\omega \in \Omega$, X_ω is an integrably bounded r.u.s.f. with respect to P_ω and it is continuous $a.s.\,[P]$,

iii) there exists $h_1 \in L^1(\Omega, \mathcal{B}_\Omega, P)$ such that $\left\|X(\omega, t)\frac{dP_\omega}{dm}(t)\right\| \leq h_1(\omega)\ a.s.\,[P]$ for every $t \in T$, and the mapping $\omega \mapsto X(\omega, t)\frac{dP_\omega}{dm}(t)$ is continuous $a.e.\,[m]$,

iv) there exists a mapping $g \in L^1([a, b], \mathcal{B}_{[a,b]}, m)$ such that for every $\omega \in \Omega$, $\|X(\omega, t)\frac{dP_\omega}{dm}(t)\| \leq g(t)\ a.e.\,[m]$ for every $\omega \in \Omega$,

v) the mapping $t \mapsto X(\omega, t)\frac{dP_t}{dP}(\omega)$ is continuous on T $a.s.\,[P]$,

vi) there exists $h_2 \in L^1(\Omega, \mathcal{B}_\Omega, P)$ such that $\|X(\omega, t)\frac{dP_t}{dP}(\omega)\| \leq h_2(\omega)\ a.s.\,[P]$ for every $t \in T$.

Let m' be a probability measure on $(T, \mathcal{B}_T)$ such that $m' \ll m$ and there exists a continuous Radon-Nikodym derivative. If for every $t \in T$, the equality

$$\frac{dP_\omega}{dm}(t) = \frac{dP_t}{dP}(\omega)\frac{dm'}{dm}(t) \quad a.s.\,[P]$$

holds, then

$$\int_\Omega \left(\int_a^t X(\omega, s)\, dP_\omega(s) \right) dP(\omega)$$

$$= \int_a^t \left(\int_\Omega X(\omega, s)\, dP_s(\omega) \right) dm'(s)$$

for every $t \in T$.

And also an unbounded version of previous theorem can be obtained.

Theorem 3.3 *Assume the conditions in Theorem 3.2 with the interval T being not necessarily bounded, and suppose that there exists $g' \in L^1(\Omega, \mathcal{B}_\Omega, P)$ such that*

$$\int_T \|X(\omega, s)\| dP_\omega(s) \leq g'(\omega)\ \ a.s.\,[P]\,.$$

Then, the following equality holds,

$$\int_\Omega \left(\int_T X(\omega, s)\, dP_\omega(s) \right) dP(\omega)$$

$$= \int_T \left(\int_\Omega X(\omega, s)\, dP_s(\omega) \right) dm'(s)\,.$$

It should be remarked that the conditions in Theorems 3.2 and 3.3 do not imply that X is an r.u.s.f. on the product measurable space as is illustrated in [24].

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4 Applied results

In this section we are using the theoretical results obtained in Section 3 to obtain applied results to the single-stage decision problem under non-product measurability conditions. Obviously, different conditions than in the Fubini-applicable case will be obtained, but it is again very important to underline that under the conditions stated here product measurability is not implied.

We first introduce the concept of fuzzy utility function considered in this communication. We will use the following notation: Θ is the state space and it will be considered an interval of $\mathbb{R}$, $\mathcal{B}_\Theta$ is the Borel σ-field on Θ, with m the Borel measure and $\mathbf{A}$ is the action space.

Definition 4.1 *A mapping $U : \Theta \times \mathbf{A} \to \mathcal{F}_c$ is said to be a fuzzy utility function on $\Theta \times \mathbf{A}$ if*

$i)$ *for every $a \in \mathbf{A}$, the projection $U_a : \Theta \to \mathcal{F}_c$ is an r.u.s.f. on $(\Theta, \mathcal{B}_\Theta)$,*

$ii)$ *for every pair $a_1, a_2 \in \mathbf{A}$, a_1 will be considered preferred or indifferent to a_2 with respect to a probability distribution ξ on $(\Theta, \mathcal{B}_\Theta)$, if $E(U_{a_1}|\xi) \geq_{\lambda,\mu} E(U_{a_2}|\xi)$ (for fixed $\lambda \in [0, 1]$ and measure μ).*

The decision problem with fuzzy utilities will be denoted by $(\Theta, \mathbf{A}, U)$.

On the other hand, it will be considered a Bayesian context, so the existence of a probability distribution π on $(\Theta, \mathcal{B}_\Theta)$, *the prior distribution*, will be assumed. Then the "value" of the decision problem will be the fuzzy value $E(U_{a^\pi}|\pi)$, where a^π is a *prior Bayes action* in the λ, μ-average sense, this is, $a^\pi \in \mathbf{A}$ verifies $E(U_{a^\pi}|\pi) \geq_{\lambda,\mu} E(U_a|\pi)$ for all $a \in \mathbf{A}$.

Similar to the case of real-valued utilities, it is useful for increasing the expected utility to incorporate *sample information*. Let $\mathbf{X}$ be a statistical experiment characterized by a probability space $(\mathbb{X}, \mathcal{B}_\mathbb{X}, P_\theta)$, where $\theta \in \Theta$, $\mathcal{B}_\mathbb{X}$ is the Borel σ-field on $\mathbb{X} \subset \mathbb{R}^k$ and the experimental distribution P_θ depends on the true unknown state θ. We will denote by P the *marginal (predictive)* distribution of the experiment.

After the experiment is performed, if $\mathbf{X} = x$ is the available sample information, the fuzzy expected utility associated with an action $a \in \mathbf{A}$ is given by $E(U_a|\pi_x)$, where π_x is the *posterior distribution* of θ given $\mathbf{X} = x$, obtained on the basis of Bayes' formula. So, a *posterior Bayes action* is any $a^{\pi_x} \in \mathbf{A}$ such that $E(U_{a^{\pi_x}}|\pi_x) \geq_{\lambda,\mu} E(U_a|\pi_x)$ for every $a \in \mathbf{A}$.

In order to generalize the choice of an action for each possible sample, the concept of *decision rule,* as a mapping from $\mathbb{X}$ to $\mathbf{A}$ satisfying several conditions (based on Theorems 3.2 and 3.3) is formalized. These conditions will allow the proper extension of the normal and extensive forms of the Bayesian analysis.

Definition 4.2 *Let $(\mathbb{X}, \mathcal{B}_\mathbb{X}, P_\theta)$ be the probability space of a statistical experiment $\mathbf{X}$ associated with the decision problem $(\Theta, \mathbf{A}, U)$. A decision rule is a mapping $d : \mathbb{X} \to \mathbf{A}$ satisfying that*

$i)$ *for every $\theta \in \Theta$, $U(\theta, d()) : \mathbb{X} \to \mathcal{F}_c$ is an integrably bounded r.u.s.f. with respect to P_θ,*

$ii)$ *for every $x \in \mathbb{X}$, $U(, d(x)) : \Theta \to \mathcal{F}_c$ is an integrably bounded r.u.s.f. with respect to π_x, moreover, it is continuous $a.s.$ $[P]$,*

$iii)$ *there exists $h_1 \in L^1(\mathbb{X}, \mathcal{B}_\mathbb{X}, P)$ such that $\left\|U(\theta, d(x))\frac{d\pi_x}{dm}(\theta)\right\| \leq h_1(x)$ $a.s.$ $[P]$ for every $\theta \in \Theta$, and the mapping $x \mapsto U(\theta, d(x))\frac{d\pi_x}{dm}(\theta)$ is continuous $a.e.$ $[m]$,*

$iv)$ *there exists $g \in L^1(\Omega, \mathcal{B}_\Omega, m)$ such that for every $x \in \mathbb{X}$, it holds that $\|U(\theta, d(x))\frac{d\pi_x}{dm}(\theta)\| \leq g(\theta)$ $a.e.$ $[m]$ for every $x \in \mathbb{X}$,*

$v)$ *the mapping $\theta \mapsto U(\theta, d(x))\frac{dP_\theta}{dP}(x)$ is continuous on Θ $a.s.$ $[P]$,*

$vi)$ *there exists $h_2 \in L^1(\mathbb{X}, \mathcal{B}_\mathbb{X}, P)$ such that $\left\|U(\theta, d(x))\frac{dP_\theta}{dP}(x)\right\| \leq h_2(x)$ $a.s.$ $[P]$ for every $\theta \in \Theta$,*

$vii)$ *there exists $g' \in L^1(\mathbb{X}, \mathcal{B}_\mathbb{X}, P)$ with $\int_\Theta \|U(\theta, d(x))\|\, d\pi_x(\theta) \leq g'(x)$.*

Now, on the one hand we can consider the *normal Bayesian analysis*. In this case we should find a *Bayes decision rule*, that is, a rule d_B such that

$$\int_\Theta \left(\int_\mathbb{X} U(\theta, d_B(x))\, dP_\theta(x) \right) d\pi(\theta)$$

$$\geq_{\lambda,\mu} \int_\Theta \left(\int_\mathbb{X} U(\theta, d(x))\, dP_\theta(x) \right) d\pi(\theta)$$

for every decision rule d. In this case, the "value" of the problem is

$$\int_\Theta \left(\int_\mathbb{X} U(\theta, d_B(x))\, dP_\theta(x) \right) d\pi(\theta). \quad (1)$$

On the other hand, we can consider the *extensive Bayesian analysis*. We should obtain for each sample information x a posterior Bayes action a^{π_x}, and consider the decision rule which associates with each x an action a^{π_x}. In this analysis, the "value" of the experiment $\mathbf{X}$ is quantified by the fuzzy expected terminal utility, defined as follows:

Definition 4.3 *Given $(\Theta, \mathbf{A}, U)$ a decision problem and $\mathbf{X} = (\mathbb{X}, \mathcal{B}_\mathbb{X}, P_\theta)$, an associated experiment, the fuzzy expected terminal utility of $\mathbf{X}$ is given by*

$$U_t(\mathbf{X}) = \int_\mathbb{X} \left(\int_\Theta U(\theta, a^{\pi_x})\, d\pi_x(\theta) \right) dP(x). \quad (2)$$

We are now showing the equivalence between the two forms of the Bayesian analysis in the sense that (1) and (2) are equal in the λ, μ-average sense. Previously, the following result for the exchange of the integrals is stated.

Theorem 4.4 *Let $(\Theta, \mathbf{A}, U)$ be a decision problem, let $\Theta \subset \mathbb{R}$ and let π be a prior probability on $(\Theta, \mathcal{B}_\Theta)$ such that $\pi \ll m$ with a continuous Radon-Nikodym derivative. Let $\mathbf{X} = (\mathbb{X}, \mathcal{B}_\mathbb{X}, P_\theta)$ be an associated experiment, and let P be the marginal distribution. For every $\theta \in \Theta$, suppose that $P_\theta \ll P$ and there exists a continuous Radon-Nikodym derivative. For every $x \in \mathbb{X}$, let π_x be the posterior distribution on $(\Theta, \mathcal{B}_\Theta)$ such that $\pi_x \ll m$ with a continuous Radon-Nikodym derivative.*

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If for every $\theta \in \Theta$, it holds that $\frac{d\pi_x}{dm}(\theta) = \frac{dP_\theta}{dP}(x)\frac{d\pi}{dm}(\theta)$ $a.s.\,[P]$, then

$$\int_{\mathbb{X}} \left(\int_{\Theta} U(\theta, d(x))\, d\pi_x(\theta) \right) dP(x)$$
$$= \int_{\Theta} \left(\int_{\mathbb{X}} U(\theta, d(x))\, dP_\theta(x) \right) d\pi(\theta)$$

whatever the decision rule $d : \mathbb{X} \to \mathbf{A}$ may be.

As a consequence, the following key result, which states the equivalence between the normal and extensive forms of Bayesian analysis, is obtained.

Theorem 4.5 *Assume the conditions of Theorem 4.4. Let us consider the mapping which associates with each sample $x \in \mathbb{X}$ a posterior Bayes action a^{π_x}. If this mapping satisfies the definition of decision rule, then it is a Bayes decision rule. Moreover, $U_t(\mathbf{X})$ is equal, in the λ, μ-average sense, to the fuzzy expected utility associated with any Bayes decision rule, this is*

$$U_t(\mathbf{X}) =_{\lambda,\mu} \int_{\mathbb{X}} \left(\int_{\Theta} U(\theta, d_B(x))\, d\pi_x(\theta) \right) dP(x)$$

Thus, the fuzzy expected terminal utility, so calculated, can be interpreted as the "value" of the decision problem once the experiment $\mathbb{X}$ is performed and one Bayes decision rule is calculated. This lead us not only to express the information of the problem by a value but also as a criterium to rank experiments in order to obtain the more informative.

5 Conclusions

By using a theoretical result about exchanging iterated integrals of $[0,1]^{\mathbb{R}}$-valued r.u.s.f., the model established in this paper provides a framework for single-stage decision problems in which both forms of Bayesian analysis (normal and extensive) are proved to be equivalent when imprecise utilities are not necessarily product measurable. Thus, these results together with those in [12, 17, 13] cover most of the situations which one can found when analyzing single-stage decision problems with imprecise utilities modeled by fuzzy random variables.

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