Non-commutative EQ-logics and their extensions

Vilém Novák, Martin Dyba

Institute for Research and Applications of Fuzzy Modelling, University of Ostrava
Ostrava, Czech Republic
Email: Vilem.Novak,martin.dyba@osu.cz

Abstract— *We discuss a formal many-valued logic called EQ-logic which is based on a recently introduced special class of algebras called EQ-algebras. The latter have three basic binary operations (meet, multiplication, fuzzy equality) and a top element and, in a certain sense, generalize residuated lattices. The goal of EQ-logics is to present a possible direction in the development of mathematical logics in which axioms are formed as identities. In this paper we propose a basic EQ-logic and three extensions which end up with a logic equivalent to the MTL-logic.*

Keywords— EQ-algebra, fuzzy equality, residuated lattice, MTL-logic, fuzzy logic.

1 Introduction

One of possible directions of the development of mathematical logic which can be tracked down to Leibnitz, Wittgenstein, and Ramsey (cf. [1]) is to develop it on the basis of identity (equality) as the principal connective. This direction has been crowned in a noble way by Henkin in [2] who developed the type theory (a higher order logic) using identity as a sole primitive constant. This is nice because it enables to treat equality between elements on the basis of the same principles not depending on their character. Thus, equality between truth values is, in principle, a relation of the same kind as an equality between other kinds of elements. Hence, we will speak about equality instead of equivalence even when speaking about truth values.

Note that there are works developing classical boolean logic on the basis of equivalence as the main (not sole) connective (a recent book presenting logic in this way is [3]). Moreover, the logic is developed there in the "equational style", i.e. proofs proceed as sequences of equations (in fact, equivalences).

This gives rise to an idea whether also fuzzy (many-valued) logic could be developed on the basis of fuzzy equality as the main connective where by fuzzy equality we mean, in fact, a fuzzy equivalence, i.e. a generalization of the classical logical equivalence. We will prefer the term "fuzzy equality", though. Recall that this idea is supported by the mentioned Henkin successful type theory (for a detailed development of this theory, see also [4]) as well as by the development of fuzzy type theory (see [5, 6]). On the other hand, unlike classical type theory, it seems impossible to have fuzzy equality a sole connective. At least conjunction is also necessary as follows from the considerations concerning the structure of the algebra of truth values. Namely, two approaches are apparent.

In the first approach, the structure of truth values is assumed to form a residuated lattice where the basic operations are, besides the lattice ones, the multiplication "$\otimes$" and its residuum "$\rightarrow$". The latter is in fuzzy logic a natural interpretation of implication which is a primary operation while the equivalence is interpreted by a biresiduation $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ which is a derived operation. Since the basic connective, in our case, should be the fuzzy equality, it seems unnatural to interpret it by a derived operation.

The second approach follows an idea presented in [7, 8] where a specific kind of algebra called EQ-algebra is developed. Unlike the residuated lattices, the basic operation in EQ-algebras is a fuzzy equality "$\sim$" while implication is derived from it. Of course, a crucial question remains, which properties such an equality should have. At first sight it seems clear that these properties should be reflexivity, symmetry, and transitivity. We are convinced, however, that the fuzzy equality should be characterized by a deeper property reflecting some basic structure so that both symmetry as well as especially transitivity would follow. The most essential is, in our opinion, the "substitution principle" stating that if we replace an object by another one equal to the former then the result is not changed; in the case of fuzzy equality, the replacement should not make the structure in concern "worse" in some reasonable sense.

What basic structure should we consider in fuzzy logic? Clearly, this is *ordering* since the basic idea standing behind fuzziness is introducing degrees to enable measuring imprecision stemming from vagueness. Since ordering is in no way included in the equality, it must be introduced explicitly. Therefore, we will consider the basic structure to be a $\wedge$-semilattice with the greatest element $\mathbf{1}$. At the same time, we must "fuse" in some way statements about ordering and equality. The fusion is in our algebra realized by a third operation of multiplication "$\otimes$" whose properties, however, should be as weak as possible because we think that it should not participate on the ordering but enable only to "fuse things together". This is the main motivation behind axiom (E4) below. The symmetry and transitivity of $\sim$ are then consequences of it. The fusion $\otimes$ needs not be commutative and, perhaps, nor even associative.

Step by step we arrive at 8 axioms which determine the EQ-algebra. In its "crude form", there is no need to introduce the smallest element $\mathbf{0}$. Morever, the axioms make it possible to have two different truth values equivalent in the degree $\mathbf{1}$ and so, the fuzzy equality generalizes classical equivalence in an essential way. From the algebraic point of view, EQ-algebras generalize, in a certain sense, residuated lattices and so, they look as interesting class of algebraic structures.

When attempting to develop a formal logical calculus it came out, surprisingly, that the crude structure though reflecting the basic principles of fuzzy equality, is not strong enough. Namely, the logic enforced adding an axiom of "goodness" (E10) (or, as a logical formula (EQ1)) below which says that

ISBN: 978-989-95079-6-8

an element a is equal to $\mathbf{1}$ always in a degree a. This axiom implies that the algebra is separated, i.e. that two elements equal in the degree $\mathbf{1}$ must be identical in the classical sense. We conclude that the logic allows only a slight generalization of the classical equivalence. Of course, the goodness axiom has many important algebraic consequences (see [9, 8]). Among them is also the fact that each good EQ-algebra gives rise to a BCK-algebra. Putting all together we expect that EQ-algebras may put a different light on residuated lattices and on their general role in algebra and logic.

This paper is an attempt to introduce a formal logic called EQ-logic[*)] whose truth values are taken from a good EQ-algebra. Several basic properties of this logic demonstrating that it behaves reasonably are proved. In comparison with residuated fuzzy logics, however, our logic seems to be more limited. For example, the deduction theorem even in its weaker form probably does not hold and we guess that its validity is equivalent with residuation. Still, it is interesting to learn how far can we go with the development of EQ-logic. Besides others, it enables us to separate properties, which stem from the fuzzy equality itself, from the properties for which residuation is necessary. It may also shed light on classical logic, especially on its equational variant ([3]).

2 Definition and fundamental properties of EQ-algebras

Definition 1
A semicopula-based EQ-algebra $\mathscr{E}$ is an algebra of type (2, 2, 2, 0), i.e.

$$\mathscr{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}\rangle\,, \tag{1}$$

where for all $a, b, c, d \in E$:

(E1) $\langle E, \wedge, \mathbf{1}\rangle$ *is a commutative idempotent monoid (i.e. $\wedge$-semilattice with top element $\mathbf{1}$) where the ordering $a \le b$ is defined in standard way as $a \wedge b = a$,*

(E2) $\otimes$ *is a binary multiplication operation isotone w.r.t. $\le$, i.e. $a \le b$ implies both $a\otimes c \le b\otimes c$ as well as $c\otimes a \le c\otimes b$ and $\mathbf{1}$ is a neutral element, i.e. $a\otimes \mathbf{1} = \mathbf{1}\otimes a = a$.*

(E3) $a \sim a = \mathbf{1}\,,$

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \le c \sim (d \wedge b)\,,$

(E5) $(a \sim b) \otimes (c \sim d) \le (a \sim c) \sim (b \sim d)\,,$

(E6) $(a \wedge b \wedge c) \sim a \le (a \wedge b) \sim a\,,$

(E7) $(a \wedge b) \sim a \le (a \wedge b \wedge c) \sim (a \wedge c)\,,$

(E8) $a \otimes b \le a \sim b\,.$

The operation $\sim$ is called a fuzzy equality.

For all $a, b \in E$ we put

$$a \to b = (a \wedge b) \sim a. \tag{2}$$

This is a derived operation called *implication*[†)].

[*)] This logic has been first introduced in Barcelona, October 2008 at the occasion of Prof. F. Esteva 65[th] birthday.

[†)] Note that this operation has been, in fact, introduced already by G. W. Leibnitz (cf. [10])

Note that axioms (E6) and (E7) can be rewritten as

$$a \to (b \wedge c) \le a \to b\,, \tag{E6'}$$

$$a \to b \le (a \wedge c) \to b\,. \tag{E7'}$$

The following theorem confirms that $\sim$ is indeed a fuzzy equality.

Theorem 1
Let $\mathscr{E}$ be an EQ-algebra. The following holds for all $a, b, c \in E$:

(a) Symmetry: $a \sim b = b \sim a\,,$

(b) Transitivity: $(a \sim b) \otimes (b \sim c) \le a \sim c\,,$

(c) Transitivity of implication: $(a \to b) \otimes (b \to c) \le a \to c\,.$

We say that the multiplication $\otimes$ is $\to$*-isotone* if

$$a \to b = \mathbf{1} \quad \text{implies} \quad a \otimes c \to b \otimes c = \mathbf{1} \tag{3}$$

for all $a, b, c \in E$. Let us remark that the $\to$-isotonicity of $\otimes$ is usually fulfilled. There exist EQ-algebras, however, which have not this property.

A semicopula-based EQ-algebra has the operation $\otimes$, in general neither commutative nor associative. If the latter is only non-commutative but associative then we will speak about a *non-commutative* EQ-algebra.

Theorem 2
Let $\mathscr{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}\rangle$ is a semicopula-based EQ-algebra and put $b \,\bar{\otimes}\, a = a \otimes b$. Then $\mathscr{E}' = \langle E, \wedge, \bar{\otimes}, \sim, \mathbf{1}\rangle$ is also a semicopula-based EQ-algebra.

The following lemmas summarizes some of the properties of EQ-algebras useful below. Its proof can be found in [9, 8].

Lemma 1
Let $\mathscr{E}$ be an EQ-algebra. For all $a, b \in E$ such that $a \le b$ it holds that

(a) $a \to b = \mathbf{1}\,,$

(b) $a \sim b = b \to a\,,$

(c) $c \to a \le c \to b$ *and* $b \to c \le a \to c\,.$

Lemma 2
Let $\mathscr{E}$ be an EQ-algebra. For all $a, b, c, d, a', b', c', d' \in E$ it holds that

(a) $a \otimes b \le a, a \otimes b \le a \wedge b, c \otimes (a \wedge b) \le (c \otimes a) \wedge (c \otimes b)\,,$

(b) $a \sim b \le a \to b, a \to a = \mathbf{1}\,,$

(c) $(a \to b) \otimes (b \to a) \le a \sim b\,,$

(d) $a = b$ *iff* $a \sim b = \mathbf{1}\,,$

(e) $a = \mathbf{1} \to a$ *and* $a \to \mathbf{1} = \mathbf{1}\,,$

(f) $a \otimes (a \sim b) \le b\,,$

(g) $b \le a \to b\,,$

(h) $((a \wedge b) \sim (c \wedge d)) \otimes (a \sim a') \otimes (b \sim b') \otimes (c \sim c') \otimes (d \sim d') \le (a' \wedge b') \sim (c' \wedge d')\,.$

ISBN: 978-989-95079-6-8

Note that if $\mathscr{E}$ is separated, then $a \to b = \mathbf{1}$ implies $a \le b$. Indeed, $a \to b = (a \wedge b) \sim a = \mathbf{1}$ implies $a \wedge b = a$, i.e. $a \le b$.

Let $\mathscr{E}$ contain also the bottom element $\mathbf{0}$. Then we put

$$\neg a = a \sim \mathbf{0}, \qquad a \in E \tag{4}$$

and call $\neg a$ a *negation* of $a \in E$.

Definition 2
Let $\mathscr{E}$ be an EQ-algebra and $a, b, c, d \in L$. We say that $\mathscr{E}$ is:

(i) separated *if for all $a \in E$,*

(E9) $a \sim b = \mathbf{1}$ *implies* $a = b$.

(ii) good *if*

(E10) $a \sim \mathbf{1} = a$.

(iii) residuated *if for all $a, b, c \in E$,*

(E10) $(a \otimes b) \wedge c = a \otimes b$ *iff* $a \wedge ((b \wedge c) \sim b) = a$.

(iv) involutive *(IEQ-algebra) if for all $a \in E$,*

(E11) $\neg\neg a = a$.

(v) prelinear *if for all $a, b \in E$,*

(E12) $\sup\{a \to b, b \to a\} = \mathbf{1}$.

(vi) lattice EQ-algebra *(ℓEQ-algebra) if it is a lattice and*

(E13) $((a \vee b) \sim c) \otimes (d \sim a) \le (d \vee b) \sim c$.

(vii) complete *if it's $\wedge$-semilattice reduct is complete.*

(viii) EQ(R)-algebra *if for all $a, b \in E$ there exist*

$$a/b = \max\{c \in E \mid a \otimes c \le b\}$$
$$a\backslash b = \max\{c \in E \mid c \otimes a \le b\}$$

(the latter equalities will be referred to as (R)-condition).

It is easy to show that good EQ-algebras are *separated*. Obviously, $\otimes$ in good EQ-algebras is $\to$-isotone. As already discussed, the "goodness property" appeared to be indispensable for logic and so, from now on, *by EQ-algebra we will always mean a good one*.

Remark 1
Let $\mathscr{E}$ be a semicopula-based EQ-algebra. A set $F \subseteq E$ is a filter if $\mathbf{1} \in F$; $a, b \in F$ implies $a \otimes b \in F$; $a, a \to b \in F$ implies $b \in F$; $a \to b \in F$ implies $(a \otimes c) \to (b \otimes c) \in F$ and $(c \otimes a) \to (c \otimes b) \in F$. Then it can be proved that a factor algebra $\mathscr{E}|F$ is a separated semicopula-based EQ-algebra. Surprizingly, there exist EQ-algebras with no proper filter and so, we cannot embed each EQ-algebra into a separated one.

The concept of EQ(R)-algebra is due to El Zekey [11]. Obviously, it inserts residuation inside EQ-algebra. In case that $E = [0, 1]$, it simply requires the operation $\otimes$ to be left-continuous.

Lemma 3
Let $\mathscr{E}$ be an IEQ-algebra. Then

(a) $\mathscr{E}$ *is good ℓEQ-algebra with join defined by*

$$a \vee b = \neg(\neg a \wedge \neg b).$$

(b) $a \le b$ *iff* $\neg b \le \neg a$.

(c) $a \sim b = \neg a \sim \neg b$.

(d) $a \otimes \neg a = \mathbf{0}$.

Theorem 3 ([11])
Let $\mathscr{E}$ be a good non-commutative EQ(R)-algebra. Then the following is equivalent:

(a) $\mathscr{E}$ *is representable (i.e. subdirectly embeddable into a product of linearly ordered good EQ(R)-algebras[‡]).*

(b) $\mathscr{E}$ *satisfies the formula*

$$(a \to b) \vee (d \to (d \otimes (c \to (b \to a) \otimes c))) = \mathbf{1}$$

for all $a, b, c, d \in E$.

3 Basic EQ-logic

In this section we introduce a propositional EQ-logic which we will call basic. This logic seems to be the simplest logic definable on the basis of EQ-algebras. Of course, we obtain more sophisticated logics when adding further axioms. From this point of view, all core (residuated) fuzzy logic in the sense of [12]) are extensions of the basic EQ-logic.

Definition 3
The language of EQ-logic consists of propositional variables $p_1, p_2, \ldots$, binary connectives $\boldsymbol{\wedge}, \boldsymbol{\&}, \equiv$ and a truth (logical) constant $\top$. Formulas are defined in the obvious way: each propositional variable is a formula, $\top$ is a formula and if A, B are formulas, then $A \boldsymbol{\wedge} B$, $A \boldsymbol{\&} B$, $A \equiv B$ are formulas. Implication is defined as a short

$$A \Rightarrow B := (A \boldsymbol{\wedge} B) \equiv A. \tag{5}$$

Let J be a language of EQ-logic, F_J a set of all formulas in the language J and $\mathscr{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}\rangle$ be a good EQ-algebra. A truth evaluation $e : F_J \longrightarrow E$ is defined as usual: if $p \in J$ is a propositional variable then $e(p) \in E$. Furthermore, we put

$$e(\top) = \mathbf{1},$$
$$e(A \boldsymbol{\wedge} B) = e(A) \wedge e(B)$$
$$e(A \boldsymbol{\&} B) = (e(A) \otimes e(B)),$$
$$e(A \equiv B) = (e(A) \sim e(B))$$

for all formulas $A, B \in F_J$.

A formula $A \in F_J$ is a tautology if $e(A) = \mathbf{1}$ for each truth evaluation $e : F_J \longrightarrow E$.

Note that our logic has a non-commutative fusion connective $\boldsymbol{\&}$ but only one implication because it is derived from equivalence (i.e. its interpretation is not residual operation adjoint with multiplication). We are convinced that this is an advantage.

[‡] A recent result of El-Zekey (personal communication) implies that the (R)-condition is unnecessary for the representability. This may lead to simplification of the structure of EQ-logics discussed below.

ISBN: 978-989-95079-6-8

3.1 Logical axioms and inference rules

The following formulas are axioms of the EQ-logic:

(EQ1) $(A \equiv \top) \equiv A$

(EQ2) $A \wedge B \equiv B \wedge A$

(EQ3) $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$

(EQ4) $A \wedge A \equiv A$

(EQ5) $A \wedge \top \equiv A$

(EQ6) $A \,\&\, \top \equiv A$

(EQ7) $\top \,\&\, A \equiv A$

(EQ8a) $((A \wedge B) \,\&\, C) \Rightarrow (B \,\&\, C)$

(EQ8b) $(C \,\&(A \wedge B)) \Rightarrow (C \,\&\, B)$

(EQ9) $((A \wedge B) \equiv C) \,\&(D \equiv A) \Rightarrow (C \equiv (D \wedge B))$

(EQ10) $(A \equiv B) \,\&(C \equiv D) \Rightarrow (A \equiv C) \equiv (D \equiv B)$

(EQ11) $(A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$

(EQ12) $(A \Rightarrow B) \Rightarrow (A \wedge C) \Rightarrow B$

Remark 2
The above axioms have been written using $\Rightarrow$ for better readability because formulas with this connective are more usual than those with $\equiv$. However, technically we apply equational-style proofs in the sense of [3], which are sequences of formulas of the form $A_1 \equiv A_2, \ldots, A_{n-1} \equiv A_n$ such that each of the individual theorems $A_i \equiv A_{i+1}$ has an independent individual proof. Therefore, a fussy presentation would require to rewrite all the axioms using $\equiv$ and the definition (5) only.

By $A[p := B]$ we denote a formula resulting from A by replacing all occurrences of a propositional variable p in A by the formula B. Then we introduce the following inference rules of EQ-logic:

$$(EA) \quad \frac{A, A \equiv B}{B} \qquad (L) \quad \frac{B \equiv C}{A[p := B] \equiv A[p := C]}$$

The rule (EA) is the *equanimity rule* and (L) is the *Leibnitz rule* for formulas (cf. [3] and elsewhere). The notion of provability is classical. A formal theory is any subset $T \subseteq F_J$. As usual, we suppose that T is defined by a set of special axioms.

3.2 Main properties

The proof of the following three lemmas is technical. They demonstrate several reasonable properties of the basic EQ-logic.

Lemma 4
The following are special derived rules:

(a) $A \equiv \top \vdash A.$ *(rule (T1))*

(b) $A \vdash A \equiv \top.$ *(rule (T2))*

(c) $A \wedge D \equiv C, A \equiv B \vdash B \wedge D \equiv C.$ *(rule (C))*

(d) $(A \equiv D) \equiv C, A \equiv B \vdash (B \equiv D) \equiv C.$ *(rule (E))*

(e) $A \,\&\, D \equiv C, A \equiv B \vdash B \,\&\, D \equiv C.$ *(rule (F1))*

(f) $D \,\&\, A \equiv C, A \equiv B \vdash D \,\&\, B \equiv C.$ *(rule (F2))*

Lemma 5
(a) $\vdash A \equiv A,$

(b) $A \equiv B \vdash B \equiv A,$

(c) $A \equiv B, B \equiv C \vdash A \equiv C,$

(d) $A, A \Rightarrow B \vdash B,$ *(Modus Ponens)*

(e) $\vdash (\top \Rightarrow A) \equiv A,$

(f) $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C,$

(g) $A, B \vdash A \,\&\, B,$

(h) $\vdash (A \equiv B) \equiv (B \equiv A),$

(i) $A \Rightarrow (B \equiv C), B \vdash A \Rightarrow C,$

(j) $\vdash (A \equiv D) \Rightarrow ((A \equiv B) \equiv (D \equiv B)),$

(k) $A \Rightarrow (B \equiv C), B \equiv D \vdash A \Rightarrow (D \equiv C),$

(l) $A \Rightarrow (B \equiv C), C \equiv D \vdash A \Rightarrow (B \equiv D),$

(m) $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B),$

(n) $A \equiv B, C \equiv D \vdash A \,\&\, C \equiv B \,\&\, D,$

(o) $A \Rightarrow B, C \Rightarrow D \vdash A \,\&\, C \Rightarrow B \,\&\, D,$

(p) $\vdash ((A \equiv B) \equiv C) \,\&(A \equiv D) \Rightarrow ((D \equiv B) \equiv C),$

(q) $\vdash (A \equiv B) \,\&(C \equiv D) \Rightarrow (A \equiv C) \equiv (B \equiv D),$

(r) $\vdash (A \Rightarrow B) \,\&(B \Rightarrow A) \Rightarrow (A \equiv B),$

(s) $\vdash B \Rightarrow (A \Rightarrow B).$

Lemma 6
(a) $\vdash (A \equiv B) \,\&(B \equiv C) \Rightarrow (A \equiv C),$

(b) $\vdash (A \,\&(A \equiv B)) \Rightarrow B$

(c) $\vdash (A \Rightarrow B) \,\&(B \Rightarrow C) \Rightarrow (A \Rightarrow C),$

(d) $\vdash (A \,\&(A \Rightarrow B)) \Rightarrow B$

(e) $\vdash B \Rightarrow (B \equiv \top),$

(f) $A \Rightarrow (B \Rightarrow C) \vdash (A \,\&\, B) \Rightarrow C,$

(g) $\vdash A \,\&\, B \Rightarrow A \equiv B,$

(h) $\vdash (A \wedge B) \Rightarrow A,$

(i) $\vdash (C \Rightarrow A) \,\&(C \Rightarrow B) \Rightarrow (C \Rightarrow (A \wedge B)).$

By the straightforward verification we can prove the following two lemmas.

Lemma 7
All axioms of EQ-logic are tautologies.

Lemma 8
The deductive rules of EQ-logic are sound in the following sense. Let $e : F_J \longrightarrow E$ be a truth evaluation:

ISBN: 978-989-95079-6-8

(a) If $e(A) = \mathbf{1}$ and $e(A \equiv B) = \mathbf{1}$ then $e(B) = \mathbf{1}$.

(b) If $e(B \equiv C) = \mathbf{1}$ then $e(A[p := B] \equiv A[p := C]) = \mathbf{1}$ for any formula A.

The following is a standard Lindenbaum-Tarski technique. Put

$$A \approx B \quad \text{iff} \quad \vdash A \equiv B, \quad A, B \in F_J.$$

It follows from Lemmas 5(a), (h) and 6(a) that $\approx$ is an equivalence on F_J. Let us denote by $[A]$ an equivalence class of A and put $\bar{E} = \{[A] \mid A \in F_J\}$. If we define

$$\begin{aligned} \mathbf{1} &= [\top], \\ [A] \wedge [B] &= [A \wedge B], \\ [A] \otimes [B] &= [A \,\&\, B], \\ [A] \sim [B] &= [A \equiv B], \end{aligned}$$

then we obtain an algebra $\bar{\mathscr{E}} = \langle \bar{E}, \wedge, \otimes, \sim, \mathbf{1} \rangle$.

Lemma 9
The algebra $\bar{\mathscr{E}}$ is a good semicopula-based EQ-algebra.

Theorem 4 (Soundness)
The basic EQ-fuzzy logic is sound.

PROOF: This is a consequence of Lemmas 7 and 8. □

Theorem 5 (Completeness)
The following is equivalent for every formula A:

(a) $\vdash A$,

(b) $e(A) = \mathbf{1}$ for every good semicopula-based EQ-algebra $\mathscr{E}$ and a truth evaluation $e : F_J \longrightarrow E$.

PROOF: The implication (a) to (b) is soundness.

(b) to (a): By Lemma 9 the algebra $\bar{\mathscr{E}}$ of equivalence classes of formulas is a good semicopula-based EQ-algebra. Thus, if (b) holds then it holds also for $e : F_J \longrightarrow \bar{E}$. If $e(A) = \mathbf{1}$ then it means that $[A] = [\top]$, i.e. $\vdash A \equiv \top$ and so, $\vdash A$ by rule (T2). □

4 Extensions of the basic EQ-logic

In this section we will briefly discuss some extensions of the basic EQ-logic.

4.1 Involutive EQ-logic

This logic (we will speak about IEQ-logic) contains falsity and is characterized by the property of double negation which leads to some simplifications.

The language J is the same as that of basic EQ-logic with the exception that $\top$ is replaced by $\bot$. Furthermore, we introduce the following shorts of formulas:

$$\top := \bot \equiv \bot, \tag{6}$$
$$\neg A := A \equiv \bot, \tag{7}$$
$$A \vee B := \neg(\neg A \wedge \neg B). \tag{8}$$

(clearly, (7) is a *negation* and (8) a disjunction.

Axioms of IEQ-logic are (EQ2)–(EQ12) plus the following ones:

(EQ13) $(A \,\&\, B) \,\&\, C \equiv A \,\&\, (B \,\&\, C)$,

(EQ14) $(A \wedge \bot) \equiv \bot$,

(EQ15) $\neg\neg A \equiv A$.

Axiom (EQ14) characterizes the basic property of $\bot$ and it can be written as $\bot \Rightarrow A$ (ex falso quodlibet). Axiom (EQ1) is in this logic provable.

A contradiction is a formula $A \,\&\, \neg A$. Then we say that a theory T is *contradictory* if $T \vdash A \,\&\, \neg A$ for some formula $A \in F_J$.

A straightforward proof will give the following (classical) theorem:

Theorem 6
A theory T is contradictory iff $T \vdash A$ for all $A \in F_{J(T)}$.

Semantics of this logic is formed by IEQ-algebras.

Theorem 7 (Completeness)
The following is equivalent for every formula A:

(a) $\vdash A$,

(b) $e(A) = \mathbf{1}$ for every IEQ-algebra $\mathscr{E}$ and a truth evaluation $e : F_J \longrightarrow E$.

4.2 EQ(R)-fuzzy logic

This logic seems to be closest to the residuated fuzzy logics. In fact, it already introduces some kind of residuated structure which enables to prove a stronger variant of the completeness theorem.

The language of this logic is that of IEQ-logic extended by two binary connectives: "\" and "/" and the shorts (6) and

$$A \vee B := ((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A).$$

Axioms are (EQ1)–(EQ14) plus the following:

(EQ16) $(A \Rightarrow B) \vee (D \Rightarrow (D \,\&\, (C \Rightarrow (B \Rightarrow A) \,\&\, C)))$

(EQ17) $C/(A \otimes B) \equiv (C/(B/A))$

(EQ18) $(A \otimes B)\backslash C \equiv ((A\backslash B)\backslash C)$

Semantics of this logic is formed by good non-commutative prelinear EQ(R)-algebras.

Using Theorem 3, the following can be proved:

Theorem 8 (Completeness)
For every formula $A \in F_J$ the following is equivalent:

(a) $\vdash A$.

(b) $e(A) = \mathbf{1}$ for every truth evaluation $e : F_J \longrightarrow E$ and every linearly ordered good non-commutative EQ(R)-algebra $\mathscr{E}$.

(c) $e(A) = \mathbf{1}$ for every truth evaluation $e : F_J \longrightarrow E$ and every good non-commutative prelinear EQ(R)-algebra $\mathscr{E}$.

ISBN: 978-989-95079-6-8

4.3 Residuated EQ-logic

This is stronger logic close to MTL-logic because of introducing the residuation property.

The language J is the same as that of basic EQ-logic. Its axioms are (EQ1)–(EQ12) plus the following:

(EQ19) $((A \mathbin{\&} B) \equiv C) \mathbin{\&} (D \equiv A) \Rightarrow (C \equiv (D \mathbin{\&} B))$

Lemma 10
The following is provable residuated EQ-logic:

(a) $\vdash A \mathbin{\&} B \equiv B \mathbin{\&} A$,

(b) $(A \mathbin{\&} B) \mathbin{\&} C \equiv A \mathbin{\&} (B \mathbin{\&} C)$,

(c) $\vdash ((A \mathbin{\&} B) \Rightarrow C) \equiv (A \Rightarrow (B \Rightarrow C))$.

By A^n we mean n-times repetition of A w.r.t. & (as usual).

Theorem 9 (Deduction)
For every theory T and formulas $A, B \in F_J$:

$$T \cup \{A\} \vdash B \quad \textit{iff} \quad \textit{there is } n \geq 1 \textit{ such that } T \vdash A^n \Rightarrow B$$

Semantics for this logic is formed by residuated EQ-algebras. If we replace the constant $\top$ in the language J by $\bot$, take the definition (6) and add axiom (EQ14) and the axiom

(EQ20) $((A \Rightarrow B) \Rightarrow C) \Rightarrow (((B \Rightarrow A) \Rightarrow C) \Rightarrow C)$

then the resulting logic is equivalent with the MTL-logic.

5 Conclusion

In this paper we introduced a class of logics based on a new algebra of truth values called EQ-algebra. This algebra is specific by introducing fuzzy equality (equivalence) as the basic operation. Accordingly, the basic connective of EQ-logic is equivalence.

It is easy to show that BCK-axioms are provable in basic EQ-logic and so, basic EQ-logic is also a BCK-one (cf. [13]). However, since modus-ponens is a derived rule in the former, translation of properties of BCK-logic to EQ-logic is not straightforward. It is an open question whether there is a formal system equivalent with basic EQ-logic with modus ponens as its sole rule.

Acknowledgment

The research was supported by the project MSM 6198898701 of the MŠMT ČR.

References

[1] F.P. Ramsey. The foundations of mathematics. *Proceedings of the London Mathematical Society*, 25:338–384, 1926.

[2] L. Henkin. A theory of propositional types. *Fundamenta Math.*, 52:323–344, 1963.

[3] G. Tourlakis. *Mathematical Logic*. New York, J. Wiley & Sons, 2008.

[4] P. Andrews. *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof.* Kluwer, Dordrecht, 2002.

[5] V. Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149:235–273, 2005.

[6] V. Novák. EQ-algebra-based fuzzy type theory and its extensions. *Logic Journal of the IGPL*, page (submitted).

[7] V. Novák. EQ-algebras in progress. In O. Castillo, editor, *Theoretical Advances and Applications of Fuzzy Logic and Soft Computing*, pages 876–884. Springer, Berlin, 2007.

[8] V. Novák and B. de Baets. EQ-algebras. *Fuzzy Sets and Systems*, page (to appear), 2009.

[9] M. El-Zekey, V. Novák, and R. Mesiar. Semicopula-based EQ-algebras. *Fuzzy Sets and Systems*, page (submitted).

[10] H.N. Castañeda. Leibniz's syllogistico-propositional calculus. *Notre Dame Journal of Formal Logic*, XVII(4):338–384, 1976.

[11] M. El-Zekey. Representable EQ(R)-algebras. page (submitted).

[12] P. Hájek and P. Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863—-880, 2006.

[13] P. Cintula, P. Hájek, and R. Horčík. Formal systems of fuzzy logic and their fragments. *Annals of Pure and Applied Logic*, 150:40–65, 2007.

ISBN: 978-989-95079-6-8