Analysis of Similarity Measures for Atanassov's Intuitionistic Fuzzy Sets

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Abstract— *We consider some existing similarity measures for Atanassov's intuitionistic fuzzy sets (A-IFSs, for short). We show that neither similarity measures treating an A-IF as a simple interval values fuzzy set, nor straightforward generalizations of the similarity measures well-known for the classic fuzzy sets work under reasonable circumstances. Next, expanding upon our previous works, we consider a family of similarity measures constructed by taking into account both all the three functions (the membership, non-membership and hesitation) describing an A-IF, and the complements of the elements we compare to each other. That is, we use all kinds and fine shades of information available. We point out their proper behavior and an intuitive appeal.*

Keywords— Atanassov's intuitionistic fuzzy sets, similarity measures.

1 Introduction

Atanassov's intuitionistic fuzzy sets (Atanassov [1], [2], [3]) – to be called A-IFSs, for short – can be viewed as a tool that may better model and process imperfect information. The use of positive and (independently) negative information that is the core of A-IFSs is natural in real life and is also well-known, advocated and studied in psychology and other social and behavioral sciences [e.g., [21], [14]]. It also attracted much attention and research interest in soft computing. It would be difficult to deal with machine learning (making use of examples and counter-examples), modeling of preferences or voting without taking into account positive and (independently) negative testimonies or opinions. Although from a mathematical point of view A-IFSs are equipotent, under some assumptions, to interval-valued fuzzy sets (cf. Atanassov and Gargov in 1989 [4]), from the point of view of solving problems (starting from the stage of collecting data), they are different as A-IFSs force the user to explicitly consider positive and negative information independently. On the other hand, while employing the interval-valued fuzzy sets, the user's attention is focused on positive information (in an interval form) only. Notably, Dubois [9] noticed recently that A-IFSs correspond to an intuition that differs from that behind the interval valued fuzzy sets.

The fact that people tend to notice and take into account only most obvious aspects (e.g. advantages only) when making decision is well known in psychology (cf. Kahneman [14]), Sutherland [21]) and may often lead to improper decisions. In this context, A-IFSs ("forcing" a decision maker to take into account both negative and positive aspects of the decisions) may be seen as belonging to modern and promising means of knowledge representation and processing.

We consider here some major similarity measures for A-IFSs. First we present a whole array of similarity measures (known from the literature) for A-IFSs viewed in terms of single intervals. Second, we consider measures that are straightforward generalization of the similarity measures for the conventional fuzzy sets. Unfortunately, both do not meet our expectations, and we provide some counter-intuitive examples. It seems to be an indirect hint that A-IFSs are not functionally equivalent to the interval valued fuzzy sets.

Next, we reconsider our (cf. Szmidt and Kacprzyk [41]) concept of a similarity measure between A-IFSs taking into account all three functions (the membership, non-membership and hesitation), and explicitly add to the above three functions the complements of the elements we compare to each other.

2 A brief introduction to A-IFSs

One of the possible generalizations of a fuzzy set in X (Zadeh [48]), given by

$$A^{'} = \{< x, \mu_{A'}(x) > | x \in X\} \tag{1}$$

where $\mu_{A'}(x) \in [0, 1]$ is the membership function of the fuzzy set $A^{'}$, is Atanassov's intuitionistic fuzzy set (Atanassov [1], [2], [3]), A-IF, A given by

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} \tag{2}$$

where: $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ such that

$$0 \underline{\leq} \mu_A(x) + \nu_A(x) \underline{\leq} 1 \tag{3}$$

and $\mu_A(x)$, $\nu_A(x) \in [0, 1]$ denote the degree of membership and a degree of non-membership of $x \in A$, respectively.

Each fuzzy set may be represented by the following A-IF

$$A = \{< x, \mu_{A'}(x), 1 - \mu_{A'}(x) > | x \in X\} \tag{4}$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{5}$$

an *intuitionistic fuzzy index* (or a *hesitation margin*) of $x \in A$ and, it expresses a lack of knowledge of whether x belongs to A or not (cf. Atanassov [3]). It is obvious that $0 \leq \pi_A(x) \leq 1$, for each $x \in X$.

The hesitation margin turns out to be important while considering the distances (Szmidt and Kacprzyk [26], [29], [39], entropy (Szmidt and Kacprzyk [31], [40]), similarity (Szmidt and Kacprzyk [41]) for the A-IFSs, etc. i.e., the measures that play a crucial role in virtually all information processing tasks.

In our further considerations we will use the following distances between fuzzy sets A, B in $X = \{x_{1,}, \ldots, x_n\}$ Szmidt

ISBN: 978-989-95079-6-8

and Baldwin [23], [24], Szmidt and Kacprzyk [29], [39]: the normalized Hamming distance

$$l_{IFS}(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|) \quad (6)$$

and the normalized Euclidean distance:

$$q_{IFS}(A,B) = (\frac{1}{2n}\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 + (\pi_A(x_i) - \pi_B(x_i))^2)^{\frac{1}{2}} \quad (7)$$

For distances (6), and (7) we have $0 \leq l_{IFS}(A,B) \leq 1$, and $0 \leq q_{IFS}(A,B) \leq 1$. Clearly these distances satisfy the conditions of the metric.

In our further considerations we will use the notion of the complement elements, which definition is a simple consequence of a complement set A^C

$$A^C = \{< x, \nu_A(x), \mu_A(x) > | x \in X\} \quad (8)$$

The use of A-IFSs instead of fuzzy sets implies the introduction of another degree of freedom (non-memberships) into the set description. Such a generalization of fuzzy sets gives us an additional possibility to represent imperfect knowledge which leads to describing many real problems in a more adequate way. Applications of intuitionistic fuzzy sets to group decision making, negotiations, voting and other situations are presented in Szmidt and Kacprzyk [25], [27], [28], [30], [32], [34], [33], [35], [38], Szmidt and Kukier [42], [43].

3 Some counter-intuitive results given by the traditional similarity measures

In the literature there is a multitude of similarity measures both for A-IFSs (Atanassov [1, 2, 3], and vague sets (Gau and Buehrer [10]) which have also been proved to be equivalent to A-IFSs (Bustince and Burillo [5]). Here we adopt the notation for A-IFSs but we will consider the measures originally introduced for vague sets, too.

Chen [6, 7] considered similarity measures between two A-IFSs A and B as

$$S_C(A,B) = 1 - \frac{\sum_{i=1}^{n}|S_A(x_i) - S_B(x_i)|}{2n} \quad (9)$$

where $S_A(x_i) = \mu_A(x_i) - \nu_A(x_i)$, $S_A(x_i) \in [-1,1]$ and $S_B(x_i) = \mu_B(x_i) - \nu_B(x_i)$, $S_B(x_i) \in [-1,1]$.

But, as Hong and Kim [11] noticed

$$\mu_A(x_i) - \nu_A(x_i) = \mu_B(x_i) - \nu_B(x_i) \Rightarrow S_C(A,B) = 1 \quad (10)$$

which is counterintuitive as, e.g., for $A = (x, 0, 0)$ and $B = (x, 0.5, 0.5)$, we have $S_C(A,B) = 1$.

To overcome the problem of S_C (9), Hong and Kim [11] proposed the similarity measures S_H and S_L

$$S_H(A,B) = 1 - (\sum_{i=1}^{n}|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)/2n \quad (11)$$

$$S_L(A,B) = 1 - \frac{1}{4n}((\sum_{i=1}^{n} S_A(x_i) - S_B(x_i)) - (\sum_{i=1}^{n}|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)) \quad (12)$$

Since $S_H(A,B)$ takes into account the absolute values, it does not distinguish the positive from negative differences, e.g., for $A = \{(x, 0.3, 0.3)\}$, $B = \{(x, 0.4, 0.4)\}$, $C = \{(x, 0.3, 0.4)\}$, and $D = \{(x, 0.4, 0.3)\}$, we obtain from (11) that $S_H(A,B) = S_H(C,D) = 0.9$ which seems counter-intuitive.

$S_L(A,B)$ also gives counter-intuitive results, e.g. for $A = \{(x, 0.4, 0.2)\}$, $B = \{(x, 0.5, 0.3)\}$, $C = \{(x, 0.5, 0.2)\}$, we obtain from (12) $S_L(A,B) = S_L(A,C) = 0.95$ which seems counter-intuitive.

The same problem like with S_H occurs with the similarity measure (cf. Li et al. [16]):

$$S_O(A,B) = 1 - (1/2n)^{0.5}(\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2)^{0.5} \quad (13)$$

Dengfeng and Chuntian [15] considered the similarity measure:

$$S_{DC}(A,B) = 1 - (1/n^{1/p})(\sum_{i=1}^{n}(|m_A(x_i) - m_B(x_i)|)^p)^{1/p} \quad (14)$$

where $m_A(x_i) = (\mu_A(x_i) + 1 - \nu_A(x_i))/2$, $m_B(x_i) = (\mu_B(x_i) + 1 - \nu_B(x_i))/2$, $1 \leq p < \infty$. Unfortunately, as for (14), the medians of two intervals are compared only, it is rather easy to point out the counter-intuitive examples, e.g., $A = (x, 0.4, 0.2)$, $B = (x, 0.5, 0.3)$, then $S_{DC}(A,B) = 1$, for each p.

Mitchell [18] modified Dengfeng and Chuntian's measure S_{DC} (14) using a statistical approach by interpreting A-IFSs as families of ordered fuzzy sets. Let $\rho_\mu(A,B)$ and $\rho_\nu(A,B)$ denote a similarity measure between the "low" membership functions μ_A and μ_B, and between the "high" membership functions $1 - \nu_A$ and $1 - \nu_B$, respectively, as:

$$\rho_\mu(A,B) = S_{DC}(\mu_A, \mu_B) = 1 - (1/n^{1/p})(\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)|)^p)^{1/p}$$

$$\rho_\nu(A,B) = S_{DC}(1 - \nu_A, 1 - \nu_B) = 1 - (1/n^{1/p})(\sum_{i=1}^{n}(|\nu_A(x_i) - \nu_B(x_i)|)^p)^{1/p}$$

Then the modified similarity measure between A and B is

$$S_{HB}(A,B) = (\rho_\mu(A,B) + \rho_\nu(A,B))/2 \quad (15)$$

Unfortunately, S_{HB} gives the same counter-intuitive results as S_H, for $p = 1$ and for one-element sets.

To overcome the drawbacks of S_{DC}, Liang and Shi [17] proposed $S_e^p(A,B)$, $S_s^p(A,B)$, $S_h^p(A,B)$ as:

ISBN: 978-989-95079-6-8

$$S_e^p(A,B) = 1 - (1/n^{1/p})(\sum_{i=1}^{n}(\phi_{\mu AB}(x_i) - \phi_{\nu AB}(x_i)|)^p)^{1/p} \quad (16)$$

where $\phi_{\mu AB}(x_i) = |\mu_A(x_i) - \mu_B(x_i)|/2$, $\phi_{\nu AB}(x_i) = |(1 - \nu_A(x_i))/2 - (1 - \nu_B(x_i))|/2$. But, for $p = 1$ and for one-element sets, $S_e^p(A,B)$=S_{HB}=S_H, which are again the same counter-intuitive results.

$$S_s^p(A,B) = 1 - (1/n^{1/p})(\sum_{i=1}^{n}(\varphi_{s1}(x_i) - \varphi_{s2}(x_i))^p)^{1/p} \quad (17)$$

where: $\varphi_{s1}(x_i) = |m_{A1}(x_i) - m_{B1}(x_i)|/2$,
$\varphi_{s2}(x_i) = |m_{A2}(x_i) - m_{B2}(x_i)|/2$,
$m_{A1}(x_i) = (\mu_A(x_i) + m_A(x_i))/2$,
$m_{A2}(x_i) = (m_A(x_i) + 1 - \nu_A(x_i))/2$,
$m_{B1}(x_i) = (\mu_B(x_i) + m_B(x_i))/2$,
$m_{B2}(x_i) = (m_B(x_i) + 1 - \nu_B(x_i))/2$,
$m_A(x_i) = (\mu_A(x_i) + 1 - \nu_A(x_i))/2$,
$m_B(x_i) = (\mu_B(x_i) + 1 - \nu_B(x_i))/2$.

S_s^p (17) avoids the problematic results obtained from S_{DC} (14) (for the intervals with equal medians) but, again, a problem of counter-intuitive results remains. For example, for $A = \{(x, 0.4, 0.2)\}$, $B = \{(x, 0.5, 0.3)\}$, $C = \{(x, 0.5, 0.2)\}$, we obtain $S_s^p(A,B) = S_s^p(A,C) = 0.95$ which seems difficult to accept.

S_h^p is given as

$$S_h^p(A,B) = 1 - (1/(3n)^{1/p})(\sum_{i=1}^{n}(\eta_1(i) + \eta_2(i) + \eta_3(i))^p)^{1/p} \quad (18)$$

where $\eta_1(i) = \phi_\mu(x_i) + \phi_\nu(x_i)$ (the same as for S_e^p),
$\eta_2(i) = m_A(x_i) - m_B(x_i))$ (the same as for S_{DC}),
$\eta_3(i) = \max(l_A(i), l_B(i)) - \min(l_A(i), l_B(i))$,
$l_A(i) = (1 - \nu_A(x_i) - \mu_A(x_i))/2$,
$l_B(i) = (1 - \nu_B(x_i) - \mu_B(x_i))/2$. But, again, there are counter-intuitive cases for this measure. For $A = (x, 0.3, 0.4)$, and $B = (x, 0.4, 0.3)$, i.e., for an element and its complement, we obtain $S_h^p(A,B) = 0.933$ (which seems to be rather too big a similarity for an element and its complement).

Hung and Yang [12] proposed the similarity measures S_{HY}^1, S_{HY}^2, S_{HY}^3 in which Hausdorff distances were employed:

$$S_{HY}^1(A,B) = 1 - d_H(A,B) \quad (19)$$

$$S_{HY}^2(A,B) = 1 - (e^{d_H(A,B)} - e^{-1})/(1 - e^{-1}) \quad (20)$$

$$S_{HY}^3(A,B) = (1 - d_H(A,B))/(1 + d_H(A,B)) \quad (21)$$

where
$d_H(A,B)$)=$\sum_{i=1}^n \max(|\mu_A(x_i) - \mu_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)|)$
Unfortunately, (19)–(21) give counter-intuitive results (implied by the calculation of $d_H(A,B)$ - cf. Szmidt [22]). For example if $A = (x, 0.4, 0.5)$, $B = (x, 0.5, 0.4)$, $C = (x, 0.5, 0.3)$, $D = (x, 0.6, 0.4)$, $E = (x, 0.6, 0.3)$, $F = (x, 0.4, 0.3)$ then $S_{HY}^1(A,B) = 0.9$ (a counter-intuitive large similarity for A and its complement as $B = A^C$), and also $S_{HY}^1(C,D) = S_{HY}^1(C,E) = S_{HY}^1(C,F) = 0.9$. Next, $S_{HY}^2(A,B) = S_{HY}^2(C,D) = S_{HY}^2(C,E) = S_{HY}^2(C,F) = 0.85$, and also $S_{HY}^3(A,B) = S_{HY}^3(C,D) = S_{HY}^3(C,E) = S_{HY}^3(C,F) = 0.85$.

A straightforward attempt to calculate the similarity between A-IFSs just by adding the non-memberships values to the existing similarity measures for fuzzy sets is due to Hung and Yang [13]. Their measures (22) and (23) are the extension of Wang's measures [46]:

$$S_{w1}(A,B) = (1/n) \sum_{i=1}^{n} \frac{\min(\mu_A(x_i), \mu_B(x_i)) + \min(\nu_A(x_i), \nu_B(x_i))}{\max(\mu_A(x_i), \mu_B(x_i)) + \max(\nu_A(x_i), \nu_B(x_i))} \quad (22)$$

But, it is easy to give counter-examples again. For example, for $A = \{(x, 0, 0.5)\}$, $B = \{(x, 0.1, 0.5)\}$, $C = \{(x, 0, 0.6)\}$, we obtain $S_{w1}(A,B) = S_{w1}(A,C) = 0.8(3)$ (for different B and C we obtain the same result), for $A = \{(x, 0, 0.5)\}$, $B = \{(x, 0.18, 0.5)\}$, $C = \{(x, 0, 0.68)\}$, we obtain $S_{w1}(A,B) = S_{w1}(A,C) = 0.735$ (again - for different B and C the same result) etc., which seems to be difficult to accept (S_{w1} is not bijective).

$$S_{w2}(A,B) = (1/n) \sum_{i=1}^{n}(1 - 0.5(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i|)) \quad (23)$$

But for $A = \{(x, 0, 0.5)\}$, $B = \{(x, 0, 0.4)\}$, $C = \{(x, 0, 0.6)\}$, we obtain $S_{w2}(A,B) = S_{w2}(A,C) = 0.95$ (again - for different B and C the same similarity), which seems to be difficult to accept (S_{w2} is not bijective).

Another straightforward extensions of fuzzy similarity measures proposed by Hung and Yang [13] for A-IFSS are measures (24), (25) and (26) (they are extensions of Pappis and Karacapilidis' [20] measures for fuzzy sets).

$$S_{pk1}(A,B) = \frac{\sum_{i=1}^n(\min(\mu_A(x_i), \mu_B(x_i)) + \min(\nu_A(x_i), \nu_B(x_i)))}{\sum_{i=1}^n(\max(\mu_A(x_i), \mu_B(x_i)) + \max(\nu_A(x_i), \nu_B(x_i)))} \quad (24)$$

For (24) the counter-intuitive examples are like for (22).

$$S_{pk2}(A,B) = 1 - 0.5(\max_i(|\mu_A(x_i) - \mu_B(x_i)|) + \max_i(|\nu_A(x_i) - \nu_B(x_i)|)) \quad (25)$$

For (25) it is easy to give counter-examples again - especially for one-element sets. For example, for $A = \{(x, 0, 0.5)\}$, $B = \{(x, 0.1, 0.5)\}$, $C = \{(x, 0, 0.6)\}$, we obtain $S_{pk2}(A,B) = S_{pk2}(A,C) = 0.95$ (for different B and C just the same result).

$$S_{pk3}(A,B) = 1 - \frac{\sum_{i=1}^n(|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)}{\sum_{i=1}^n(|\mu_A(x_i) + \mu_B(x_i)| + |\nu_A(x_i) + \nu_B(x_i)|)} \quad (26)$$

But for $A = \{(x, 0, 0.5)\}$, $B = \{(x, 0, 0.26)\}$, $C = \{(x, 0, 0.965)\}$, we obtain $S_{pks}(A,B) = S_{pk3}(A,C) = 0.68$ (again, for different B and C the same similarity), which seems to be difficult to accept.

ISBN: 978-989-95079-6-8

3.1 Why the measures presented may yield counter-intuitive results?

It is worth noticing that all the above measures were constructed to satisfy the following conditions:

$$S(A,B) \in [0,1] \quad (27)$$
$$S(A,B) = 1 \Longleftrightarrow A = B \quad (28)$$
$$S(A,B) = S(B,A) \quad (29)$$
$$If\, A \subseteq B \subseteq C, \quad then \quad S(A,C) \leq S(A,B)\ and \quad S(A,C) \leq S(B,C) \quad (30)$$

Conditions (27)–(29) are obvious. The problem lies in (30) as this condition is meant as:

$$A \subset B \text{ iff } \forall x \in X,\ \mu_A(x) \leq \mu_B(x)\ and\ \nu_A(x) \geq \nu_B(x) \quad (31)$$

Unfortunately, (31) is not constructive and operational for A-IFSs as for many cases it can not be used. For example, for the elements $(x : (\mu,\nu,\pi))$: x_1: $(0.12, 0.4, 0.48)$ and x_2: $(0.1, 0.3, 0.6)$ we can not come to a conclusion. Moreover, element x: $(0, 0, 1)$ seems to be always beyond consideration in the sense of (31) which is very specific, and mostly practically irrelevant.

Furthermore, most of the similarity measures shown above are in fact similarity measures comparing just two intervals (each interval representing one of the A-IFSs under comparison). But we should bear in mind that elements of A-IF are described via the membership and non-membership function and the hesitation margin. In other words, in the terms of intervals, we have both the membership in an interval, and the non-membership in an interval so that we should represent an A-IF via two (not one) intervals.

Considering the representation of A-IFSs as single intervals implies some problems while calculating distances. Distances used in the (counter-intuitive) similarity measures mentioned in the previous section are calculated without taking into account the hesitation margins as the membership and non-membership functions only are taken into account. The counter-intuitive results obtained in such a case are in Szmidt and Kacprzyk [29], [39], Szmidt [22].

4 Some examples of intuitively justified and operational similarity measures

First we recall the measure of similarity between A-IFSs presented by Szmidt and Kacprzyk [37], [36]).

In the simplest situations we calculate the similarity of any two elements X and F belonging to an A-IF (A-IFSs). The proposed measures indicate if X is more similar to F or to F^C, where F^C is the complement of F. In other words, the proposed measures answer the question: is X more similar or more dissimilar to F?

Definition 1

$$Sim_{rule}(X,F) = \frac{l_{IF}(X,F)}{l_{IF}(X,F^C)} \quad (32)$$

where: $l_{IFS}(X,F)$ is a distance from $X(\mu_X,\nu_X,\pi_X)$ to $F(\mu_F,\nu_F,\pi_F)$,
$l_{IFS}(X,F^C)$ is a distance from $X(\mu_X,\nu_X,\pi_X)$ to $F^C(\nu_F,\mu_F,\pi_F)$,
F^C is a complement of F, distances $l_{IFS}(X,F)$ and $l_{IFS}(X,F^C)$ are calculated from (6).

For (32) we have

$$0 \underline{\leq} Sim_{rule}(X,F) \underline{\leq} \infty \quad (33)$$

$$Sim_{rule}(X,F) = Sim_{rule}(F,X)$$

The similarity has typically been assumed to be symmetric. Tversky [44], however, has provided some empirical evidence that the similarity should not always be treated as a symmetric relation. We stress this to show that a similarity measure (32) may have some features which can be useful in some situations but are not welcome in others (see Cross and Sudkamp [8], Wang et al. [47], Veltkamp [45]).

It is obvious (cf. Szmidt and Kacprzyk [36]) that the formula (32) can also be stated as

$$Sim_{rule}(X,F) = \frac{l_{IFS}(X,F)}{l_{IFS}(X,F^C)} = \frac{l_{IFS}(X^C,F^C)}{l_{IFS}(X,F^C)} = \frac{l_{IFS}(X,F)}{l_{IFS}(X^C,F)} = \frac{l_{IFS}(X^C,F^C)}{l_{IFS}(X^C,F)} \quad (34)$$

It is worth noticing that
– $Sim_{rule}(X,F) = 0$ means the identity of X and F.
– $Sim_{rule}(X,F) = 1$ means that X is to the same extent similar to F and F^C (i.e., values bigger than 1 mean a closer similarity of X and F^C to X and F).
– When $X = F^C$ (or $X^C = F$), i.e. $l_{IFS}(X,F^C) = = l_{IFS}(X^C,F) = 0$ means the complete dissimilarity of X and F (or in other words, the identity of X and F^C), and then $Sim_{rule}(X,F) \rightarrow \infty$.
– When $X = F = F^C$ means the highest possible entropy (see [31]) for both elements F and X i.e. the highest ”fuzziness” – not too constructive a case when looking for compatibility (both similarity and dissimilarity).
In other words, while applying the measure (32) to analyze the similarity of two objects, one should be interested in the values $0 \underline{\leq} Sim_{rule}(X,F) < 1$.

The proposed measure (32) was constructed for selecting objects which are more similar than dissimilar [and well-defined in the sense of possessing (or not) attributes we are interested in]. In Szmidt and Kacprzyk [37] it was shown that a measure of similarity defined as above, (32), between $X(\mu_X,\nu_X,\pi_X)$ and $F(\mu_F,\nu_F,\pi_F)$ is more powerful than a simple distance between them. The following conclusion were drawn:

– when a distance between two (or more) objects/elements, or sets, is large, then it means for sure that the similarity does not occur.
– when a distance is small, we can say nothing for sure about similarity just on the basis of a distance between two objects [when not taking into account complements of the objects as in (32)]. The distance between objects can be small and the compared objects can be more dissimilar than similar.

We have shown on a simple example (cf. Szmidt and Kacprzyk [37]) that the measure (32) gives reasonable results when applied to assessing agreement in a group of experts. The only disadvantage of the proposed measure is that it does

ISBN: 978-989-95079-6-8

Table 1: Example results obtained from the similarity measures (35)–(38)

1	2	3	4	5
$X = (\mu, \nu, \pi)$ $F = (\mu, \nu, \pi)$	$(0.3, 0.4, 0.3)$ $(0.4, 0.3, 0.3)$	$(0.4, 0.2, 0.4)$ $(0.5, 0.3, 0.2)$	$(0.4, 0.2, 0.4)$ $(0.5, 0.2, 0.3)$	$(0, 0, 0)$ $(0.5, 0.5, 0)$
Sim_1	0	0.6	0.75	0.5
Sim_2	0	0.43	0.6	0.33
Sim_3	0	0.72	0.88	0.6
Sim_4	0	0.48	0.65	0.38

not follow the range of the usually assumed values for the similarity measures. But it is possible to construct a whole array of similarity measures following the philosophy, and preserving the advantages of the measure (32), and whose numerical values are consistent with the common scientific tradition (i.e. belonging to $[0, 1]$). For example:

$$Sim_1(X,F) = Sim_1(l_{IFS}(X,F), l_{IFS}(X,F^C)) = 1 - \frac{l_{IFS}(X,F)}{l_{IFS}(X,F) + l_{IFS}(X,F^C)} \tag{35}$$

$$Sim_2(X,F) = Sim_2(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{1 - f(l_{IFS}(X,F), l_{IFS}(X,F^C))}{1 + f(l_{IFS}(X,F), l_{IFS}(X,F^C))} \tag{36}$$

$$Sim_3(X,F) = Sim_3(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{1 - f(l_{IFS}(X,F), l_{IFS}(X,F^C))^2}{1 + f(l_{IFS}(X,F), l_{IFS}(X,F^C))^2} \tag{37}$$

$$Sim_4(X,F) = Sim_4(l_{IFS}(X,F), l_{IFS}(X,F^C)) = \frac{e^{-f(l_{IFS}(X,F), l_{IFS}(X,F^C))} - e^{-1}}{1 - e^{-1}} \tag{38}$$

where

$$f(l_{IFS}(X,F), l_{IFS}(X,F^C)) = = \frac{l_{IFS}(X,F)}{l_{IFS}(X,F) + l_{IFS}(X,F^C)} \tag{39}$$

and $0 \leq f(l_{IFS}(X,F), l_{IFS}(X,F^C)) \leq 1$.

The measures (35) – (38) give intuitive results. Some examples, being troublesome for other measures, are presented in Table 1. It is worth noticing that each measure assigns similarity equal 0 for an element $(0.3, 0.4, 0.3)$ and its complement $(0.4, 0.3, 0.3)$. In general, similarity measures (35) – (38) satisfy the following properties:

$$Sim_i(X,F) \in [0,1], \tag{40}$$

$$Sim_i(X,X) = 1, \tag{41}$$

$$Sim_i(X,X^C) = 0, \tag{42}$$

$$Sim_i(X,F) = Sim_i(F,X), \tag{43}$$

for $i = 1, \ldots, 4$.

The similarity measures introduced in this section assess similarity of any two elements (X and F) belonging to an intuitionistic fuzzy set (or sets). The counterpart similarity measures for A-IFSs A and B containing n elements each, are:

$$Sim_k(A,B) = \frac{1}{n}\sum_{i=1}^{n} Sim_k(l_{IFS}(X_i,F_i), l_{IFS}(X_i,F_i^C)) \tag{44}$$

for $k = 1, \ldots, 4$.

5 Conclusions

We considered two groups of similarity measures between A-IFSs. First, we dealt with similarity measures constructed as if an A-IFSs was equal to a simple interval valued fuzzy set, or similarity measures being straightforward generalizations of those well known for the fuzzy sets. Unfortunately, in some situations, both approaches give counter-intuitive results. Second, we considered similarity measures accounting for all three functions describing an A-IF (the membership, non-membership, and hesitation margin) which is different from viewing an A-IF as a single interval. Next, we also took into account the complements of the elements compared. That is, we employed all kinds and fine shades of information available. It seems that these last measures are the most promising because, first of all, they help avoid some strong counter-intuitive results. This is crucial for both theory and applications.

References

[1] Atanassov K. (1983), Intuitionistic Fuzzy Sets. VII ITKR Session. Sofia (Centr. Sci.-Techn. Libr. of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[2] Atanassov K. (1986) Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems, 20, 87–96.

[3] Atanassov K. (1999), Intuitionistic Fuzzy Sets: Theory and Applications. Springer-Verlag.

[4] Atanassov K. and Gargov G. (1989), Interval-valued intuitionistic fuzzy sets. Fuzzy sets and Systems, 31 (3), 343–349.

[5] Bustince H., Burillo P. (1996) Vague sets are intuitionistic fuzzy sets. Fuzzy Sets and Systems 67, 403–405.

[6] Chen S. M. (1995) Measures of similarity between vague sets. Fuzzy Sets and Systems 74(2), 217–223.

[7] Chen S. M. (1997) Similarity measures between vague sets and between elements. IEEE Trans. Syst. Mn Cybernet. 27(1), 153–158.

[8] Cross V. and Sudkamp T. (2002) Similarity and Compatibility in Fuzzy Set Theory. Physica-Verlag.

[9] Dubois D. On degrees of truth, partial ignorance and contradiction. Magdalena L., Ojeda-Aciego M., Verdegay J.M. (eds): Proc. IPMU08, pp. 31–38.

[10] Gau W.L., Buehrer D.J. (1993) Vague sets. IEEE Trans. Systems Man Cybernet 23, 610–614.

[11] Hong D.H. and Kim C. (1999) A note on similarity measures between vague sets and between elements. Inform Science 115, 83–96.

ISBN: 978-989-95079-6-8

[12] Hung W-L. and Yang M-S. (2004) Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. Pattern recognition Letters 25, 1603–1611.

[13] Hung W.L and Yang M.S. (2008) On similarity measures between intuitionistic fuzzy sets. International Journal of Intelligent Systems 23 (3): 364–383.

[14] Kahneman D. (2002) Maps of bounded rationality: a perspective on intuitive judgment and choice. Nobel Prize Lecture.

[15] Li D.F, Cheng C.T. (2002) New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern Recognition Letters 23, 221–225.

[16] Li Y., Zhongxian C., Degin Y. (2002) Similarity measures between vague sets and vague entropy. J. Computer Sci. 29(12), 129–132.

[17] Liang Z. and Shi P. (2003) Similarity measures on intuitionistic fuzzy sets. Pattern Recognition Lett. 24, 2687–2693.

[18] Mitchell H.B. (2003) On the Dengfeng-Chuntian similarity measure and its application to pattern recognition. Pattern Recognition Lett. 24, 3101–3104.

[19] Montero J., Gómez D. and Bustince H. (2007): *On the relevance of some families of fuzzy sets*. Fuzzy Sets and Systems 158, 2429–2442.

[20] Pappis C. P. and Karacapilidis N. (1993) A comparative assessment of measures of similarity of fuzzy values. Fuzzy Sets and Systems 56, 171–174.

[21] Sutherland S. (1994) Irrationality. The Enemy Within. Penguin Books.

[22] Szmidt E. (2006) On a problem with the distances based on the Hausdorff metric. Report RB/52/2006, IBS PAN.

[23] Szmidt E. and Baldwin J. (2003) New Similarity Measure for Intuitionistic Fuzzy Set Theory and Mass Assignment Theory. Notes on IFSs, 9 (3), 60–76.

[24] Szmidt E. and Baldwin J. (2004) Entropy for Intuitionistic Fuzzy Set Theory and Mass Assignment Theory. Notes on IFSs, 10(3), 15-28.

[25] Szmidt E. and Kacprzyk J. (1996c) Remarks on some applications of intuitionistic fuzzy sets in decision making, Notes on IF, 2(3), 22–31.

[26] Szmidt E. and Kacprzyk J. (1997) On measuring distances between intuitionistic fuzzy sets, Notes on IF, 3(4), 1–13.

[27] Szmidt E. and Kacprzyk J. (1998a) Group Decision Making under Intuitionistic Fuzzy Preference Relations. IPMU'98, 172–178.

[28] Szmidt E. and Kacprzyk J. (1998b) Applications of Intuitionistic Fuzzy Sets in Decision Making. EUSFLAT'99, 150–158.

[29] Szmidt E. and Kacprzyk J. (2000) Distances between intuitionistic fuzzy sets, Fuzzy Sets and Systems, 114(3), 505–518.

[30] Szmidt E. and Kacprzyk J. (2000) On Measures on Consensus Under Intuitionistic Fuzzy Relations. IPMU 2000, 1454–1461.

[31] Szmidt E., Kacprzyk J. (2001) Entropy for intuitionistic fuzzy sets. Fuzzy Sets and Systems, 118 (3), 467–477.

[32] Szmidt E. and Kacprzyk J. (2001) Analysis of Consensus under Intuitionistic Fuzzy Preferences. Proc. Int. Conf. in Fuzzy Logic and Technology. Leicester, UK, 79–82.

[33] Szmidt E. and Kacprzyk J. (2002) An Intuitionistic Fuzzy Set Based Approach to Intelligent Data Analysis (an application to medical diagnosis). In A. Abraham, L.Jain, Kacprzyk (Eds.) Recent Advances in Intelligent Paradigms and Applications. Springer-Verlag, 57-70.

[34] Szmidt E. and Kacprzyk J. (2002) Analysis of Agreement in a Group of Experts via Distances Between Intuitionistic Fuzzy Preferences. IPMU 2002, Annecy, France, 1859–1865.

[35] Szmidt E. and J. Kacprzyk J. (2002b) An Intuitionistic Fuzzy Set Based Approach to Intelligent Data Analysis (an application to medical diagnosis). In A. Abraham, L. Jain, J. Kacprzyk (Eds.): Recent Advances in Intelligent Paradigms and Applications. Springer-Verlag, 57–70.

[36] Szmidt E. and Kacprzyk J. (2004) Similarity of intuitionistic fuzzy sets and the Jaccard coefficient. IPMU 2004, 1405–1412.

[37] Szmidt E., Kacprzyk J. (2004) A Concept of Similarity for Intuitionistic Fuzzy Sets and its use in Group Decision Making. 2004 IEEE Conf. on Fuzzy Systems, Budapest, 1129–1134.

[38] Szmidt E. and Kacprzyk J. (2006) An Application of Intuitionistic Fuzzy Set Similarity Measures to a Multi-criteria Decision Making Problem. ICAISC 2006, LNAI 4029, Springer-Verlag, 314–323.

[39] Szmidt E. and Kacprzyk J. (2006) Distances Between Intuitionistic Fuzzy Sets: Straightforward Approaches may not work. 3rd Int. IEEE Conf. Intelligent Systems IS06, 716–721.

[40] Szmidt E. and Kacprzyk J. (2007). Some problems with entropy measures for the Atanassov intuitionistic fuzzy sets. Applications of Fuzzy Sets Theory. LNAI 4578, 291–297. Springer-Verlag.

[41] Szmidt E. and Kacprzyk J. (2007a). A New Similarity Measure for Intuitionistic Fuzzy Sets: Straightforward Approaches may not work. 2007 IEEE Conf. on Fuzzy Systems, 481–486.

[42] Szmidt E. and Kukier M. (2006). Classification of Imbalanced and Overlapping Classes using Intuitionistic Fuzzy Sets. 3rd International IEEE Conference on Intelligent Systems IS'06, London, 2006, 722-727.

[43] Szmidt E. and Kukier M. (2008) A New Approach to Classification of Imbalanced Classes via Atanassov's Intuitionistic Fuzzy Sets. In: Hsiao-Fan Wang (Ed.): Intelligent Data Analysis : Developing New Methodologies Through Pattern Discovery and Recovery. Idea Group, 85–101.

[44] Tversky A. (1977). Features of similarity. Psychol. Rev., 84, 327–352.

[45] Veltkamp R.C. (2001) Shape matching: similarity measures and algorithms. Proc. Shape Modelling International, Genova, Italy, IEEE Press, 188–197.

[46] Wang W.J. (1997) New similarity measures on fuzzy sets and on elements. Fuzzy Sets and Systems, 85, 305–309.

[47] Wang X., De Baets B., and Kerre E. (1995). A comparative study of similarity measures. Fuzzy Sets and Systems, 73 (2), 259–268.

[48] Zadeh L.A. (1965). Fuzzy sets. Information and Control, 8, 338–353.