Some results on Lipschitz quasi-arithmetic means

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Abstract— *We present in this paper some properties of k-Lipschitz quasi-arithmetic means. The Lipschitz aggregation operations are stable with respect to input inaccuracies, what is a very important property for applications. Moreover, we provide sufficient conditions to determine when a quasi–arithemetic mean holds the k-Lipschitz property and allow us to calculate the Lipschitz constant k.*

Keywords— k-Lipschitz aggregation functions, quasi-arithmetic means, stability, triangular norms.

1 Introduction

Aggregation of several input values into a single output value is an indispensable tool in many disciplines and applications such as decision making [1], pattern recognition, expert and decision support systems, information retrieval, etc [2]. There is a wide range of aggregation functions which provide flexibility to the modeling process, including different types of aggregation functions. There are several recent books that provide details of many aggregation methods [3, 4, 5, 6, 7].

For applications it is important to design aggregation functions that are stable with respect to small perturbations of inputs (e.g., due to input inaccuracies). Such aggregation functions need not only be continuous, but Lipschitz continuous [8]. Kernel and 1-Lipschitz aggregation functions have been studied in [9, 10, 11]. It is known, for instance, that 1-Lipschitz triangular norms are copulas [12, 3, 7]. More recently, k-Lipschitz t-norms and t-conorms were studied [13, 14, 15]. k-Lipschitz t-norms do not increase the perturbation of inputs by more than a factor of k, which is suitable for many applications.

There are many other generated functions constructed similarly to the Archimedean triangular norms with the help of additive generators. In this article we examine quasi-arithmetic means and establish conditions under which these functions are Lipschitz or not Lipschitz.

Firstly, in Section 2 we recall some basic notions to develop the rest of the work. Section 3 contains the main results involving quasi–arithmetic means. At the end we provide some conclusions.

2 Preliminaries and related works

We restrict ourselves to aggregation functions defined on $[0, 1]^n$.

Definition 1 *A function $f : [0, 1]^n \to [0, 1]$ is called an aggregation function if it is monotone non-decreasing in each variable and satisfies $f(0, \dots, 0) = 0$, $f(1, \dots, 1) = 1$.*

Now, we will pay attention to a special class of aggregation function, -the class of weighted quasi–arithmetic means-, for this we need to consider a continuous strictly monotone function $g : [0, 1] \to [-\infty, \infty]$, which we call a *generating function* or generator. Of course, g is invertible, but it is not necessarily a bijection (i.e., its range may be $Ran(g) \subset [-\infty, \infty]$). Other two examples of generated functions are Archimedean t-norms and t-conorms. Further there exists a class of uninorms, known as *representable uninorms* or *generated uninorms*, that can also be built by means of additive generators. Further, a vector $\vec{w} = (w_1, \dots, w_n)$ is called a weighting vector if $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 2 *For a given generating function g, and a weighting vector $\vec{w}$, the weighted quasi-arithmetic mean is the function*

$$M_{\vec{w},g}(\vec{x}) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right). \quad (1)$$

From this definition, we have the following particular quasi–arithmetic means:

Arithmetic mean	$M(\vec{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$
Geometric mean	$G(\vec{x}) = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$
Harmonic mean	$H(\vec{x}) = n(\sum_{i=1}^{n} \frac{1}{x_i})^{-1}$
Power mean	$M_r(\vec{x}) = \frac{1}{n}(\sum_{i=1}^{n} x_i^r)^{\frac{1}{r}}$, if $r \neq 0$ and $M_0(\vec{x}) = G(\vec{x})$.

Another class of aggregation operators is the following

Definition 3 *Let $g : [0, 1] \to [-\infty, \infty]$ be a continuous strictly monotone function and let $\vec{w}$ be a weighting vector. The function*

$$GenOWA_{\vec{w},g}(\vec{x}) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_{(i)})\right) \quad (2)$$

is called a generalized OWA (also known as ordered weighted quasi-arithmetic mean [4]). As for OWA, $x_{(i)}$ denotes the i-th largest value of $\vec{x}$.

Another aggregation operator that include the previous one for the case of a symmetric fuzzy measure is the generalized discrete Choquet integral which is defined as follows

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Definition 4 *Let $g : [0,1] \to [-\infty, \infty]$ be a continuous strictly monotone function. The generalized Choquet integral with respect to a fuzzy measure v is the function*

$$C_{v,g}(\vec{x}) = g^{-1}\left(C_v(g(\vec{x}))\right),$$

where C_v is the discrete Choquet integral with respect to v and $g(\vec{x}) = (g(x_1), \ldots, g(x_n))$.

Now, we consider the crucial concept of this work

Definition 5 *An aggregation function f is called Lipschitz continuous if there is a positive number k, such that for any two vectors $\vec{x}, \vec{y}$ in the domain of definition of f:*

$$|f(\vec{x}) - f(\vec{y})| \leq kd(\vec{x}, \vec{y}), \quad (3)$$

where $d(\vec{x}, \vec{y})$ is a distance between $\vec{x}$ and $\vec{y}$. The smallest such number k is called the Lipschitz constant of f (in the distance d). We shall call such functions k-Lipschitz.

Typically the distance is chosen as a p-norm $d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||_p$, with $||\vec{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, for finite p, and $||\vec{x}||_\infty = \max_{i=1,\ldots,n} |x_i|$. In this work we concentrate on the 1–norm.

Definition 6 *A function f is called locally Lipschitz continuous on Ω if for every $x \in \Omega$ there exists a neighbourhood $D(x)$ such that f restricted to $D(x)$ is Lipschitz.*

Of course, duality w.r.t. standard negation preserves Lipschitz property (and the Lipschitz constant). It is easy to see that if an aggregation function A is k–Lipschitz, it is also m–Lipschitz for any $m \geq k$. Also any convex combination of $k-$Lipschitz aggregation functions $f = \alpha f_1 + \beta f_2$, $\alpha + \beta = 1, \alpha, \beta \geq 0$, is $k-$Lipschitz.

The class of k–Lipschitz t–norms, whenever $k > 1$, has been already characterized (see [13]). Note that 1–Lipschitz t–norms are copulas, see, e.g., [3, 5]. A strictly decreasing continuous function $g : [0,1] \to [0,1]$ with $g(1) = 0$ is an additive generator of a 1–Lipschitz Archimedean t–norm if and only if g is convex.

The k–Lipschitz property implies continuity of the t–norm. Recall that a continuous t–norm can be represented by means of an ordinal sum of continuous Archimedean t–norms, and that a continuous Archimedean t–norm can be represented by means of a continuous additive generator [3, 16]. Characterization of all k–Lipschitz t–norms can be reduced to the problem of characterization of all Archimedean k–Lipschitz t–norms.

Definition 7 *Let $g : [0,1] \to [0, +\infty]$ be a strictly monotone function and let $k \in]0, +\infty[$ be a real constant. Then g will be called k–convex if*

$$g(x + k\varepsilon) - g(x) \leq g(y + \varepsilon) - g(y)$$

holds for all $x \in [0,1[, y \in]0,1[$, with $x \leq y$ and $\varepsilon \in]0, \min(1-y, \frac{1-x}{k})]$.

Obviously, if $k = 1$ the function g is convex.

Observe that, a k–convex monotone function is also continuous in $]0,1[$, as was earlier. A decreasing function g can be k–convex only for $k \geq 1$. Moreover, when a decreasing function g is k–convex, it is also m–convex for all $m \geq k$. In the case of a strictly increasing function $g^\star$, it can be k-convex only for $k \leq 1$. Moreover, when $g^\star$ is k–convex, it is m–convex for all $m \leq k$.

Considering $k \geq 1$ and a strictly decreasing function g, we provide the following characterization given in [13].

Proposition 1 *Let $T : [0,1]^2 \to [0,1]$ be an Archimedean t–norm and let $g : [0,1] \to [0, +\infty], g(1) = 0$ be an additive generator of T. Then T is k–Lipschitz if and only if g is k–convex.*

Another useful characterizations are the following

Corollary 1 *(Y.-H. Shyu) [17] Let $g : [0,1] \to [0, \infty]$ be an additive generator of a t–norm T which is differentiable on $]0,1[$ and let $g'(x) < 0$ for $0 < x < 1$. Then T is k–Lipschitz if and only if $g'(y) \geq kg'(x)$ whenever $0 < x < y < 1$.*

Corollary 2 *Let $T : [0,1]^2 \to [0,1]$ be an Archimedean t–norm and let $g : [0,1] \to [0, \infty]$ be an additive generator of T such that g is differentiable on $]0,1[\backslash S$, where $S \subset [0,1]$ is a discrete set. Then T is k–Lipschitz if and only if $kg'(x) \leq g'(y)$ for all $x, y \in [0,1], x \leq y$ such that $g'(x)$ and $g'(y)$ exist.*

The following useful results follow from Corollary 1, with it we can determine whether a given piecewise differentiable t–norm is k–Lipschitz.

Corollary 3 *Let $T : [0,1]^2 \to [0,1]$ be an Archimedean t–norm and let $g : [0,1] \to [0, \infty]$ be its additive generator differentiable on $]0,1[$, and $g'(t) < 0$ on $]0,1[$. If*

$$\inf_{t \in]x,1[} g'(t) \geq k \sup_{t \in]0,x[} g'(t)$$

holds for every $x \in]0,1[$ then T is k–Lipschitz.

Corollary 4 *Let $g : [0,1] \to [0, \infty]$ be a strictly decreasing function, differentiable on $]0,a[\cup]a,1[$. If g is k-convex on $[0,a[$ and on $]a,1]$, and if*

$$\inf_{t \in]a,1[} g'(t) \geq k \sup_{t \in]0,a[} g'(t),$$

then g is k–convex on $[0,1]$.

Remark 1 *Generated uninorms are not k–Lipschitz since that they are not continuous at (0,1) and (1,0) (in the binary case). Nullnorms are aggregation functions related to t–norms and t–conorms. In this case, it is clear that a nullnorm V is k–Lipschitz if and only if the underlying t-norm and t–conorm are k–Lipschitz, and the k–Lipschitz constant of V is the maximum of Lipschitz constants of the underlying t–norm and t–conorm.*

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3 Quasi-arithmetic means

Consider a univariate continuous strictly monotone function $g : [0, 1] \to [-\infty, \infty]$, called generator.

For a given g, the quasi-arithmetic means are defined in Definition 2, and are denoted by M_g. We start with bivariate quasi-arithmetic means of such operations. Quasi-arithmetic means are continuous if and only if $Ran(g) \neq [-\infty, \infty]$ [18]. Moreover, its generator is not defined uniquely, i.e., if $g(t)$ is a generating function of some weighted quasi-arithmetic mean, then $ag(t) + b$, $a, b \in \Re$, $a \neq 0$ is also a generating function of the same mean provided $Ran(g) \neq [-\infty, \infty]$. For this reason, one can assume that g is monotone increasing, as otherwise we can simply take $-g$.

We shall consider two cases: I) $g(0) = 0$, $g(1) = 1$, and II) $g(0) = -\infty, g(1) = 0$. Of course, by duality we also cover the case $g(1) = \infty, g(0) = 0$, and by using appropriate linear transformations, all generators can be reduced to the mentioned cases.

Let us make some preliminary remarks on convexity. As opposed to the case of convex additive generators of t-norms, where the resulting t-norms are 1-Lipschitz, convexity of the generator g does not play any role by itself for quasi-arithmetic means. Since both g and $-g$ are generators of the same mean, and obviously when g is convex $-g$ is concave, convexity of g by itself does not lead to the Lipschitz condition. Also note that $g(x) = -\ln(x)$ is a convex generator of the geometric mean $G(x, y) = \sqrt{xy}$, which is not Lipschitz. Further, even if g is convex and increasing or convex and decreasing, this does not imply Lipschitz condition either: note that $g_d(x) = 1 - g(1 - x)$ is a generator of a quasi-arithmetic mean dual to the one generated by g, and Lipschitz condition is preserved under duality. If g is convex increasing, then g_d is convex decreasing and vice versa. Thus we will look for a different condition.

3.1 *Case of finite generators*

We start with the case I) of g finite. First, let us show that g must be Lipschitz on [0,1]. For short, we will denote $M_{\vec{w},g}$ by M_g.

Lemma 1 *Let g be finite locally Lipschitz and continuously differentiable except at a point $a \in [0; 1]$. Then M_g is not k–Lipschitz for any k.*

Proof. Suppose that M_g is k-Lipschitz, which means it is differentiable almost everywhere in its domain (Rademacher's theorem), and we must have

$$\frac{\partial M_g}{\partial x}(x, y) \leq k,\ x, y \neq a$$

whenever such a derivative exists, and similarly for the other partial derivative. Since M_g is symmetric, only the derivative with respect to x is needed.

$$\frac{\partial M_g}{\partial x} = \frac{1}{g'(M_g(x, y)))} \cdot \frac{1}{2}g'(x) \leq k.$$

Since g is strictly increasing we must have $\frac{1}{2}g'(x) \leq kg'(M_g(x, y))$ for all $x, y \in [0, 1]$ such that $x \neq a$ and $M_g(x, y) \neq a$ or

$$\frac{1}{2}g'(x) \leq k \cdot \inf_{y\in[0,1]} g'(M_g(x, y)). \tag{4}$$

Since g is finite, M_g does not have an absorbing element. Let $\lim_{x\to a} g'(x) = \infty$.

$\exists y \neq a : z = M_g(a, y) \neq a$ such that $g'(z) \leq M < \infty$ (because g is locally Lipschitz). Then inequality (4) fails, because we can always choose such $x \neq a$ that $g'(x) > 2kM$, which would give us

$$kM < \frac{1}{2}g'(x) \leq kg'(z) \leq kM,$$

which is false. Then $\frac{\partial M_g}{\partial x} > k$, hence M_g is not Lipschitz. □

Now, since g is Lipschitz on $[0, 1]$, it is differentiable almost everywhere, which means that the left- and right-derivatives exist in $[0, 1]$. We start with the case of g differentiable on $[0, 1]$, and then adapt it to g differentiable almost everywhere by using left- and right-derivatives. Let M_g be k-Lipschitz. Then we must have

$$\frac{\partial M_g}{\partial x}(x, y) \leq k.$$

Following the same procedure as in Lemma 1, we get the condition

$$\frac{1}{2}g'(x) \leq k \cdot \inf_{y\in[0,1]} g'(M_g(x, y)) \tag{5}$$

for all $x \in [0, 1]$. Finally, by using left- and right- derivatives g'_-, g'_+ we obtain a general condition for non-smooth increasing generators

$$\frac{1}{2}g'_-(x) \leq k \inf_{z\in[M(x,0),M(x,1)]} g'_-(z) \tag{6}$$
$$\frac{1}{2}g'_+(x) \leq k \inf_{z\in[M(x,0),M(x,1)]} g'_+(z)$$

for all $x \in]0, 1[$, and only one of the above inequalities for $x = 0$ and $x = 1$.

Remark 2 *If g is finite and concave increasing, then it is sufficient to check*

$$\frac{1}{2}g'(x) \leq kg'(M(x, 1)) = k(g' \circ g^{-1}) \left(\frac{g(x)}{2} + \frac{1}{2}\right),$$

(and similarly for left- and right-derivatives if g is not smooth). If g is finite and convex increasing, it is sufficient to check

$$\frac{1}{2}g'(x) \leq kg'(M(x, 0)) = k(g' \circ g^{-1}) \left(\frac{g(x)}{2}\right).$$

Let us provide some examples of Lipschitz and non-Lipschitz quasi-arithmetic means.

Example 1 *If g is linear (M_g is the arithmetic mean), $g'(x) = const$, and M_g is k-Lipschitz for $k = \frac{1}{2}$.*

Example 2 *If $g(x) = x^p, p > 1$ (M_g is a power mean $M_{[p]}$), $g'(x) = px^{p-1}$, and M_g is k-Lipschitz for $k = \left(\frac{1}{2}\right)^{\frac{1}{p}}$. It follows from*

$$\frac{1}{2}px^{p-1} \leq kp\left(\frac{x^p}{2}\right)^{\frac{p-1}{p}} = kpx^{p-1}\left(\frac{1}{2}\right)^{\frac{p-1}{p}}.$$

Example 3 *If $g(x) = x^p, 0 < p < 1$ (M_g is a power mean $M_{[p]}$), M_g is not Lipschitz by Lemma 1.*

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3.2 *Case of generators infinite at 0*

Now we turn to the case II), g increasing with $g(0) = -\infty$, which entails that 0 is the absorbing element of M_g. We have an analogue of Lemma 1. The proof is similar, except that it fails for $a = 0$, hence the modification.

Let g be finite locally Lipschitz and continuously differentiable except at a point a 2 [0; 1]. Then Mg is not kLipschitz for any k

Lemma 2 *Let g be locally Lipschitz and continuously differentiable except at a point $a\ \in]0,1]$. Then M_g is not k-Lipschitz for any k.*

Let g be finite locally Lipschitz and continuously differentiable except at a point $a \in [0;1]$.

For $x \in]0,1]$ we have condition (6), to which we add condition

$$\frac{1}{2}\lim_{x\to 0^+}\frac{g'_+(x)}{g'_+(M(x,y))} \le k \tag{7}$$

for any fixed $y \in]0,1]$. This condition may or may not be satisfied depending on the rate at which $M(x,y)\to 0$ as $x\to 0$. The choice of $y > 0$ is irrelevant as $g(y)$ is finite and disappears under the limit.

Example 4 *If $g(x) = -x^p, p < -1$ (M_g is a power mean $M_{[p]}$), $g'(x) = -px^{p-1}$, and M_g is k-Lipschitz for $k = \left(\frac{1}{2}\right)^{\frac{1}{p}}$. Differentiating M_g*

$$\begin{array}{lcl}
\frac{\partial M_g}{\partial x} & = & \frac{1}{p}\left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}-1}\frac{p}{2}x^{p-1}\\
 & = & \left(\frac{1}{2}\right)^{\frac{1}{p}} x^{-p(\frac{1}{p}-1)}(x^p+y^p)^{\frac{1}{p}-1}\\
 & = & \left(\frac{1}{2}\right)^{\frac{1}{p}}(1+x^{-p}y^p)^{\frac{1}{p}-1}.\\
\textit{Given } p<-1, & & \\
k & = & \left(\frac{1}{2}\right)^{\frac{1}{p}}\lim\limits_{x\to 0}(1+x^{-p}y^p)^{\frac{1}{p}-1} = \left(\frac{1}{2}\right)^{\frac{1}{p}}.
\end{array}$$

Example 5 *Let $M_{[p]}$, $-1 < p < 0$ be a power mean with a generator given by $g(x) = -x^p = -x^{-\frac{1}{q}},\ q > 1$). The Lipschitz constant will be $k = \sup\frac{\partial M}{\partial x} = 2^q$. To see this*

$$\begin{array}{ll}
\frac{\partial M_g}{\partial x} & = \frac{-q}{2}\left(\frac{x^{-\frac{1}{q}}+y^{-\frac{1}{q}}}{2}\right)^{-q-1}\cdot\left(-\frac{1}{q}x^{-\frac{1}{q}-1}\right)\\
 & = 2^q x^{\frac{1}{q}(-q-1)}(x^{-\frac{1}{q}}+y^{-\frac{1}{q}})^{-q-1}\\
 & = 2^q\left(1+\left(\frac{x}{y}\right)^{\frac{1}{q}}\right)^{-q-1}.\\
k & = \sup\left\{2^q\left(1+\left(\frac{x}{y}\right)^{\frac{1}{q}}\right)^{-q-1}\right\}\\
 & = 2^q = \left(\frac{1}{2}\right)^{\frac{1}{p}}.
\end{array}$$

Condition 7 deals with the asymptotic behavior of the additive generators near 0. Its direct verification for a given g may be difficult. In the remainder of this section we will establish two sufficient conditions that guarantee that a quasi-arithmetic mean is not Lipschitz (although it is continuous). These conditions are easier to verify, and they provide a tool for a quick screening of additive generators with respect to their suitability for applications. One sufficient condition involves an inequality on the derivatives of the inverse of an additive generator. The other condition is that a decreasing additive generator cannot decrease slower than a certain rate (1/polynomial) when $x\to 0$. We will express this rate through the growth of an auxiliary function $1/g^{-1}$, for which the growth is expressed in traditional terms (e.g., polynomial) when $x\to\infty$. First, two simple auxiliary results.

Lemma 3 *If two functions f, g are continuous and differentiable at $x = 0$ and, $f(0) = g(0)$ and $f(x)\ge g(x)$ for $x > 0$, then $f'(0)\ge g'(0)$.*

Proof: Follows directly from the definition of the derivative.

The next result is a well-known condition for comparability of quasi-arithmetic means, see, e.g., [19].

Theorem 1 *Let g_1, g_2 be the generators of quasi-arithmetic means M_{g_1} and M_{g_2},and g_1 decreasing. Then $M_{g_1}\le M_{g_2}$ if and only if $g_1\circ g_2^{-1}$ is convex.*

Theorem 2 *Let g be an increasing (decreasing) twice continuously differentiable on $]0,1]$ generator of a quasi-arithmetic mean M_g where $g^{-1} = h$, and $\lim\limits_{x\to 0} g(x) = -\infty$ ($\lim\limits_{x\to 0} g(x) = +\infty$). If $h'^2 - hh''\ge 0$ then M_g is not Lipschitz.*

Proof. We will show that $M_{[p]}\le M_g$ for any $-1 < p < 0$ decreasing, and hence by Lemma 3 is not Lipschitz. If $x^p\circ g^{-1}$ is convex, for $-1 < p < 0$ by Theorem 1, with $g_1(x) = x^p$, $M_{[p]}\le M_g$. Let us show that $(x^p\circ h)''\ge 0$.

$$\begin{array}{ll}
(x^p\circ h)' & = ph^{p-1}h'\\
(x^p\circ h)'' & = p(p-1)h^{p-2}h'^2 + ph^{p-1}h''\\
 & = ph^{p-2}\left((p-1)h'^2 + hh''\right)\ge 0.
\end{array}$$

Given $ph^{p-2} < 0$ for $p < 0, h > 0$, convexity will hold if for all $p < 0$

$$(1-p)h'^2 - hh''\ge 0. \tag{8}$$

Therefore $h'^2 - hh''\ge 0$ implies $(x^p\circ h)''\ge 0$ and $M_{[p]}\le M_g$, and by Lemma 3 the Lipschitz constant of M_g is greater than that of $M_{[p]}$, which is $2^{-\frac{1}{p}}$, and $p\to 0^-$. □

Remark 3 *The generator g can be either increasing or decreasing. Clearly when changing g to $-g$, we change $h(x)$ to $h(-x)$. Then $h^{'}$ changes the sign but $h^{''}$ does not, hence the inequality in Theorem 2is the same for either increasing or decreasing generators.*

Example 6 *Using the geometric mean M_g, take $g(x) = \ln x$ with $h(x) = h^{'}(x) = h^{''}(x) = e^x$. Then $(h'^2 - hh'')(x) = e^{2x} - e^{2x} = 0$. Therefore M_g is not Lipschitz.*

For the sake of convenience, we will formulate our next result for decreasing additive generators satisfying $g(0) = \infty$. To obtain the respective condition on the increasing generators, we simply invert the sign of g.

Theorem 3 *Let $h = g^{-1}$ be the inverse of a decreasing generator g of a quasi-arithmetic mean M_g. If the function $\tilde{h} = \frac{1}{h}$ grows faster than any power $x^q, q > 0$, then M_g is not Lipschitz.*

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Proof. Fix y so that $g(y) = h^{-1}(y) = 0$, which is always possible (we remind that g is defined up to an arbitrary linear transformation).

$$\begin{aligned}\lim_{x\to 0}\frac{\partial M_g(x,y)}{\partial x} &= \lim_{x\to 0}\frac{dh\left(\frac{h^{-1}(x)}{2}\right)}{dx},\\ &= \lim_{x\to 0}\frac{1}{2}h'\left(\frac{h^{-1}(x)}{2}\right)(h^{-1})'(x)\\ &= \frac{1}{2}\lim_{x\to 0}h'\left(\frac{h^{-1}(x)}{2}\right)\frac{1}{h'(h^{-1}(x))}.\end{aligned}$$

Let $z = \frac{h^{-1}(x)}{2}$. Then

$$\lim_{x\to 0}\frac{\partial M_g(x,y)}{\partial x} = \frac{1}{2}\lim_{z\to\infty}\frac{h'(z)}{h'(2z)}.$$

Since h decreases faster than the power function $p(z) = Cz^r$, l'Hôpital's rule gives

$$0 = \lim_{z\to\infty}\frac{h(z)}{p(z)} = \lim_{z\to\infty}\frac{h'(z)}{p'(z)} = \lim_{z\to\infty}\frac{h'(2z)}{p'(2z)}.$$

For convenience of notation take p such that $p'(z) = \frac{1}{z^q}$. Then $p'(z) = p'(2z)2^q$.

$$\begin{aligned}\lim_{x\to 0}\frac{\partial M_g(x,y)}{\partial x} &= \frac{1}{2}\lim_{z\to\infty}\frac{h'(z)}{h'(2z)} = \frac{1}{2}\lim_{z\to\infty}\frac{h'(z)}{h'(2z)}\frac{p'(2z)2^q}{p'(z)}\\ &= 2^{q-1}.\end{aligned}$$

Since q can be arbitrarily large, the derivative is unbounded and M_g is not Lipschitz.

□

Example 7 *Let the generator be $g(x) = -\ln x$ as in Example 6. Clearly its inverse is $\exp(-x)$, and the auxiliary function $\frac{1}{h(x)} = \exp(x)$, which grows faster than any polynomial, hence the corresponding geometric mean is not Lipschitz.*

Further take any power of the logarithm $g(x) = (-\ln x)^r, r > 1$. The auxiliary function $\frac{1}{h(x)} = \exp(x^{\frac{1}{r}})$, it grows faster than a polynomial, hence the resulting mean is not Lipschitz either. Note that this quasi-arithmetic mean is related to the Aczél-Alsina family of t-norms [3, 5] by the equation

$$M_g = (T_r^{AA})^{\frac{1}{\sqrt[r]{2}}},$$

which shows directly that M_g is not Lipschitz ($f(x) = M_g(x,1) = T_r^{AA}(x,1)^{\frac{1}{\sqrt[r]{2}}} = x^{\frac{1}{\sqrt[r]{2}}}$ is not Lipschitz).

Example 8
Consider the generator $g(x) = (-\ln x)^2, \tilde{h}(x) = e^{\sqrt{x}}$. From the previous example, $r = 2$ and we know the resulting mean is not Lipschitz, however this would not have been apparent from the application of Theorem 2, as

$$\begin{aligned}h'^2 - hh'' &= \frac{1}{4x}e^{-2\sqrt{x}} - \frac{1}{4x}e^{-2\sqrt{x}} - \frac{1}{4\sqrt{x^3}}e^{-2\sqrt{x}}\\ &= -\frac{1}{4\sqrt{x^3}}e^{-2\sqrt{x}} < 0.\end{aligned}$$

3.3 Weighted quasi-arithmetic means

We adapt conditions (6) and (7) for the case of unequal weights. For this we take partial derivatives with respect to all arguments. The Lipschitz constant is the largest, hence we have conditions

$$\begin{aligned}g'_-(x) &\le \frac{k}{\max w_i}\min_z g'_-(z) \qquad (9)\\ g'_+(x) &\le \frac{k}{\max w_i}\min_z g'_+(z)\end{aligned}$$

where
the minimum for z is over $[M(x,0,\ldots,0), M(x,1,\ldots,1)]$, and

$$\lim_{x\to 0^+}\frac{g'_+(x)}{g'_+(M(x,c,\ldots,c))} \le \frac{k}{\max w_i} \qquad (10)$$

with $c \in]0,1]$.

Conditions (9) and (10) can also be used for symmetric means in the multivariate case, where $\max w_i = \frac{1}{n}$. It is clearly seen that the higher the number of variables, the smaller is the Lipschitz constant, if it exists.

Remark 4 *Similar, results can be obtained for generalized OWAs and generalized Choquet integrals.*

4 Conclusions

k-Lipschitz aggregation functions are important for applications because they can control the changes in the outputs due to input inaccuracies, to a fixed factor of k. k-Lipschitz triangular norms and conorms have been already characterized by k-convex additive generators, however no analogous results were available for quasi-arithmetic means. We have found verifiable conditions which guarantee that an aggregation function is $k-$Lipschitz for a given k, or alternatively, not Lipschitz. We also presented various examples of both Lipschitz and non-Lipschitz aggregation functions. Our results will benefit those who design aggregation functions for practical applications, as they allow one to make an informed choice on suitability of specific functions for these applications.

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