On classical, fuzzy classical, quantum, and fuzzy quantum systems

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Abstract*—In this paper we consider physical systems and the concept of their states in the context of the theory of fuzzy sets and systems. In section 1 we give a brief sketch on the fundamental difference between the theories of classical physics and quantum mechanics. In section 2 and 3 we introduce very shortly systems and their states in classical and quantum mechanics, respectively. Section 4 presents the concept of fuzzy systems. We propose to fuzzify the classical systems in section 5 and quantum systems in section 6. In section 7 we start to consider a fuzzy interpretation of the uncertainty principle.*

Keywords— Fuzzy sets and systems, quantum mechanics, system theory, philosophy.

1 Introduction

Due to the scientific revolution brought about by the discovery of quantum mechanics in the first third of the 20th century, a basic change took place in the relationship between the exact scientific theory of physics and the phenomena observed in basic experiments. Systems of quantum mechanics − "quantum systems" − do not behave like systems of classical theories in physics – their elements are not particles and they are not waves, they are different. This change led to a new mathematical conceptual fundament in physics.

Niels Bohr, Max Born, Louis de Broglie, Paul A. M. Dirac, Werner Heisenberg, Pacual Jordan, Wolfgang Pauli, Erwin Schrödinger, John von Neumann and others introduced new objects and theoretical terms to the new mathematical theory of atomic physics and rather quantum physics, then so-called "quantum mechanics" that differ significantly from those in classical physics. Their properties are completely new and not comparable with those of observables in classical theories such as Newton's mechanics or Maxwell's electrodynamics.

The new theoretical term is the quantum mechanical state function ψ that is an element of the abstract Hilbert space H. The theory of quantum mechanics is completely abstract: it is a theory of mathematical state functions that have no exact counterpart in reality. This means that per se ψ is not observable but, nonetheless, we can experiment with a quantum mechanical object having a state function in order to measure its position value, and we can also experiment with this object in order to measure its momentum value. However, we cannot conduct both experiments simultaneously and thus are not able to get both values for the same point in time. But we can predict these values as outcomes of experiments at this point in time. Since predictions are targeted on future events, we cannot valuate them with the logical values "true" or "false," but must use probabilities. The probability distribution to measure a certain position, e.g. value $\boldsymbol{r}$ (t) = (x (t), y (t), z (t)) at point t in time is given by $|\psi(\boldsymbol{r}, t)|^2$ and the probability distribution to measure a certain momentum value $\boldsymbol{p}$ at time point t is given by $|\psi(\boldsymbol{p}, t)|^2$, where $\psi(\boldsymbol{x}, t)$ or $\psi(\boldsymbol{p}, t)$, are representtations of the abstract Hilbert space element ψ in the position or momentum representation respectively.

2 Classical Systems

In classical physics, the state of a "system" or an "object" is represented by a set of observables. For example, in Newtonian mechanics the state of an object (a particle with mass m) is given by the pair of 3-component vector values of the object's position vector $\boldsymbol{r}$ and its momentum vector $\boldsymbol{p}$. These two vectors imply all other properties of the object that are relevant in the Newtonian theory of mechanics. We can formulate that the state of a physical object is the collection of all the object's properties P_i. In order to represent these properties P_i in terms of the physical theory, we must determine the formally possible functions F_i in this mechanical theory, and in order to know the object's properties at a given point in time t, we must measure the values of these functions F_i. Thus, the representation of the "state of a classical object" is related to the measurement process or the perception process of the observer.

Due to the possible errors of measurement and the systematic errors occurring in every experiment, we can attribute their probability of this being the real value to all measured values of observables. Thus, the state of an object in Newtonian mechanics is given by the pair of the probability distributions of position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$.

3 Quantum Systems

3.1 The concept of state in quantum mechanics

The state of a quantum mechanical system is much more difficult to determine than that of classical systems as we cannot measure sharp values for both variables simultaneously. This is the meaning of Heisenberg's uncertainty principle.

However, we can experiment with quantum mechanical objects in order to measure a position value, and we can also experiment with these systems in order to measure their momentum value. But: we cannot conduct both experiments

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simultaneously and thus are not able to get both values for the same point in time respectively. We can predict these values as outcomes of experiments at this point in time. Since predictions are targeted on future events, we cannot valuate them with logical values "true" or "false", but with probabilities.

Accordingly, in quantum mechanics, we have to use a modified concept of the state: the state of a quantum mechanical system consists of the probability distributions of all the object's properties that are formally possible in this physical theory.

- Max Born ([1, 2]) proposed an interpretation of this non-classical peculiarity of quantum mechanics – the quantum mechanical wave function is a "probability-amplitude": The absolute square of its value equals the probability of it having a certain position or a certain momentum if we measure the position or momentum respectively. The higher the probability of the position value, the lesser that of the momentum value and vice versa.
- In 1932, John von Neumann published the *Mathematical Foundations of Quantum Mechanics* [3], in which he defined the quantum mechanical wave function as a one-dimensional subspace of an abstract Hilbert space, which is defined as the state function of a quantum mechanical system (or object). Its absolute square equals the probability density function of it having a certain position or a certain momentum in the position or momentum representation of the wave function respectively.

Unfortunately there is no joint probability distribution for events in which both variables have a certain value simultaneously, as there is no classical probability space that comprises these events. Such pairs would describe classical states. Thus, the quantum mechanical system's state function embodies the probabilities of all properties of the object, but it delivers no joint probability distribution for all these properties. Therefore we claim here: "We need a radically different kind of mathematics, the mathematics of […] quantities which are not describable in terms of [classical] probability distributions.[1]

3.2 *Quantum logic and Quantum probability theory*

After the establishment of quantum mechanics there appeared some approaches to achieve a new logic and later also a new probability theory to handle quantum mechanical propositions and quantum mechanical events.

- 1936, Garett Birkhoff and John von Neumann proposed the introduction of a "quantum logic", as the lattice of quantum mechanical propositions is not distributive, and therefore not Boolean [5].
- In 1963, George Whitelaw Mackey attempted to provide a set of axioms for the propositional system of predictions of experiment's outcomes. He was able to show that this system is an orthocomplemented partially ordered set. [6]

In these logico-algebraic approaches, the "probabilities" of evaluating the predictions of the properties of a quantum mechanical system do not satisfy Kolmogorov's well-known axioms. The double-slit experiment shows that they are not additive and together with their non-distributivity it is indicated that the probabilistic structure of quantum mechanics is more complicated than that of the classical probability space as it was defined by Kolmogorov.

- Already in the 1960s, the philosopher and statistician Patrick Suppes discussed the "probabilistic argument for a non-classical logic of quantum mechanics" [7, 8]. He introduced the concept of a "quantum mechanical σ-field" as an "orthomodular partial ordered set" covering the classical σ-fields as substructures.
- In the 1980s, a "quantum probability theory" was proposed and developed by Stanley Gudder and Imre Pitowski [9, 10].[2]

The quantum mechanical lattice of predictions is Suppes' "quantum mechanical σ-field", which can be restricted to a Boolean lattice corresponding to a given observable. The quantum probabilities became classical probabilities again, only applying to predictions of compatible observables.

The theory of Fuzzy sets and systems pertains to "quantities which are not describable in terms of probability distributions" and theory of Quantum mechanics pertains to quantities which are not describable in terms of classical probability distributions.

Quantum logic and Quantum probability theory represent important approaches to manage quantum mechanical uncertainties within limits of usual mathematics and the developments in the last decades show numerous and also very difficult results. Thus, quantum logic and quantum probability theory are new theories in classical mathematics that became more and more complex. On the other hand there was the new theory of fuzzy sets and systems available at the same time and the question arose in the 1980s and 1990s of whether fuzzy sets could be useful in the interpretation of quantum mechanics. However, at that time this approach was not successful. The disappointing results may have stemmed from the fact that fuzzy set theory was not as well accepted as a mathematical tool the as it is today and from a lack of interest in using the new theory on the part of theoretical physicists. Moreover, until recently there was also no interest in fuzzy set theory in the philosophy of science. But now the theory of fuzzy sets is broadly accepted, particularly in applied sciences and technology and in the history of science the theory of fuzzy sets attracts

[1] This is analogue to Zadeh's requirement: ". . . we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions" in his 1962 article [4], p. 857.

[2] These developments regarding a theory of probabilistic structures of quantum mechanics became very complex, as the reader can see in the authors Ph. D thesis [11].

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attention [12, 13, 14]. In physics, the results of new experiments (Alain Aspect's test of Bell's inequality in 1982 [15, 16] and Anton Zeilinger's experiments on quantum teleportation since 1997 [17] have sparked a new debate on the interpretation of quantum mechanics and there is growing interest in the theory of fuzzy sets in the field of scientific history [29]. In the next sections we will evolve some ideas towards a "fuzzy view" on quantum systems.

4 Fuzzy Systems

In his talk on "A New View on System Theory," for the *Symposium on System Theory* that took place in Brooklyn in April 1965, Lotfi Zadeh defined fuzzy systems as follows:

Definition: A system S is a fuzzy system if input $u(t)$, output $y(t)$, or state $x(t)$ of S or any combination of them ranges over fuzzy sets. ([18], p. 33)[3]

He explained that "these concepts relate to situations in which the source of imprecision is not a random variable or a stochastic process but rather a class or classes which do not possess sharply defined boundaries." ([18], p. 29

Eight years later, in "Outline of a New Approach to the Analysis of Complex Systems and Decision Processes," he introduced "linguistic variables" that are variables whose values may be sentences in a specific natural or artificial language. [19] To illustrate, the values of the linguistic variable "age" might be expressible as *young*, *very young*, *not very young*, *somewhat old*, *more or less young*.

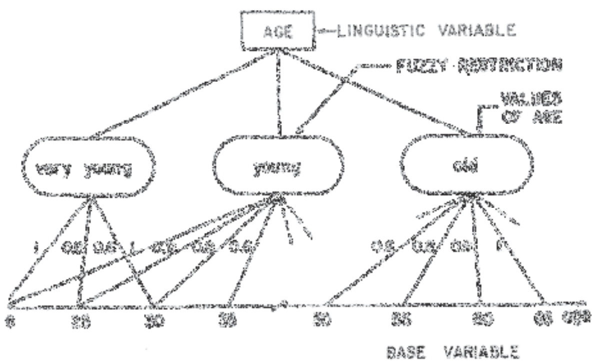

Figure 1: Example of the linguistic variable "Age" [20].

These values are formed with the label *old*, the negation *not*, and the hedges *very*, *somewhat*, and *more or less*. In this sense the variable "age" is a linguistic variable (see Fig. 1).

Linguistic variables became a proper tool for reasoning without exact values. Since in many cases, it is either impossible or too time-consuming (and therefore too expensive) to measure or compute exact values, the concept of linguistic variables has been successful in many fuzzy application systems, e.g., in control and (medical) decision making. In the next section we seek to apply the concept of linguistic variables in quantum mechanics where exact values of observables do not exist. The situation is that outcomes of a physicist's experiments have to be values of observables, i.e., an observing physicist assigns sharp values (and their classical probability distributions of the classical variables (e.g. its position $\boldsymbol{r}$ or its momentum $\boldsymbol{p}$) due to the possible errors of measurement and the systematic errors occurring in every experiment)[4] to a quantum theoretical object. This value is not sufficient to determine the quantum object's state. This is only one representation – among many others – and none of these representations of the state of the quantum system is complete!

5 Fuzzy Classical Systems

In this section a new approach to the interpretation of uncertainty in quantum mechanics using the methodology of fuzzy sets and systems is outlined.

We will interpret the "fuzzy state variable" of a physical system as a vector of linguistic variables instead of numerical variables. This interpretation yields to the interpretation of the "fuzzy state" of a classical physical object as a pair of the two linguistic variables – position and momentum

In a manner similar to Zadeh's extension of systems to fuzzy systems, the definition of a linguistic variable operating on a fuzzy set, and assignment of membership degrees and elements of the term set of the linguistic variable, the "fuzzy state" of a physical system is interpreted as a vector of linguistic variables instead of numerical variables.

A concrete system a has a certain number of properties P^i, $i \in \{1, \ldots, n\}$. In classical physics we attach classical numerical variables (observables) $V_P{}^i$ to these properties that can be measured. To use the methods of fuzzy set theory, now we attach can find also a linguistic variable $LV_P{}^i$ to representing the property P^i. These linguistic variables operate on fuzzy sets and assign membership degrees and elements of a term set, for example:

$$T(LV_P{}^i) = \{\text{very small, small, big, very big, … etc .}\}$$

We can imagine the n-tuple $\boldsymbol{LVP_n} = \langle LV_P{}^1, LV_P{}^2, \ldots, LV_P{}^n \rangle$ to be a "linguistic vector" in an n-dimensional Cartesian space. The value $\boldsymbol{LVP_n}(a, t)$ for a system a at time point t is called the "linguistic state" or "fuzzy state" of this system at this time. During this time, the linguistic state of a system moves in the "linguistic state space" or "fuzzy state space" $\Sigma_n{}^L(\mathrm{a}) = \{\boldsymbol{LVP_n}(\mathrm{a}, t) \mid t \in T\}$ of the system.

In the case of a classical particle in Newtonian mechanics, the "fuzzy state" is the pair (2-tupel) (LV_r, LV_p) of the two linguistic variables position LV_r and momentum LV_p that operate on fuzzy sets and assign membership degrees and elements of a term set, for example:

$$T(LV_r) = \{\text{very small, small, null, big, very big, … etc.}\}$$

and

$$T(LV_p) = \{\text{very small, small, null, big, very big, … etc.}\}$$

Usually the shape of the fuzzy set's membership functions is subjectively chosen or dependant on the problem. In a very special case, the membership function may have the shape of the classical Gaussian probability distribution and thus the

[3] We assume that the reader is familiar with the basic principles of the theory of fuzzy sets.

[4] In the next sections we sometimes will omit this completion about the classical errors of measurement and the systematic errors.

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fuzzy state variable yields the probabilities of measuring the observables position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$ due to the calculation of errors. However, in general, membership functions of fuzzy sets do not represent probability distributions of measurement errors or randomness, but more general uncertainties that are deeply rooted in the absence of the theoretical concept's strict boundaries.

We know already that classical concepts such as position and momentum, having strict boundaries in Newtonian mechanics, do not have such boundaries in the theory of quantum mechanics. Therefore this pair of concepts does not match the quantum mechanical state variable – to represent the quantum mechanical "state", position and momentum are in use with some uncertainty. This concept of uncertainty in quantum mechanics is often represented by classical probability, but, in the strict sense, the quantum mechanical uncertainty is different from the concept of classical probability.

6 Fuzzy Quantum Systems

Let's try to extend our approach of "fuzzy states" of systems to quantum systems to include the assumption that the classical theoretical concepts are not the right concepts, but that we have no better concepts to interpret the outcomes of classical experiments. Thus, we can use fuzzy sets and linguistic variables to convert classical observables to systems of quantum mechanics.

In the case of a quantum system, the "fuzzy quantum state" is an infinite dimensional vector $\boldsymbol{LVP_\infty}$ *(a,t)* of linguistic variables in the abstract Hilbert space, with an infinite tuple $\boldsymbol{LVP_\infty} = \langle LV_P^1, LV_P^2, \ldots, LV_P^n, \ldots\rangle$ of linguistic variables LV_i, but not all linguistic variables LV_i and LV_j are compatible, i.e. they are in an uncertainty relation with each other, e.g., $LV_I = LV_{\boldsymbol{r}}$ (position observable) and $LV_j = LV_{\boldsymbol{p}}$ (momentum observable).

In general, at one point in time we can measure one of these LV_i and this measurement reduces the membership function to a numerical value of this observable. This is an effect that is known in usual quantum mechanics as the "collapse" of the quantum mechanical state function.

Let's use the fuzzy state concept for quantum systems! After the measurement of an observable, say V_P^i, there is no "collapse" of the fuzzy quantum state. We still have it's complete representation as the infinite tuple of linguistic variables $\boldsymbol{LVP_\infty} = \langle LV_P^1, LV_P^2, \ldots, LV_P^n, \ldots\rangle$. However, after the measurement of observable V_P^i, this component of the fuzzy quantum state is not any longer a linguistic but a numeric term: the measurement, just a number (associated with a unit).

7 Fuzzy Uncertainty Principle

First we consider the classical uncertainty principle that was found by Heisenberg in 1925. This uncertainty principle states that the values of certain pairs of conjugate variables[5] (e.g. position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$) cannot both be known with arbitrary precision. That is, the more precisely one variable is known, the less precisely the other is known. We already noticed that this is not the uncertainty of the measurement of particular observables of a system.

Heisenberg's uncertainty principle was very often formulated in usual mathematics (classical probability theory and statistics) as follows: every quantum state has the property that the *root-mean-square deviation* of the position $\boldsymbol{r}$ from its mean (the standard deviation of the $\boldsymbol{r}$-distribution) times the root-mean-square deviation of the momentum $\boldsymbol{p}$ from its mean (the standard deviation of $\boldsymbol{p}$) can never be smaller than a small fixed fraction of Planck's constant:

$$\Delta r \cdot \Delta p \geq \tfrac{\hbar}{2}, \qquad (1)$$

where $\Delta r = \sqrt{\langle (r - \langle r \rangle)^2 \rangle}$ and $\Delta p = \sqrt{\langle (p - \langle p \rangle)^2 \rangle}$.

Any measurement of the position with accuracy $\Delta\boldsymbol{r}$ collapses the quantum state making the standard deviation of the momentum $\Delta\boldsymbol{p}$ larger than $\frac{\hbar}{2}$.

This view is using classical probability theory what is adequate for each one of the two variables $\boldsymbol{r}$ and $\boldsymbol{p}$ but what is not appropriate to the combination of the two "pictures". There is no joint probability distribution for the two variables $\boldsymbol{r}$ and $\boldsymbol{p}$ and therefore the product of their *root-mean-square deviations* has no meaning in classical probability theory. That is, what is meant by the logico-algebraic result that there is no Boolean lattice of quantum mechanics.

Let's consider the uncertainty principle in our approach of fuzzy quantum states! In our view position and momentum, $\boldsymbol{r}$ and $\boldsymbol{p}$, are associated with linguistic variables, $LV_{\boldsymbol{r}}$ and $LV_{\boldsymbol{p}}$. Hence we can consider $\Delta r = \sqrt{\langle (r - \langle r \rangle)^2 \rangle}$ and $\Delta p = \sqrt{\langle (p - \langle p \rangle)^2 \rangle}$ as being linguistic variables as well.

Because Heisenberg's uncertainty relation (1) is a proposition of the possible values of $\Delta\boldsymbol{r}\cdot\Delta\boldsymbol{p}$, we can also consider this product of the two linguistic variables $\Delta\boldsymbol{r}$ and $\Delta\boldsymbol{p}$ as being a linguistic variable $\boldsymbol{S}$ ("action"[6]).

Heisenberg's inequality claims that $\boldsymbol{S} = \Delta\boldsymbol{r}\cdot\Delta\boldsymbol{p}$ can never be smaller than $\frac{\hbar}{2}$, i.e. the minimal value of $\boldsymbol{S}$ equals $\frac{\hbar}{2}$, but this result was never derived by Heisenberg himself. Moreover, in his famous 1930 lecture in Chicago he refined his principle to the following inequality [21]:

$$\Delta x \Delta p \gtrsim h. \qquad (2)$$

The formulation (1) of the uncertainty relations was proved by E. H. Kennard in 1927 but in this prove it was postulated that $\Delta\boldsymbol{r}$ and $\Delta\boldsymbol{p}$ are the standard deviations of position and momentum, therefore this formulation uses classical probability theory [21]

What does this mean? – The value of the linguistic variable $\boldsymbol{S}$ at a given point of time is the composition of the values of the linguistic variables $\Delta\boldsymbol{r}$ and $\Delta\boldsymbol{p}$ but to get both of the two

[5] In physics, conjugate variables are pair of variables mathematically defined in such a way that they become Fourier transform duals of one-another.

[6] "Action" is the name of this particular quantity (product of position and momentum or of energy and time) in physics.

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values is not possible in the theory of quantum mechanics. (Remark: To compute the value of $\Delta\boldsymbol{r}$ or $\Delta\boldsymbol{p}$ we have to measure the value of $\boldsymbol{r}$ or $\boldsymbol{p}$ respectively.)

Thus, it is only possible to know one of the linguistic values of $\Delta\boldsymbol{r}$ or $\Delta\boldsymbol{p}$ and therefore it is not possible to compute the value of the linguistic variable $\boldsymbol{S} = \Delta\boldsymbol{r}\cdot\Delta\boldsymbol{p}$.

8 Conclusions

During the last decade the literature on combining the theories of quantum mechanics and fuzzy sets was growing, e.g. [23- 29].[7] Of course there might be many more papers on the meeting of the two theories but unfortunately – as one of the reviewers to this paper, "it seems to be sometimes the application of ideas from quantum theory to fuzzy sets than the other way around". This paper was written to plea for applications of ideas from Fuzzy set theory to quantum mechanics. Of course, we gave only some preliminary ideas to establish a fuzzy approach to physics and particularly to quantum mechanics.

We presented a rough idea of what the methodologies of fuzzy sets and systems, along with linguistic variables, could contribute to the project to represent phenomena in physics that can not be represented in classical mechanics. Quantum mechanics is a very successful theory to represent these phenomena but there are difficulties with the using of the concept of probability. It seems that the concepts of fuzzy sets and linguistic variables can be more appropriate for the representation of quantum phenomena in the framework of quantum mechanics than the classical concept of probability. Our definition of the fuzzy state of a quantum mechanical system avoids the difficulties that arise in classical probability theory in attempts to define this state. On the other hand, with this fuzzy approach new difficulties arise, e.g. with the experimental side of physics and also with its interpretative side. Of course, scientists should take such problems seriously. At the other hand, these are the usual and well-known problems of all fuzzy-approaches in the field of science and technology of the last 40 year. From the point of view of a historian and philosopher of science this is also a big challenge!

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