Two Results About Optimization of Fuzzy Variable Functions.

Andrés David Báez-Sánchez
Ph.D. Student
Department of Applied Mathematics
Institute of Mathematics, Statistics and Scientific Computation, Campinas State University
Campinas, SP. , Brazil
Email: davidbaez@ime.unicamp.br

Abstract— *We discuss some optimization problems for fuzzy variable functions and show two interesting results. First result is related to conditions for existence of global optimal solutions for some general class of optimization problems on $\mathbb{E}^n$, the space of compacts, upper semicontinuous and normal fuzzy sets of $\mathbb{R}^n$. The second result is about preservation of local optimality points, via Zadeh's principle of extension.*

Keywords— Coercitivity of fuzzy variable functions, compacity in fuzzy metric spaces, fuzzy optimization, optimization of fuzzy variable functions, principle of extension.

1 Introduction

In most optimization problems, the coefficients, parameters and variable decisions involved are assumed to be precise elements of some real space $\mathbb{R}^n$. However in many real situations there is not such certainty about those values and is necessary to consider some models or methods to deal with optimization problems under uncertainty. Two main approaches have been deeply developed when we consider uncertainty in the parameters, in the constraints or in the values of the objective function, these are stochastic programming and fuzzy optimization. We may refer [1] or [2] for stochastic programming and [3], [4] or [5] for an overview of fuzzy optimization.

By the other hand, in connection with applications and modelling using fuzzy theory, a different kind of fuzzy optimization problems has been appeared in which some of the decision variables are itself fuzzy, this means, optimization of fuzzy variable functions. Some situations like this can be found in problems of depend or constraints programming as is shown in [6] or [7]), in estimation and regression models for fuzzy random variables as in [8], [9], [10], [11] or [12], or in problems of optimal control for fuzzy differential equations as in [13] or [14].

In most of the cases those fuzzy variable optimization problem was faced successfully using some *ad hoc* techniques or properties, some heuristic algorithm, or considering just specific classes of fuzzy numbers such as triangular or L-R fuzzy numbers. We believe that fuzzy variable optimization problems have not been thoroughly studied yet, and they deserve a systematic study in itself, because a general approach to this issue can leave a fertile field for their application and theoretical developments.

In this work we are dealing with fuzzy quantities in $\mathbb{R}^n$, specifically we are considering $\mathbb{E}^n$ the spaces of compacts, upper semicontinuous and normal fuzzy sets of $\mathbb{R}^n$. It is well known that $\mathbb{E}^n$ with usual arithmetic between fuzzy numbers is not a vector space, so many of general results in optimization theory are not availables immediately on this context, precisely because most of them use the vector space structure. Taking advantage of the special characteristics and properties of $\mathbb{E}^n$ we will be able to show two interesting theoretical results in optimization of fuzzy variable functions.

In section 2 we set the two problems under consideration and we recall some basic notation and definitions. In section 3 we recall some results about compacity in $\mathbb{E}^n$. Section 4 presents our first main result related to conditions for existence of optimal global solutions for some general class of optimization problems on $\mathbb{E}^n$. In section 5 we recall the Zadeh's principle of extension and some related properties and in section 6 we present our second main result related to preservation of local optimality points, via Zadeh's principle of extension.

2 Preliminaries

We are going to consider first the following optimization problem:

$$\min f(u) \qquad \text{subject to} \qquad u \in D \tag{1}$$

with $D \subset \mathbb{E}^n$ and $f : \mathbb{E}^n \to \mathbb{R}$. Later we will consider a given function $f : \mathbb{R}^n \to \mathbb{R}$ and study the problem of optimal local points of the function $\hat{f} : \mathbb{E}^n \to \mathbb{E}$ defined by Zadeh's principle of extension from f.

Besides of the space $\mathbb{E}^n$ we will also use the metric space $K(\mathbb{R}^n)$ of compacts subsets of $\mathbb{R}^n$ with the Hausdorff metric h defined by $h(A, B) = \max\{d(A, B), d(B, A)\}$ where A, B are compacts subsets of $\mathbb{R}^n$, $d(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} d(a, b)$.

Let u be an element of $\mathbb{E}^n$. For every $\alpha \in (0, 1]$ the α-level of u is defined by $u_\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$ and $u_0 = supp(u) = \overline{\{x \in \mathbb{R}^n : u(x) > 0\}}$.

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The supremum metric d_∞ on $\mathbb{E}^n$ is defined in terms of Hausdorff metric between α-level sets:

$$d_\infty(u,v) = \sup_{\alpha\in[0,1]} h(u_\alpha, v_\alpha),$$

where $u, v \in \mathbb{E}^n$.

3 Compacity in $\mathbb{E}^n$

Definition 3.1. *A set $U \subset \mathbb{E}^n$ is said to be* ***compact-supported*** *if there is a compact subset K of $\mathbb{R}^n$ such that $supp(u) \subset K$ for every $u \in U$.*

If a subset U of $\mathbb{E}^n$ is compact-supported then is bounded. We explain this in the next theorem to be used later.

Theorem 3.2. *Let U be a subset of $\mathbb{E}^n$. If U is not compact-supported then exist a sequence $\{u^k\} \subset U$ such that $||u^k|| := d_\infty(u^k, 0) \to \infty$ as $k \to \infty$.*

Proof: If U is not compact-supported then for every compact set $K \subset \mathbb{R}^n$ there exist some $u^K \in U$ such that $supp(u^K) \not\subset K$. If we take $K = \overline{B_k(0)}$ the closed ball with center 0 an radius k, then for each $k \in \mathbb{N}$ there is a $u^k \in U$ such that $supp(u^k) \not\subset \overline{B_k(0)}$.

We have that $d(0, \overline{B_k(0)}) = 0$ and $d(\overline{B_k(0)}, 0) = k$ so $h(0, \overline{B_k(0)}) = k$, and by the other hand we have that at least one element x of $supp(u^k)$ is not an element of $\overline{B_k(0)}$ so $d(x, 0) > k$ and this implies that $d(supp(u^k), 0) = d((u^k)_0, 0) > k$ and furthermore $h((u^k)_0, 0) > k$.

By definition of supremum metric we have that

$$d_\infty(u^k, 0) = \sup_{\alpha\in[0,1]} h((u^k)_\alpha, 0) > k,$$

so, $||u^k|| := d_\infty(u^k, 0) \to \infty$ as $k \to \infty$ □

We consider also the level-set mapping $u_{(\cdot)} : [0,1] \to K(\mathbb{R}^n)$ which associate with each α the level-set u_α.

Definition 3.3. *A set $D \subset \mathbb{E}^n$ is called* ***equicontinuous*** *if the family of level-sets functions $\{u_{(\cdot)} : u \in D\}$ is a family of equicontinuous functions of [0,1] in $K(\mathbb{R}^n)$.*

The following is a well known result of metric spaces writen in terms of our fuzzy optimization problem:

Theorem 3.4. *Let D be a compact subset of $\mathbb{E}^n$ and f a continuous function on D. Then exists an optimal global solution for problem 1.*

An also well known corollary it follows, but first we need an additional notation. For $c \in \mathbb{R}$ and $f : D \to \mathbb{R}$ the set $L_{f,D}(c)$ is defined by

$$L_{f,D}(c) := \{x \in D : f(x) \le c\}$$

Corollary 3.5. *If f is continuous and exist $c \in \mathbb{R}$ such that $L_{f,D}(c)$ is not empty and compact then exists an optimal global solution for problem 1.*

Proof: If $L_{f,D}(c)$ is not empty and compact then previous theorem implies that the function f reaches a minimum at some $x^* \in L_{f,D}(c)$ when f is restricted to $L_{f,D}(c)$. By definition if $x \notin L_{f,D}(c)$ then $f(x) > c$. But $f(x^*) \le c < f(x)$ so x^* is an optimal global solution of problem 1 □

So far what we have just write are old compacity results. Those results can be easy to understand but maybe hard to apply because until now we have not talked about the meaning of compacity on $(\mathbb{E}^n, d_\infty)$.

An interesting characterizations of compacity in $(\mathbb{E}^n, d_\infty)$ in terms of concepts previously defined is the following.

Theorem 3.6. *A subset D of $(\mathbb{E}^n, d_\infty)$ is relatively compact if D is compact-supported and equicontinuous.*

Proof: See Corollary 10 in [15] □

Corollary 3.7. *If a subset D of $(\mathbb{E}^n, d_\infty)$ is closed, compact-supported and equicontinuous then D is compact.*

4 A Coercitivity Condition

In this section we will prove our first main result. We need to recall some definitions.

Definition 4.1. *A sequence $\{x^k\} \subset \mathbb{E}^n$ is called critic in relation to a set D if $\{x^k\} \subset D$ and $||x^k|| := d_\infty(0, x^k) \to \infty$ or $x^k \to x$ with $x \in \overline{D}$ but $x \notin D$.*

Definition 4.2. *A function f is coercive in a set D if for all sequence $\{x^k\}$ critics in relation to D, $\limsup_{k\to\infty} f(x^k) = +\infty$.*

Previous ones are well known definitions on normed vector spaces but even if $\mathbb{E}^n$ is not exactly a vector space, this concepts can be used in this context because of the special characteristics of $\mathbb{E}^n$. Let's see how this can be done.

Theorem 4.3. *If f is continuous and coercive in D and exist $c \in \mathbb{R}$ such that $L_{f,D}$ is not empty and equicontinuous, then problem 1 has an optimal global solution.*

Proof: We will show that under this hypothesis $L_{f,D}(c)$ is compact and then we use corollary 3.5.

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If $L_{f,D}(c)$ is not compact then because corollary 3.7 and hypothesis, $L_{f,D}(c)$ is not closed or is not compact-supported.

If $L_{f,D}(c)$ is not closed, as we assume that $L_{f,D}(c)$ is not empty, then exist a sequence $\{u^k\} \subset L_{f,D}(c)$ such that $u^k \to u$ and $x \notin L_{f,D}(c)$. If $u \in D$ then continuity of f in D and the fact that $f(u^k) \leq c$ for all $k = 0, 1, ...$, allow us to conclude $f(u) \leq c$ but $u \notin L_{f,D}(c)$ by definition of u, so $u \notin D$.

This means that $\{u^k\}$ is a critic sequence in relation to D and by coercitivity of f in D we have that $f(u^k) \to \infty$, which is a contradiction because $f(u^k) \leq c$.

Now, if $L_{f,D}(c)$ is not compact-supported then for theorem 3.2 there is a sequence $\{u^k\} \subset L_{f,D}(c)$ such that $||u^k|| := d_\infty(u^k, 0) \to \infty$ and again because of coercitivity of f we conclude that $f(u^k) \to \infty$ which is again a contradiction because $f(u^k) \leq c$.

So $L_{f,D}(c)$ has to be compact and the desired result it follows from corollary 3.5 □

5 Zadeh's Principle of Extension and Ordering in $\mathbb{E}$

We recall now for the case of $f : \mathbb{R}^n \to \mathbb{R}$ one of the cornerstones of fuzzy theory.

Zadeh's Principle of Extension: Given a function $f : \mathbb{R}^n \to \mathbb{R}$ we define the Zadeh's extension of f by:

$$\hat{f} : \mathbb{E}^n \to \mathbb{E},$$

$$\hat{f}(u)(x) = \begin{cases} \sup_{z \in f^{-1}(x)} u(z), & \text{if } f^{-1}(x) \neq \emptyset, \\ 0, & \text{if } f^{-1}(x) = \emptyset, \end{cases}$$

for each $u \in \mathbb{E}^n$.

Next theorem is now a well known result and establishes conditions on f in order to $\hat{f}$ be a well-defined and continuous function.

Theorem 5.1. *If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous then $\hat{f} : (\mathbb{E}^n, d_\infty) \to (\mathbb{E}, d_\infty)$ is a well-defined continuous function and*

$$[\hat{f}(u)]_\alpha = f(u_\alpha),$$

for all $u \in \mathbb{E}^n$ and $\alpha \in [0, 1]$.

Proof: Proof of this result is an easy adaptation of theorems 2.1 and 3.3 at [16] and a famous result of Nguyen at [17] □

A usual partial order on $\mathbb{E}^n$ is defined by α-cuts inclusion in the following way:

$$u \leq v \Leftrightarrow u_\alpha \subseteq v_\alpha \quad \text{for all } \alpha \in [0, 1]],$$

with $u, v \in \mathbb{E}^n$.

Nevertheless, even that this order has been used succesfully for theoretical developments as in [18], it is still a little unsatisfactory at least in the case $n = 1$ because in this case this order is not a generalization of usual order on $\mathbb{R}$. In fact, number 2 is not longer greater than 1, they are not even comparable using this order.

There is, however, a generalization of natural order on $\mathbb{R}$ that we are going to consider now. Before, is necessary to note that if $u, v \in \mathbb{E}$ this implies that all their α-cuts are bounded closed interval of real line, this means that for all $\alpha \in [0, 1]$

$$u_\alpha = [a_\alpha, b_\alpha], v_\alpha = [c_\alpha, d_\alpha]$$

with $a_\alpha, b_\alpha, c_\alpha, d_\alpha \in \mathbb{R}$.

Definition 5.2. *Given $u, v \in \mathbb{E}$ and $\alpha \in [0, 1]$ we consider $u_\alpha = [a_\alpha, b_\alpha]$ and $v_\alpha = [c_\alpha, d_\alpha]$. We define a partial order relation on $\mathbb{E}$ by:*

$$u \leq_1 v \Leftrightarrow \quad a_\alpha \leq c_\alpha \quad \textit{and} \quad b_\alpha \leq d_\alpha$$

for all $\alpha \in [0, 1]$.

It is easy to show that previous order is in fact a partial order on $\mathbb{E}$ and that it is indeed a generalization of natural order on $\mathbb{R}$.

6 Local Optimality Preservation

Now we are going to prove our second main result. We are still considering the order defined in definition 5.2 and the supremum metric on $\mathbb{E}^n$

Theorem 6.1. *If $f : \mathbb{R}^n \to R$ is continuous and has a local minimum (maximum) at x^* then $\hat{f} : (\mathbb{E}^n, d_\infty) \to (\mathbb{E}, d_\infty, \leq_1)$ has also a local minimum (maximum) at x^*.*

Proof: We will just make the proof for local minimum. The other part is identical.

If f has a local minimum at x^* then exist a closed ball $B_r(x^*) \subset \mathbb{R}^n$ such that $f(x^*) \leq f(x)$ for all $x \in B_r(x^*)$.

Because $\hat{f}$ is just an extension of f to $\mathbb{E}^n$ and the order $\leq_1$ is a generalization or usual order of $\mathbb{R}$ to $\mathbb{E}$ then obviously $\hat{f}(x^*) \leq_1 \hat{f}(x)$ for all $x \in B_r(x^*)$ so we need to find a neighborhood of x^* in $(\mathbb{E}^n, d_\infty)$ with the same property.

We consider simply the closed ball $\tilde{B}_r(x^*) \subset (\mathbb{E}^n, d_\infty)$ with the same radius and center than the previous one, this means

$$\begin{aligned} \tilde{B}_r(x^*) =& \{u \in \mathbb{E}^n : d_\infty(u, x^*) \leq r\} \\ =& \{u \in \mathbb{E}^n : \sup_{\alpha \in [0,1]} h(u_\alpha, x^*) \leq r\} \\ =& \{u \in \mathbb{E}^n : \forall \alpha \in [0, 1] \quad h(u_\alpha, x^*) \leq r\} \\ \tilde{B}_r(x^*) =& \{u \in \mathbb{E}^n : \forall \alpha \in [0, 1] \quad u_\alpha \subset B_r(x^*),\} \end{aligned} \tag{2}$$

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and we have that $B_r(x^*) \subset \tilde{B}_r(x^*)$ when elements of $B_r(x^*)$ are considered as fuzzy as well.

If $u \in \tilde{B}_r(x^*)$, we have that for all $\alpha \in [0,1]$, $u_\alpha \subset B_r(x^*)$ so $f(u_\alpha) \subset f(B_r(x^*))$ and by definition of $B_r(x^*)$ this implies that for all $\alpha \in [0,1]$, $f(x^*) \leq y$ for all $y \in f(u_\alpha)$.

By theorem 5.1 we have that $[\hat{f}(u)]_\alpha = f(u_\alpha)$ so, for all $\alpha \in [0,1]$

$$f(x^*) \leq y \quad \text{for all} \quad y \in [\hat{f}(u)]_\alpha. \tag{3}$$

If we write $[\hat{f}(u)]_\alpha$ as an interval, this means, $[\hat{f}(u)]_\alpha = [a_\alpha, b_\alpha]$ for some $a_\alpha, b_\alpha \in \mathbb{R}$ the previous expression in the specific case of a_α and b_α becomes

$$\forall \alpha \in [0,1], \quad f(x^*) \leq a_\alpha \quad \text{and} \quad f(x^*) \leq b_\alpha. \tag{4}$$

Because $f(x^*) = \hat{f}(x^*)$ and because $f(x^*)$ considered as a fuzzy set has α-level sets always equals to $[f(x^*), f(x^*)]$, the previous considerations and the expression 4 implies that for $u \in \tilde{B}_r(x^*)$

$$\hat{f}(x^*) \leq_1 \hat{f}(u),$$

so $\hat{f}$ has a local minimum at x^*. $\square$.

7 Conclusions and Future Work

In this work has been shown that a classical result of vector space optimization about coercitivity can be extended also for optimization problems on $\mathbb{E}^n$, even if this spaces is not a vector space. We believe that many other results can be generalized in this way and we will continue to study this subject and its applications.

This kind of problem can be seen also as general metric spaces optimization problems, a subject that is lately getting attention again (See [19]), however because of the fact that $\mathbb{E}^n$ is indeed more than just a metric space (after all it has some arithmetic structure) we will try to take advantage of its particular structure to obtain better results and applications.

We also believe that there are interesting questions about the way that a fuzzy extension of a function and orderings are related to optimal points and others qualitative characteristics of a function. We will keep study this subject deeply and we hope to find interesting interpretations and future applications.

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