Confidence regions for the mean of a fuzzy random variable

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Abstract— *The aim of this paper is to extend the classical problem of confidence interval estimation for the mean of a random variable to the case of a fuzzy random variable. The key idea consists in considering a confidence region defined as a ball w.r.t. a given metric, which is centered in the sample mean and whose radius is determined via bootstrapping. The developed approach is illustrated both by means of simulated examples using C and R and by means of a practical example consisting of courses evaluations.*

Keywords— Fuzzy random variables, confidence interval, confidence region, expectation, bootstrap.

1 Introduction

Inferential procedures for handling fuzzy information have gained in importance in the literature during the last years. Hypothesis testing problems concerning fuzzy random variables (hereafter FRV for short) in different situations (see, for instance, [9, 13, 14, 5, 6, 3]) as well as consistency properties of the considered estimators, i.e. the associated point-wise estimation problems, have been studied extensively (see, for instance, [12, 10, 16, 7]). Nevertheless, the problem of confidence interval estimation for FRVs has not received too much attention. In this communication we will focus on the the problem of estimating by a confidence interval the expected value of an FRV in Puri & Ralescu's sense [15].

Initially, there are two main difficulties in connection with this estimation problem: the well-known lack of ordering and the lack of linearity of the space of fuzzy sets. The lack of ordering is not crucial - indeed this difficulty has been faced previously in some classical situations (as for instance in the determination of confidence bands for the linear regression). On the other hand the lack of linearity makes it senseless to attempt to express the confidence interval in the standard way (sample mean plus/minus a given quantity depending on the variability and the sample size). Consequently, we will speak about confidence regions instead of confidence intervals in the following, keeping in mind that these regions cannot be specified simply by means of an upper and lower bound.

It is well-known that if X is a real-valued random variable with mean μ and finite variance, then, based on a random sample $X_1, \ldots, X_n$ of n independent random variables having the same distribution as X, a confidence interval with confidence level $100(1-\beta)\%$ for μ, can be determined by

$$IC_n = [\overline{X} - \delta, \overline{X} + \delta],$$

whereby $\overline{X}$ is the sample mean and $\delta = \delta(X_1, \ldots, X_n)$ is so that $P(\mu \in IC_n) = 1 - \beta$. The keys are, firstly, that this confidence interval can be seen as a ball with respect to the Euclidian distance which is centered in the sample mean $\overline{X}$ and has radius δ and, secondly, that in practice δ can be computed via bootstrapping. In other words: classical confidence intervals for the mean μ can equivalently be seen as balls, having the sample mean as center and a suitable radius.

Having this is in mind, our goal is to consider a *good* metric (that is, both easy to handle and easy to interpret) on the space of all fuzzy sets and, on the basis of this metric, to define confidence regions as balls centered in the sample mean and with a given radius which is empirically determined by applying bootstrap procedures. By the comments made before this technique surely is a natural extension of the classical procedure for real random variables.

The rest of the manuscript is organized as follows. In Section 2 we will recall some preliminary concepts concerning FRVs. At the beginning of Section 3 we propose an initial algorithm to determine a confidence ball for the mean of an FRV by using bootstrap. After checking its capability by means of various simulations made in C and R we finally present an improvement of the algorithm. In Section 4 we will apply the developed procedure to a concrete fuzzy sample consisting of courses evaluations. Finally, some concluding remarks, next developments and open problems are gathered in Section 5.

2 Preliminaries

Fuzzy random variables in Puri and Ralescu's sense [15] are an extension of both random variables and random (convex compact) sets. They allow us to model situations in which an imprecise (fuzzy) value is associated with the outcome of a stochastic experiment. In order to give an exact definition of FRVs we need some preliminary notions.

$\mathcal{K}_c(\mathbb{R})$ will denote the family of all non-empty compact intervals. $\mathcal{F}_c(\mathbb{R})$ denotes the class of fuzzy numbers, i.e. functions $U : \mathbb{R} \to [0,1]$, whose α-levels U_α fulfill $U_\alpha \in \mathcal{K}_c(\mathbb{R})$ for all $\alpha \in [0,1]$ whereby, as usual, $U_\alpha := [\underline{U}_\alpha, \overline{U}_\alpha] := \{x \in \mathbb{R} | U(x) \geq \alpha\}$ for every $\alpha \in (0,1]$ and $U_0 := [\underline{U}_0, \overline{U}_0]$ is defined as the topological closure of $\cup_{\alpha \in (0,1]} U_\alpha$.

The class $\mathcal{F}_c(\mathbb{R})$ is usually endowed with a semilinear structure that is compatible with Zadeh's Extension Principle (see [17]) - i.e. with a *sum* (extending the Minkowsky addition of intervals), defined by $(U+V)_\alpha = U_\alpha + V_\alpha$ for all $\alpha \in [0,1]$, and with a *product by a scalar*, $(aU)_\alpha = aU_\alpha$ for all $\alpha \in [0,1]$ and $a \in \mathbb{R}$. For every $U \in \mathcal{F}_c(\mathbb{R})$ mid_U and spr_U are defined to be real-valued functions on $[0,1]$ such that $mid_U(\alpha) := (\underline{U}_\alpha + \overline{U}_\alpha)/2$ and $spr_U(\alpha) := (\overline{U}_\alpha - \underline{U}_\alpha)/2$

ISBN: 978-989-95079-6-8

for every $\alpha \in [0,1]$.
A very useful metric on $\mathcal{F}_c(\mathbb{R})$ from the statistical point of view is the one introduced by Bertoluzza *et al* [1]. It depends on two weighting probability measures W and φ, defined on $([0,1], \mathcal{B}_{[0,1]})$ ($\mathcal{B}_{[0,1]}$ being the Borel σ-field on $[0,1]$). For any two fuzzy numbers $U, V \in \mathcal{F}_c(\mathbb{R})$ the D_W^φ-distance is defined as:

$$\sqrt{\int_{[0,1]^2} \left[t(\underline{U}_\alpha - \underline{V}_\alpha) + (1+t)(\overline{U}_\alpha - \overline{V}_\alpha)\right]^2 \, dW(t)d\varphi(\alpha)}$$

Thereby φ is assumed to have a strictly increasing distribution function on $[0,1]$ (and therefore has the whole interval $[0,1]$ as support) and W is non-degenerate (i.e. it is not a Dirac measure). A especially useful subclass of choices for W, on that we will concentrate in the sequel is that for which $\int_0^1 t dW(t) = 1/2$ (which in fact is equivalent to assuming D_W^φ to be invariant to rigid motions on $\mathbb{R}$). In this case it is easy to see that $(D_W^\varphi(U,V))^2$ can be rewritten as follows:

$$(D_W^\varphi(U,V))^2 = \int_{[0,1]} (\text{mid}\,_U(\alpha) - \text{mid}\,_V(\alpha))^2 d\varphi(\alpha)$$

$$+\theta_W \int_{[0,1]} (\text{spr}\,_U(\alpha) - \text{spr}\,_V(\alpha)^2 d\varphi(\alpha)$$

That is, the squared D_W^φ-distance of $U, V \in \mathcal{F}_c(\mathbb{R})$ is the sum of the squared φ-weighted L_2-distances between the mids and between the spreads, where the parameter $\theta_W \in (0,1]$ allows us to weight the relative importance of the distance between mids w.r.t. the distance between spreads (in particular, when W is chosen as the Lebesgue measure λ then $\theta_W = 1/3$).

Having this, we can now return to the definition of an FRV mentioned before. Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. an FRV $\mathcal{X}$ is a mapping $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$, which is Borel-measurable with respect to the D_W^φ-metric. This definition is different from the original one by Puri and Ralescu, however in many cases and in particular in the cases we will consider in the sequel coincides with the original one (see [15, 11, 2]).
The FRV $\mathcal{X}$ is said to be integrably bounded whenever $\max\{|\min X_0|, |\max X_0|\} \in L^1(\Omega, \mathcal{A}, P)$. If $\mathcal{X}$ is an integrably bounded FRV then its *expected value (or mean)* is defined as the unique fuzzy number $\mathbb{E}(\mathcal{X}) \in \mathcal{F}_c(\mathbb{R})$ such that $\big(\mathbb{E}(\mathcal{X})\big)_\alpha$ is the Aumman integral of the random set $\mathcal{X}_\alpha$ for all $\alpha \in [0,1]$ (see [15]). In particular,

$$\big(\mathbb{E}(\mathcal{X})\big)_\alpha = [\mathbb{E}(\min \mathcal{X}_\alpha), \mathbb{E}(\max \mathcal{X}_\alpha)]$$

holds for all $\alpha \in [0,1]$ if $\mathcal{X}$ is integrably bounded.
If, in addition, $\max\{|\min \mathcal{X}_0|, |\max \mathcal{X}_0|\} \in L^2(\Omega, \mathcal{A}, P)$, then the (W,φ)-*variance* of $\mathcal{X}$ (see [12, 10]) is defined as

$$\text{Var}(\mathcal{X}) = \mathbb{E}([D_W^\varphi(\mathcal{X}, \mathbb{E}(\mathcal{X}))]^2). \qquad (1)$$

Remark: The main reasons for using the above mentioned D_W^φ-metric are that (1) it has more good intuitive properties (see [1]) than the usual L_p metrics d_p (see [4]) do not have, and (2) it can be expressed as an inner product on the Hilbert space of all square integrable functions on $\mathcal{H} = L^2([0,1] \times \{-1,1\})$ (via using the concept of support function).

3 Confidence regions for the mean of an FRV

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space, and consider an FRV $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ verifying $\max\{|\min \mathcal{X}_0|, |\max \mathcal{X}_0|\} \in L^2(\Omega, \mathcal{A}, P)$. As mentioned in the introduction, our aim is to calculate a confidence ball for the expectation $\mu = \mathbb{E}(\mathcal{X}) \in \mathcal{F}_c(\mathbb{R})$ on the basis of n independent observations $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ of $\mathcal{X}$.
We will denote by $\overline{\mathcal{X}}$ and $\widehat{S}$ the associated sample fuzzy mean and sample (W,φ)-deviation respectively, that is,

$$\overline{\mathcal{X}} = \frac{1}{n}\sum_{i=1}^n \mathcal{X}_i, \quad \widehat{S} = \sqrt{\frac{1}{n}\sum_{i=1}^n [D_W^\varphi(\mathcal{X}_i, \overline{\mathcal{X}})]^2}. \qquad (2)$$

Following the approach mentioned in the introduction, for a given significance level $\beta \in (0,1)$ we define the confidence ball CR_β with respect to the D_W^φ-metric as

$$CR_\beta = B(\overline{\mathcal{X}}, \delta) = \{U \in \mathcal{F}_c(\mathbb{R}) | D_W^\varphi(U, \overline{\mathcal{X}}) \le \delta\}, \qquad (3)$$

whereby the radius δ has to verify the coverage condition

$$P(\mu \in B(\overline{\mathcal{X}}, \delta)) = P(D_W^\varphi(\mu, \overline{\mathcal{X}}) \le \delta) = 1 - \beta. \qquad (4)$$

Due to the non-existence of sufficiently general parametric models for FRVs in general it is not possible to find δ fulfilling the coverage condition. Nevertheless we may choose δ as the $(1-\beta)$-quantile of the distribution of $D_W^\varphi(\mu, \overline{\mathcal{X}})$, which in turn can be approximated by the corresponding bootstrap $(1-\beta)$-quantile. In fact, given a sample $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ of $\mathcal{X}$ we propose to proceed as follows:

Algorithm 3.1

Step 1. *Fix the significance level* $\beta \in (0,1)$ *and the number* B *of bootstrap replications.*

Step 2. *Obtain* B *bootstrap samples* $(\mathcal{X}_1^{b\star}, \ldots, \mathcal{X}_n^{b\star})$ $(b = 1, \ldots, B)$ *from* $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$*, and for each one compute its corresponding sample mean* $\overline{\mathcal{X}}^{b\star}$*.*

Step 3. *Compute the distance between the sample mean and each bootstrap sample mean,* $d_b^\star = D_W^\varphi(\overline{\mathcal{X}}, \overline{\mathcal{X}}^{b\star})$*, for each* $b = 1, \ldots, B$*.*

Step 4. *Choose* δ *as one of the* $(1-\beta)$*-quantiles of the sample* $(d_1^\star, \ldots, d_B^\star)$ *(that is, choose* δ *so that at least the* $100(1-\beta)\%$ *of the computed distances are smaller or equal than* δ *and at least the* $100\beta\%$ *of the computed distances are greater or equal than* δ*).*

In order to analyze the quality and accuracy of the confidence regions obtained by means of this algorithm we have made simulations with different sample sizes n. We have chosen φ as the Lebesgue measure on $[0,1]$ and used two different weight distributions W (in fact $\theta_W = 1/3$, which corresponds to the Lebesgue measure, and $\theta_W = 9/10$ that gives more weight to the spreads, were used). Each simulation corresponds to 10000 iterations with 1000 bootstrap replications and a significance level β of 0.05.
For the simulation of general FRVs the approach introduced in [8] has been applied and $n_0 = 101$ equally spaced alpha-levels ($\alpha_i = (i-1)/(n_0-1)$, $i = 1, \ldots, n_0$) were considered. In short, this approach consists essentially of the following three steps (for more details see [8]):

ISBN: 978-989-95079-6-8

SIM1 **Decomposition:**
Fix the (future) expectation $V \in \mathcal{F}_c(\mathbb{R})$, an index $n_0 \in \mathbb{N}$ and α-levels $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{n_0} = 1$. Let V^c be the mid-point of the 1-level of V, $V^l := V^c - \min(V_0)$ the total left spread and $V^r := \max(V_0) - V^c$ the total right spread. Furthermore define

$$L_\alpha = \begin{cases} \{0\} & \text{if } V^l = 0, \\ [(\min(V_\alpha) - V^c)/V^l, 0] & \text{if } V^l \neq 0 \end{cases}$$

$$R_\alpha = \begin{cases} \{0\} & \text{if } V^r = 0, \\ [0, (\max(V_\alpha) - V^c)/V^r] & \text{if } V^r \neq 0 \end{cases}$$

for every $\alpha \in [0,1]$ and set $F^l(t) := \max(-L_{1-t})$ and $F^r(t) := \max(R_{1-t})$ for every $t \in [0,1]$ (in fact F^l and F^r are distribution function but we do not use their stochastic interpretation).

SIM 2 **Discretization:**
Define $p_1^l = F^l(\alpha_1)$ and $p_i^l = F^l(\alpha_i) - F^l(\alpha_{i-1})$ for all $i = 2, \ldots, n_0$, and $p_1^r = F^r(\alpha_1)$ and $p_i^r = F^r(\alpha_i) - F^r(\alpha_{i-1})$ for all $i = 2, \ldots, n_0$ respectively. Let $B_x \in \mathcal{F}_c(\mathbb{R})$ denote the fuzzy set fulfilling $(B_x)_\alpha = [0, \mathbf{1}_{[\alpha,1]}(x)]$ for every $\alpha \in [0,1]$ and set

$$V_{n_0} = V^c + \sum_{i=1}^{n_0} -B_{1-\alpha_i} V^l p_i^l + \sum_{i=1}^{n_0} B_{1-\alpha_i} V^r p_i^r$$

SIM 3 **Stochastic Perturbation**
Consider a $(2n_0 + 1)$-dimensional random vector

$$\mathcal{Y} = (C_0, C_1^l, \ldots, C_{n_0}^l, C_1^r, \ldots, C_{n_0}^r) : \Omega \to \mathbb{R} \times [0,\infty)^{2n_0}$$

of coefficients for the 'approximating' simulated FRV $\mathcal{X}^\star$ as random perturbation of $(V^c, V^l p_1^l, \ldots, V^l p_{n_0}^l, V^r p_1^r, \ldots, V^r p_{n_0}^r)$ in such a way that

$$\mathbb{E}(C_0, C_1^l, \ldots, C_{n_0}^l, C_1^r, \ldots, C_{n_0}^r) = \\ = (V^c, V^l p_1^l, \ldots, V^l p_{n_0}^l, V^r p_1^r, \ldots, V^r p_{n_0}^r)$$

holds and set

$$\mathcal{X}^\star := C^0 + \sum_{i=1}^{n_0} -B_{1-\alpha_i} C_i^l + \sum_{i=1}^{n_0} B_{1-\alpha_i} C_i^r \qquad (5)$$

This procedure generates a sample of size 1 of the FRV $\mathcal{X}$ - for a sample of size n SIM 3 simply has to be repeated n times.

Remark: To be precise, in the simulations we have made we have used perturbations from the same distributions, i.e. $C_1^l, \ldots, C_{n_0}^l$ are independent and distributed as a fixed D^l, and $C_1^r, \ldots, C_{n_0}^r$ are independent and distributed as a fixed D^r (these distributions are also mentioned in the tables below).

In particular we made simulations for three FRVs $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ with different expectations - the above-mentioned parameters defining these FRVs are listed in Table 1. Moreover for the FRV $\mathcal{Z}$ (1) the mean, (2) examples of the simulated samples of $\mathcal{Z}$, (3) the distance between the mean V and one sample mean and (4) the empirical distribution function of the quantity $d_b^\star$ are depicted in Figure 1 - Figure 4.

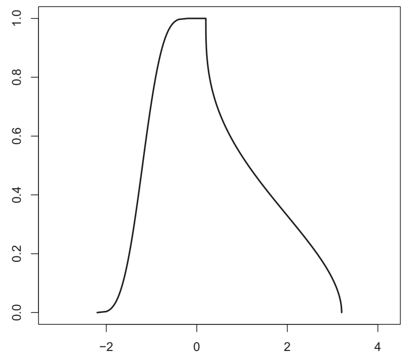

Figure 1: Expectation V of the FRV $\mathcal{Z}$

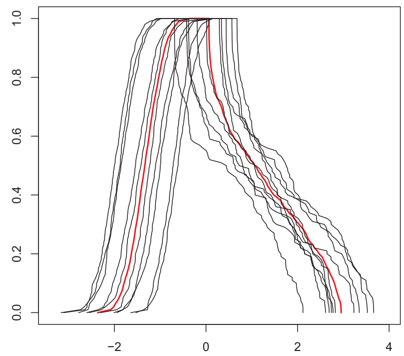

Figure 2: Simulated sample and sample mean (in red)

Table 1: Parameters defining the FRVs for the simulation

	$V = [V^c, V^l, V^r, F^l, F^r]$	$C = [C^0, D^l, D^r]$
$\mathcal{X}$	[0,1,1,Beta(1,1),Beta(1,1)]	$[U_{(-\frac{1}{2},\frac{1}{2})}, U_{(0,2)}, U_{(0,2)}]$
$\mathcal{Y}$	[4,2,2,Beta(0.5,1),Beta(3,2)]	$[U(0,8), \chi_1, \chi_2/2]$
$\mathcal{Z}$	[0,2.2,3.2,MixL,MixR)]	$[U(-1,1), U(0,2), Ex(1)]$

Thereby MixL denotes a mixture distribution consisting of a point mass in 0 with weight 0.083 and a Beta(0.3,0.3)-distribution with weight 0.9167, and MixR denotes a mixture distribution consisting of a point mass in 0 with weight 0.067 and a Beta(3,2)-distribution with weight 0.933.

In Table 2 and Table 3 the percentage of confidence regions (among the 10,000 simulated ones) containing the populational mean in each situation is collected.

Table 2: Percentage of accurate confidence regions using Algorithm 3.1 - Part 1

	$n = 10$			$n = 30$		
θ_W	$\mathcal{X}$	$\mathcal{Y}$	$\mathcal{Z}$	$\mathcal{X}$	$\mathcal{Y}$	$\mathcal{Z}$
1/3	89.86	90.03	90.15	93.58	93.25	93.80
9/10	89.89	89.69	0.00	93.07	93.64	93.64

ISBN: 978-989-95079-6-8

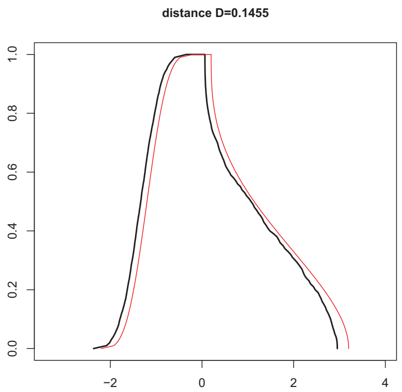

Figure 3: D_W^{φ}-distance of mean and sample mean ($\theta_W = 1/3$)

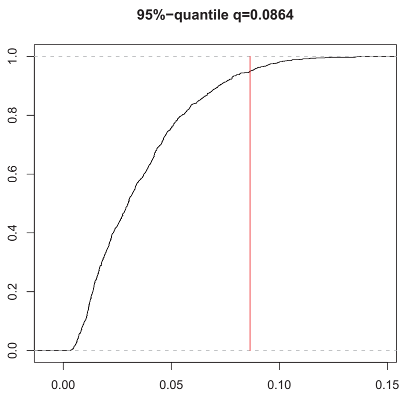

Figure 4: Empirical distribution function of $d_b^{\star}$ and 95%-quantile

Table 3: Percentage of accurate confidence regions using Algorithm 3.1 - Part 2

	$n = 100$		
θ_W	$\mathcal{X}$	$\mathcal{Y}$	$\mathcal{Z}$
$1/3$	94.51	94.83	0.00
$9/10$	94.54	94.65	0.00

Taking a look at the results in Table 2 and Table 3 it seems that (irrespective of the underlying distribution and the selected value for the parameter θ_W), the accuracy of the confidence regions generated by means of Algorithm 3.1 is sufficient for either moderate or large sample sizes (30-100 observations), whereas the results are not satisfactory for small sample sizes (10 observations). As usual, the bigger the sample size, the closer the empirical accuracy to the theoretical one.

In order to improve the results, especially for small sample sizes, we are going to modify the proposed method. The modification we make is the usual one in classical statistics - we define the radius of the confidence ball to be directly proportional to the (estimated) population variability (standardization). For this purpose, define a new confidence ball w.r.t. the D_W^{φ}-metric as follows

$$CR_{\beta} = B(\overline{\mathcal{X}}, \widehat{S}\delta) = \{U \in \mathcal{F}_c(\mathbb{R}) | D_W^{\varphi}(U, \overline{\mathcal{X}})/\widehat{S} \leq \delta\}, \quad (6)$$

whereby the radius δ has to verify the coverage condition

$$P(\mu \in B(\overline{\mathcal{X}}, \widehat{S}\delta)) = P(D_W^{\varphi}(\mu, \overline{\mathcal{X}})/\widehat{S} \leq \delta) = 1 - \beta. \quad (7)$$

For the same reasons mentioned in the previous case we again use bootstrapping. In this case the variability of the population, from which the bootstrap sample is taken, is completely known (denoted by $\widehat{S}$). Consequently, one possibility consists in approximating the distribution of $D_W^{\varphi}(\mu, \overline{\mathcal{X}})/\widehat{S}$ by means of $D_W^{\varphi}(\overline{\mathcal{X}}, \overline{\mathcal{X}^{\star}})/\widehat{S}$, which, however, leads exactly to the procedure proposed in Algorithm 3.1. Another possibility is to reestimate the variability of the bootstrap population instead of using the exact known value $\widehat{S}$. We will see that this alternative approach leads to better results.
In fact, the proposed procedure works as follows:

Algorithm 3.2

Step 1. *Fix the significance level $\beta \in (0, 1)$ and the number B of bootstrap replications.*

Step 2. *Obtain B bootstrap samples $(\mathcal{X}_1^{b\star}, \ldots, \mathcal{X}_n^{b\star})$ $(b = 1, \ldots, B)$ from $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$, and for each one compute its corresponding sample mean $\overline{\mathcal{X}}^{b\star}$ and its sample deviation $\widehat{S}^{b\star}$.*

Step 3. *Compute the distance between the sample mean and each bootstrap sample mean, and calculate $d_b^{\star} = D_W^{\varphi}(\overline{\mathcal{X}}, \overline{\mathcal{X}}^{b\star})/\widehat{S}^{b\star}$ for each $b = 1, \ldots, B$.*

Step 4. *Choose δ as one of the $(1 - \beta)$ quantiles of the sample $(d_1^{\star}, \ldots, d_B^{\star})$.*

In Table 4 the percentage of accurate confidence regions using Algorithm 3.2 in the same situations as in the preceding simulating study is collected. As it was expected the accuracy of the confidence regions obtained by using the second approach is much better than the one of the first approach. Indeed, the procedure can be applied also for small sample sizes ($n = 10$) and seems to be a little bit conservative (with an empirical confidence level greater than the theoretical one).

Table 4: Percentage of accurate confidence regions using Algorithm 3.2 - Part 1

	$n = 10$			$n = 30$		
θ_W	$\mathcal{X}$	$\mathcal{Y}$	$\mathcal{Z}$	$\mathcal{X}$	$\mathcal{Y}$	$\mathcal{Z}$
$1/3$	96.31	95.18	95.70	95.38	95.17	95.42
$9/10$	96.17	95.77	95.57	94.89	95.26	95.27

Table 5: Percentage of accurate confidence regions using Algorithm 3.2 - Part 2

	$n = 100$		
θ_W	$\mathcal{X}$	$\mathcal{Y}$	$\mathcal{Z}$
$1/3$	94.93	95.06	0.00
$9/10$	94.94	94.92	0.00

ISBN: 978-989-95079-6-8

4 Case Study

In this section we will apply the proposed methodology in order to obtain a confidence ball for the mean overall rating of the II Summer Course organized by the European Centre for Soft Computing in July 2008. A survey regarding different aspects of this summer course was presented to the students at the end of the course. Among other questions, the students where asked about the overall rating of the course. The answers where collected directly as fuzzy sets. In fact, a scale ranging from 0 (minimum rating) to 100 (maximum rating) was presented (see Figure 5). The students were asked to draw a trapezoidal fuzzy set R representing their rating by fixing two intervals, the 0-level and the 1-level. Firstly the 0-level was chosen in such a way that their perception about the overall rating was surely not outside of this interval. Finally the 1-level was chosen as an interval in which they thought their overall rating would be contained.

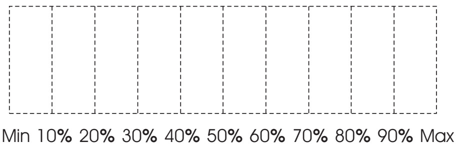

Figure 5: Scale for the made questionary

The overall rating of 33 students were collected, edited in Table 6 and the sample mean was calculated.

Applying Algorithm 3.1 and Algorithm 3.2 with $B = 10000$ bootstrap iterations and a significance level $\beta = 0.05$ we obtained the following results for the confidence balls listed in Table 7. The centre of the balls in each case is the sample mean $\overline{\mathcal{X}}$ (see Figure 6).

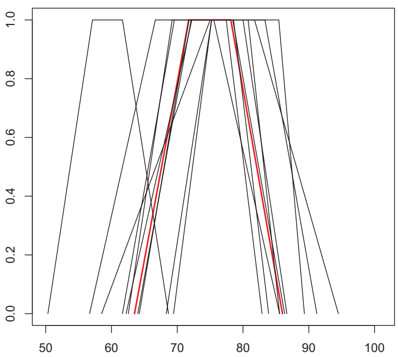

Figure 6: Thn elements of the fuzzy sample and the sample mean (red)

Figures 7 and Figures 8 depict the empirical distribution functions for the quantities $d_b^\star$ in Algorithm 3.1 and Algorithm 3.2 as well as the corresponding 95%-quantiles (for the case $\theta_W = 1/3$).

Remark: Having calculated the 95%-confidence balls for the mean overall rating, one question that naturally arises is, how this confidence balls really look like. Going back to the definition of the D_W^φ-metric it is clear that $U \in B(V, \delta)$ (ball with respect to D_W^φ), for instance, does not imply any restrictions on the support of U - in fact, the support can be

Table 6: Overall ratings of the II Summer Course.

Student	$\min R_0$	$\min R_1$	$\max R_1$	$\max R_0$
1	58,50	75,00	81,75	94,50
2	64,22	71,78	78,44	86,33
3	69,44	75,22	80,78	85,56
4	64,00	72,11	85,44	89,33
5	61,67	69,56	83,33	91,22
6	56,67	66,67	75,56	85,56
7	62,22	72,22	80,00	86,67
8	50,33	57,11	61,67	68,67
9	68,33	75,22	77,44	82,89
10	62,56	69,22	78,56	83,89
11	82,22	87,33	88,11	93,44
12	50,33	60,00	64,44	74,44
13	45,56	56,67	70,00	81,11
14	69,13	75,00	89,00	94,00
15	58,89	68,89	73,11	83,11
16	51,11	62,22	63,33	77,78
17	65,00	72,50	78,50	86,25
18	51,89	61,22	68,22	86,56
19	45,89	51,67	59,44	66,33
20	74,11	80,56	88,78	95,11
21	67,89	76,22	78,33	85,56
22	74,17	85,00	88,33	97,33
23	72,22	75,00	80,44	85,33
24	51,11	60,00	71,11	78,89
25	55,56	65,56	68,89	77,22
26	76,11	80,89	90,11	93,44
27	69,89	75,11	78,00	83,22
28	80,00	90,00	90,00	100
29	71,29	79,14	82,86	89,57
30	63,78	70,89	79,44	86,67
31	63,33	74,44	85,56	95,56
32	60,00	72,11	72,11	79,89
33	78,00	83,00	87,80	92,60
Mean	63,50	71,74	78,15	86,00

Table 7: Results of the case study - calculated radius

		Radius	
θ_W	$\hat{S}$	Algorithm 3.1	Algorithm 3.2
$1/3$	8.41	2.8239	0.3587
$9/10$	8.61	2.8224	0.3476

arbitrarily big as long as (loosely speaking) not too many[1] α-levels are arbitrarily big. The reason for this behaviour lies in the fact that the D_W^φ-metric (because of the integration involved) averages over the distances of the α-levels (the same holds for most metrics on $\mathcal{F}_c(\mathbb{R})$). Nevertheless the condition $U \in B(V, \delta)$ (ball with respect to D_W^φ) surely implies strong restrictions on U - if for instance the spread of V is small for all α, then the spread of U can not be too big for many α's. As an example, the trapezoidal fuzzy number $(73, 73, 78, 78)$ (representation as in Table 6) is not contained in the calculated confidence Ball $B(\overline{\mathcal{X}}, 2.8239)$ since the mean spread is

[1]"many" in this context has to be understood as "for a set of big Lebesgue measure"

ISBN: 978-989-95079-6-8

too small.

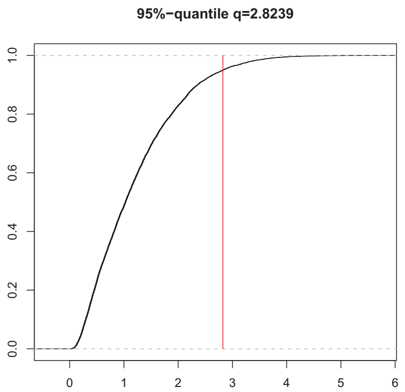

Figure 7: Empirical distribution function of $d_b^{\star}$ and 95%-quantile (Algoritm 3.1)

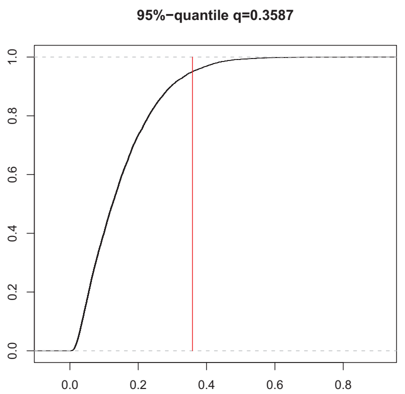

Figure 8: Empirical distribution function of $d_b^{\star}$ and 95%-quantile (Algoritm 3.1)

5 Conclusions and open points

In this paper we have proposed two algorithms for calculating confidence balls for the mean of an FRV - we used the sample mean as centre of the ball and calculated the radius of the ball via bootstrapping. Since we already have implemented the developed procedures (and others) in C and R our aim is to write a general R-package for statistics with fuzzy data, which could then be freely used by everybody working with R.

The simulations indicate that the second method is quite accurate, in the sense of achieving the nominal significance level for moderate and even small samples. Nevertheless, it is essential to analyze theoretically this approach in order to soundly justify its usefulness.

We have focused on a kind of percentile bootstrapping for the estimations, however it can be interesting to consider other kinds of bootstrap approaches.

Finally, for a future work we propose to analyze a possible modification of the metrics to make the estimation as reasonable as possible, by avoiding to include in the confidence region fuzzy sets which are close to the sample mean, but are not too realistic.

Acknowledgment

The research in this communication has been partially supported by the Grant from the Spanish Ministry of Education and Science MTM2006-07501. This financial support is gratefully acknowledged.

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