# Possibility and Necessity Evaluations based on Ordinal Comparability

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***Abstract***— *Subjective uncertainty is one of the most essential subjects for evaluation that was the reason Fuzzy theory was proposed. Related research areas are widely spread such as decision making, data analysis, information retrieval, psychology, and human computer interaction and so on. As a basis for various researches mathematical interpretation of evaluation methods occupies an important position. In Possibility theory, possibility and necessity evaluations have been utilized as ways for evaluating a vague achievement in an ambiguous situation. However, mathematical interpretations for these evaluations have been insufficient. In this article, possibility and necessity evaluations are examined based on the ordinal comparability in social choice theory. Equivalent conditions to the evaluations are deduced so that axiomatic interpretations for the evaluations become possible, and order relations related to the two evaluations are deduced.*

***Keywords***— evaluation, measure, necessity, ordinal comparability, ordinal utility, possibility.

## 1 Introduction

In Possibility theory [1, 2, 3, 4, 5], subjective value of attainment or satisfaction of an evaluation item in state $x$ is represented by the membership function $\mu(x)(0 \leqq \mu(x) \leqq 1)$. And occurrence degree of subjective possibility is represented by the membership function $\pi(x)(0 \leqq \pi(x) \leqq 1)$. Tuple of these functions, $(\mu(x), \pi(x))$, can be regarded as a representation of state $x$ with two kinds of uncertainties (vagueness and ambiguity) and be applied for evaluation and decision making.

An evaluation value for $(\mu, \pi)$ is defined as follows:

**Definition 1**

$$\Pi(\mu, \pi) = \sup_x \min\left(\mu(x), \pi(x)\right) \tag{1}$$

This can be regarded as an extension of possibility measure for fuzzy sets. This is also interpreted as a fuzzy integral for fuzzy set on possibility measure.

In Possibility theory, necessity measure is defined as dual for possibility evaluation:

**Definition 2**

$$N(\mu, \pi) = \inf_x \max\left(\mu(x), 1 - \pi(x)\right) \tag{2}$$

These two evaluations (possibility and necessity) also can be interpreted as optimistic (positive) and pessimistic (negative) evaluations respectively.

In this paper, we investigate generalized possibility and necessity evaluations. As seen in these definitions, generally in Fuzzy theory, *max* operation and *min* operation are used frequently. For example, they are used in the synthetic rule of fuzzy reasoning.

Moreover, some restrictions, e.g., *normality*, are assumed on membership functions in many cases. A continuity and convexity are sometimes assumed on membership functions. [3].

**Definition 3** *A membership function $\pi(x)$ is normal means that there exists $x$ such that $\pi(x) = 1$.*

In order to perform the mathematical interpretation on possibility and necessity evaluations and to examine the validity of the evaluations, it is necessary to consider on the property of functions $\mu(x)$ and $\pi(x)$ determined subjectively. For example, whether it is a cardinal utility function or ordinal utility function.

In Possibility theory, considerations for improving qualitative possibilistic criteria are discussed. Axiomatic approach has been made in order to examine decision theoretic foundation [6]. Generalized concordance rules that request comparisons within possible values for components and comparisons between sets of attributes is proposed [7]. The injection of lexicographic ingredients in the possibilistic criteria has been studied [8].

It is necessary to consider the conditions at the time of unifying the functions and performing synthetic evaluation, e.g., evaluation level comparability on evaluation items. In social choice theory or voting theory, the comparability of utility functions was introduced with ordinal number based on extended ordering [9, 10]. and examination has been made about how synthetic evaluation should be performed [11].

This paper newly defines possibility and necessity evaluations based on ordinal comparability and deduces equivalent conditions to them. Therefore, the applicable conditions of possibility and necessity evaluations are clarified. Moreover, relations between these two evaluations are discussed.

Hereafter, Section 2 examines necessary and sufficient condition of the evaluations based on extended ordering. In Section 3, interpretations of the conditions for evaluations, possibility, and necessity evaluation, are discussed. Section 4 considers order relations between possibility evaluation and necessity evaluation.

## 2 Extended ordering and preference relation

We denote that $X$ is a set of states or a set of alternatives. We define *preference relation* as a binary relation.

**Definition 4** *Preference relation $R$ on $X$ ($x, y \in X, xRy$: $x$ is preferred to $y$) is a binary relation which is reflective*

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*($\forall x \in X : xRx$), transitive ($\forall x, y, z \in X : (xRy \wedge yRz) \rightarrow xRz$). and total ($\forall x, y \in X : (x \neq y) \rightarrow (xRy \vee yRx)$). Strong preference relation $P$ and indifference relation $I$ are defined by: $xPy \equiv (xRy \wedge \neg(yRx))$, and $xIy \equiv (xRy \wedge yRx)$.*

We denote $H$ as a set of evaluation items. In social choice theory [9], $X$ is called *a set of social states*, and $H$ is called *a set of individuals*. In this paper, we make fundamental investigations on preference relation, so that the number of elements of the set $H$ is restricted to two.

We consider $(x, i) \in X \times H$. A binary relation $\tilde{R}$ on $X \times H$ is assumed to be a preference relation. This relation is called *extended ordering* [9]. In similar meaning, it is also called extended sympathy [10].

**Definition 5** *Preference relation $(x, i)\tilde{R}(y, j)$ means that an evaluation item $i$ on a state $x$ is preferred to $j$ on a state $y$. Strong preference relation $\tilde{P}$ and indifference relation $\tilde{I}$ on $X \times H$ are defined by: $(x, i), (y, j) \in X \times H$ and $\tilde{R}$, $(x, i)\tilde{P}(y, j) \equiv ((x, i)\tilde{R}(y, j) \wedge \neg((y, j)\tilde{R}(x, i)))$, $(x, i)\tilde{I}(y, j) \equiv ((x, i)\tilde{R}(y, j) \wedge (y, j)\tilde{R}(x, i))$.*

Here, we consider generalized function $f : \tilde{R} \rightarrow R$, which corresponds the preference relation $\tilde{R}$ ($\tilde{P}$ and $\tilde{I}$) on $X \times H$ to the preference relation $R$ ($P$ and $I$) on $X$. The function $f$ is called *Generalized Social Welfare Function* (GSWF) in social choice theory [9].

In the following section, higher evaluation on state $x$ is denoted as $2(x) \in \{1, 2\}$ and lower evaluation is denoted as $1(x) \in \{1, 2\}$. This means that $(x, 2(x))\tilde{R}(x, 1(x))$.

**Definition 6** *GSWF $f^P$ is defined as follows. For all $x, y \in X$,*
*(1) if for all $i \in \{1, 2\}$ $(x, i(x))\tilde{I}(y, i(y))$, then $xIy$,*
*(2) if $(x, 1(x))\tilde{I}(y, 1(y))$ and $(x, 2(x))\tilde{P}(y, 2(y))$, then $xPy$,*
*(3) if $(x, 1(x))\tilde{P}(y, 1(y))$, then $xPy$.*

GSWF $f^P$ corresponds to *leximin rule*, because we compare preference from lower level to higher level [9, 11].

Next, we consider following five conditions on the two preference relations $\tilde{R}$ and $R$. We denote $(x, i)\tilde{R}(y, i)\tilde{R}(z, i)$ for short as $(x, i)\tilde{R}(y, i)$ and $(y, i)\tilde{R}(z, i)$. We denote for $\tilde{P}$ and $\tilde{I}$ in similar way.

**(C1)** *Preference relation $\tilde{R}$ can take any logically possible relation.*

**(C2)** *If the restrictions of $\tilde{R}$ and $\tilde{R}'$ on any pair in $X$ are the same, then the restrictions of $R$ and $R'$ on that pair are also same.*

**(C3)** *For any $x, y$ in $X$, if $(x, 1)\tilde{R}(y, 1) \wedge (x, 2)\tilde{R}(y, 2)$, then $xRy$. If one of the two relations is a strong preference relation $\tilde{P}$, then $xPy$.*

**(C4)** *For any $x, y$ in $X$, if $(y, 2)\ \tilde{P}\ (x, 2)\ \tilde{R}\ (x, 1)\ \tilde{P}\ (y, 1)$ or if $(y, 1)\ \tilde{P}\ (x, 1)\ \tilde{R}\ (x, 2)\ \tilde{P}\ (y, 2)$, then $xPy$.*

**(C5)** *For any $x, y$ in $X$, if $(x, 1)\tilde{I}(y, 2) \wedge (x, 2)\tilde{I}(y, 1)$ and $\neg((x, 1)\tilde{I}(x, 2))$, then $xIy$.*

We note that if $(x, 1)\tilde{R}(y, 1) \wedge (x, 2)\tilde{R}(y, 2)$ and $(y, 1)\tilde{R}(x, 1) \wedge (y, 2)\tilde{R}(x, 2)$, then $xIy$ by (C3). Furthermore we note that the conditions (C3), (C4), and (C5) are mutually independent.

For GSWF $f^P$ and the five conditions, following theorem is stated [12].

**Theorem 1** *Assume that $|X| > 2$, only GSWF that satisfies (C1), (C2), (C3), (C4), and (C5) is $f^P$.*

Next, from the view of symmetry, we consider a different condition only for (C4) in the above five conditions. We define the following converse condition (C4').

**(C4')** *For any $x, y$ in $X$, if $(x, 2)\ \tilde{P}\ (y, 2)\ \tilde{R}\ (y, 1)\ \tilde{P}\ (x, 1)$ or if $(x, 1)\ \tilde{P}\ (y, 1)\ \tilde{R}\ (y, 2)\ \tilde{P}\ (x, 2)$, then $xPy$.*

In this case we consider the followings.

**Definition 7** *GSWF $f^N$ is defined as follows. For all $x, y \in X$,*
*(1) for any $i \in \{1, 2\}$, if $(x, i(x))\tilde{I}(y, i(y))$, then $xIy$,*
*(2) if $(x, 2(x))\tilde{I}(y, 2(y))$ and $(x, 1(x))\tilde{P}(y, 1(y))$, then $xPy$,*
*(3) if $(x, 2(x))\tilde{P}(y, 2(y))$, then $xPy$.*

For GSWF $f^N$ and the five conditions, following theorem is stated.

**Theorem 2** *Assume $|X| > 2$, only GSWF that satisfies conditions (C1), (C2), (C3), (C4'), (C5) is $f^N$.*

By the above two theorems, conditions equivalent to two GSWFs, $f^P$ and $f^N$, were obtained. Based on these results we can discuss what kind of conditions are assumed when each GSWF is applied for evaluation.

## 3 Interpretation of possibility and necessity evaluation based on extended ordering

### 3.1 Ordinal utility function

In this section, as an application of the knowledge obtained in the previous section to Possibility theory, we consider the case that $H = \{\mu, \pi\}$. In the case we define GSWF $f^P$ as a possibility evaluation.

We can say from (1) and the definition of $f^P$ that if $xPy$ for $f^P$, then $\min(\mu(x), \pi(x)) \geqq \min(\mu(y), \pi(y))$.

In the following, we discuss the interpretation of possibility evaluation based on the theorem obtained in the previous section.

We consider that $(x, i) \in X \times H$ corresponds to $(x, \mu)$ and $(x, \pi)$. The relations between the preference relation $\tilde{R}$ and membership functions are defined as follows;
$(x, \mu)\tilde{R}(y, \mu) \equiv \mu(x) \geqq \mu(y)$,
$(x, \pi)\tilde{R}(y, \pi) \equiv \pi(x) \geqq \pi(y)$,
$(x, \mu)\tilde{R}(x, \pi) \equiv \mu(x) \geqq \pi(x)$,
$(x, \pi)\tilde{R}(x, \mu) \equiv \pi(x) \geqq \mu(x)$.

We denote strong preference relation and indifference relation as "$>$" and "$=$" respectively.

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We should note that membership functions $\mu$ and $\pi$ can be regarded as **ordinal level comparable utility functions** based on extended ordering.

### 3.2 Possibility evaluation

In this subsection we discuss the conditions in Theorem 1.

Conditions (C1) and (C2) are naturally acceptable conditions.

Condition (C3) (if $\mu(x) \geqq \mu(y)$ and $\pi(x) \geqq \pi(y)$, then $xRy$) is so-called Pareto condition, so that this is naturally acceptable for the functions $\mu$ and $\pi$.

However, (C4) (if $\mu(y) > \mu(x) \geqq \pi(x) > \pi(y)$ or $\pi(y) > \pi(x) \geqq \mu(x) > \mu(y)$, then $xRy$) means that we must compare two evaluation elements of $\mu$ and $\pi$ on one state.

And condition (C5) (if $\mu(x) = \pi(y)$ and $\pi(x) = \mu(y)$, then $xIy$) means that in indifferent two states, two evaluation elements of $\mu$ and $\pi$ can be exchanged as same.

Therefore conditions (C4) and (C5) for the functions $\mu$ and $\pi$ are restrictions on two evaluations, then these restrictions are strongly imposed on setting these two membership functions.

If conditions (C1), (C2), (C3), (C4), (C5) are acceptable for $\mu$ and $\pi$, then by Theorem 1, we can utilize the function $f^P$ as a preference functions on $x, y \in X$.

On the other hand, in the case that we use the function $f^P$ as a preference relation, the conditions (C1), (C2), (C3), (C4), (C5) should be satisfied.

Consequently, for $x, y \in X$ we clarify the conditions for evaluation of (1) based on comparing $\min(\mu(x), \pi(x))$ and $\min(\mu(y), \pi(y))$.

### 3.3 Necessity evaluation

Similarly we can discuss necessity evaluation. In case of necessity, for $\pi$ in (2) we can see $1 - \pi(x)$. Therefore, as a converse order relation we defined an ordinal utility function $\pi^*(x)$ as follows.

**Definition 8** *$\pi^*(x)$ is defined by:*

$$\pi^*(x) < \pi^*(y) \quad \equiv \quad \pi(x) > \pi(y), \tag{3}$$
$$\pi^*(x) = \pi^*(y) \quad \equiv \quad \pi(x) = \pi(y). \tag{4}$$

Similarly as possibility evaluation we can discuss the interpretation of necessity evaluation based on Theorem 2 obtained in the previous section.

We consider that $(x, i) \in X \times H$ corresponds to $(x, \mu)$ and $(x, \pi^*)$ in the same way as possibility. The relation between the preference relation $\tilde{R}$ and membership functions are defined as follows;

$(x, \mu)\tilde{R}(y, \mu) \equiv \mu(x) \geqq \mu(y)$,
$(x, \pi^*)\tilde{R}(y, \pi^*) \equiv \pi^*(x) \geqq \pi^*(y)$,
$(x, \mu)\tilde{R}(x, \pi^*) \equiv \mu(x) \geqq \pi^*(x)$,
$(x, \pi^*)\tilde{R}(x, \mu) \equiv \pi^*(x) \geqq \mu(x)$.

We denote strong preference relation and indifference relation as "$>$" and "$=$" respectively.

We should note that membership functions $\mu$ and $\pi^*$ can be regarded as ordinal comparable utility functions based on extended ordering. As similar with possibility, we can discuss $\mu$, $\pi^*$, and conditions (C1), (C2), (C3), (C4'), (C5).

## 4 Order relation between possibility evaluation and necessity evaluation

In this section, as discussed in earlier sections we assume that $H = \{\mu, \pi\}$ and that extended ordering on $(x, \mu)$ and $(x, \pi)$ satisfies conditions (C1), (C2), (C3), (C4), and (C5) for preference relations $\tilde{R}$ and $R$.

**Definition 9** *$x^P \in X$ is defined by: for all $y(\neq x^P) \in X$, $x^P Ry$.*

Similarly we assume that $H = \{\mu, \pi^*\}$ and that extended ordering on $(x, \mu)$ and $(x, \pi^*)$ satisfies conditions (C1), (C2), (C3), (C4'), and (C5) for preference relations $\tilde{R}^*$ and $R^*$. We discriminate $\tilde{R}, R$ and $\tilde{R}^*, R^*$.

**Definition 10** *$x^N \in X$ is defined by: for all $y(\neq x^N) \in X$, $yR^*x^N$.*

We should recall that $\mu(x)$ and $\pi(x)$ are ordinal functions.

In Possibility theory, optimistic evaluation and pessimistic evaluation are discussed. In this section, order relation between possibility evaluation and necessity evaluation is discussed.

We obtain the following result with respect to $\mu(x^P)$ and $\mu(x^N)$.

**Lemma 3**

$$\mu(x^P) \geqq \mu(x^N). \tag{5}$$

(Proof)
Assuming $\mu(x^P) < \mu(x^N)$ we deduce contradiction. In this case, $x^P \neq x^N$. If $\pi(x^P) \leqq \pi(x^N)$, then $(x^N, \mu)\tilde{P}(x^P, \mu)$ and $(x^N, \pi)\tilde{R}(x^P, \pi)$. Therefore, by (C3) of GSWF $f^P$, $x^N Px^P$. This contradicts the definition of $x^P$. Next, we consider the case that $\pi(x^P) > \pi(x^N)$. In this case, $\pi^*(x^P) < \pi^*(x^N)$ and $\mu(x^P) < \mu(x^N)$, then $(x^N, \mu)\tilde{P}^*(x^P, \mu)$ and $(x^N, \pi)\tilde{P}^*(x^P, \pi)$. By (C3) of GSWF $f^N$, $x^N P^* x^P$, so that this contradicts the definition of $x^N$.

For order relations with respect to $\max(\mu(x^N), \pi^*(x^N))$ and $\min(\mu(x^P), \pi(x^P))$, we obtain the following results.

**Lemma 4** *If $\pi^*(x) > \pi(x)$, then*

$$\max(\mu(x^N), \pi^*(x^N)) > \min(\mu(x^P), \pi(x^P)). \tag{6}$$

**Lemma 5** *If there exists $x'$ which satisfies the condition $\pi(x') \geqq \mu(x') \geqq \pi^*(x')$, then*

$$\min(\mu(x^P), \pi(x^P)) \geqq \mu(x') \geqq \max(\mu(x^N), \pi^*(x^N)). \tag{7}$$

(Proof)
By the definition of $x^P$, we have $\min(\mu(x^P), \pi(x^P)) \geqq \min(\mu(x'), \pi(x')) \geqq \mu(x')$. Similarly with respect to $x^N$, $\max(\mu(x^N), \pi^*(x^N)) \leqq \max(\mu(x'), \pi^*(x')) \leqq \mu(x')$. Consequently we obtain the result.

Next we discuss in the framework of conventional Possibility theory, i.e., we consider $\pi^*(x) = 1 - \pi(x)$. $x^P$ is most preferable on $X$ with respect to GSWF $f^P$. Then $x^P$ takes the optimal value of (1) in Definition 1. Similarly, $x^N$ is most unpreferable on $X$ with respect to GSWF $f^N$. Then $x^N$ takes the optimal value of (2) in Definition 2.

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Some assumptions are sometimes taken in Possibility theory. In this section we assume that the membership function $\pi(x)(0 \leqq \pi(x) \leqq 1)$ satisfies normality condition, i.e., there exists the $x' \in X$ such that $\pi(x') = 1$. Then we obtain the following result by Lemma 5.

**Theorem 6** *If $\pi(x)$ is normal, then*

$$\Pi(\mu, \pi) \geqq N(\mu, \pi). \quad (8)$$

(Proof) By the definition of normality there exists $x'$ that $\pi^*(x') = 0$ and $\pi(x') = 1$, and $0 \leqq \mu(x') \leqq 1$, then the prerequisite condition of Lemma 5 is satisfied.

## 5 Conclusion

This paper described the mathematical foundation of subjective uncertainty. Axiomatic interpretations on possibility evaluation and necessity evaluation, and the relation between these evaluations were examined based on the extended ordering in social choice theory.

The obtained results are as follows:

In section 2, based on extended ordering we obtained the equivalent conditions for two kinds of GSWF $f^P$ and $f^N$; conditions (C1), (C2), (C3), (C4), (C5), and conditions (C1), (C2), (C3), (C4'), (C5).

In section 3, we examined generalized possibility evaluation and necessity evaluation, and discussed the conditions obtained in section 2.

In section 4, we defined $x^P$ and $x^N$, and discussed the order relations between possibility evaluation and necessity evaluation, and obtained some results on the relations.

By these results obtained in this paper, mathematical interpretations and applicable conditions of the evaluations became clear. These are important when the evaluations are applied in various practical situations.

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