Repairing infeasibility in fuzzy goal programming

Mariano Jiménez[1], Amelia Bilbao[2], Mar Arenas[2], Mª Victoria Rodríguez-Uría[2]

1 Dpto. de Economía Aplicada I, Universidad del País Vasco,
Plaza Oñati 1. 20018 San Sebastián. Spain
2 Dpto. de Economía Cuantitativa, Universidad de Oviedo,
Avda. del Cristo s/n. 33006 Oviedo. Spain
E-mail: mariano.jimenez@ehu.es , ameliab@uniovi.es , mariamar@uniovi.es , vrodri@uniovi.es

Abstract*—The problem of solving a multi-objective programming problem, by assuming that the decision maker has fuzzy goals for each of the objectives, is addressed. In this case a fuzzy goal programming problem has to be solved. But when we have several objectives it can be a difficult task for the decision maker to determine coherent aspiration levels, so it is possible that the tolerance thresholds be inconsistent with each other, therefore obtaining an infeasible problem. We present a procedure in order to repair the infeasibility. We rebuild the unattainable fuzzy goals keeping their relaxation levels as small as possible while maintaining, at the same time, a good balance vis-à-vis the degree of achievement of the others goals.*

Keywords—, Fuzzy goal programming, Goal programming, Infeasibility, Multi-objective programming.

1 Introduction

Let a multi-objective programming (MOP) problem with k objective functions $z_i(x) = c_i x$, $i = 1,...,k$ be

$$\text{(MOP)} \quad \text{Min} \quad z(x) = (z_1(x), z_2(x),..., z_k(x))$$
$$\text{S. t.} \quad x \in X$$

Where $X = \{x \in \mathbb{R}^n \,|\, Ax \geq b,\, x \geq 0\}$, $c_i = (c_{i1},...,c_{in}) \in \mathbb{R}^n$, $i = 1,...,k$; $b = (b_1,...,b_m) \in \mathbb{R}^m$ and A is a $\mathbb{R}^{m \times n}$ matrix.

For the sake of simplicity we suppose that all the objectives should be minimized. However, the procedure demonstrated is easily extendable were the case to include some maximizing objectives.

In (MOP) it is unlikely that all objectives will simultaneously achieve their optimal value. Therefore in practice the decision maker (DM) chooses a satisfying solution, according to the aspiration level fixed for each objective.

Assuming that the DM proposes imprecise aspiration levels such as, "$z_i(x)$ should be essentially less than or equal to some value g_i", model (MOP) can be written as

$$\text{(FGP)} \quad \text{Find } x$$
$$\text{Such that} \quad z_i(x) \lesssim g_i \quad i = 1,2,...,k$$
$$x \in X$$

Each expression $c_i x \lesssim g_i$ is represented by a fuzzy set called fuzzy goal, whose membership function, $\mu_i(z_i)$, $\mu_i : \mathbb{R} \to [0,1]$, provides the satisfaction degree λ_i to which the i^{th} fuzzy inequality is satisfied. In order to define the membership function $\mu_i(z_i)$ the DM has to provide the tolerance margins g_i+t_i that she/he is willing to accept. So $\mu_i(z_i)$ should be equal to 1 if $z_i \leq g_i$, strictly monotone decreasing from 1 to 0 over the interval $(g_i, g_i + t_i)$ and equal to 0 if $z_i \geq g_i + t_i$.

Models like this are named fuzzy goal programming (FGP) problems. (FGP) should be considered as an auxiliary model in order to solve (MOP).

As it has been widely seen, a FGP problem, using the fuzzy decision max-min of Bellman and Zadeh [1] and introducing the auxiliary variable λ, adopts the following formulation [2]

$$\text{(B-Z)} \quad \text{Max } \lambda$$
$$\text{S.t.} \quad 0 \leq \lambda \leq \mu_i(z_i(x)) \quad i = 1,\ldots,k$$
$$x \in X \text{ , } \lambda \leq 1$$

But when there are several objective functions it can be a difficult task for the DM to determine coherent aspiration levels. Therefore it is possible that some tolerance thresholds will be inconsistent with each other or with the crisp constraints, some of them being unattainable (membership degree=0). In this case (B-Z) is infeasible. Chinneck [3] summarizes the state of the art in algorithms related to infeasibility in optimization. Most published research has been focused on the diagnosis of infeasibility. Little investigation has been made in infeasibility resolution and very limited literature can be found in infeasibility resolution using fuzzy approach. León and Liern [5] use a fuzzy sets approach to relax constraints to repair infeasibility in crisp mathematical programming problems. Gupta et al. [6] take a similar fuzzy approach to finding a best approximate solution to an infeasible mathematical programming problem. In this paper we propose a procedure in order to solve a FGP should the above described drawback occur. We demonstrate the

ISBN: 978-989-95079-6-8

operativeness of our approach by means of a numerical example.

2 Repairing infeasibility

Let us assume that (B-Z) is infeasible because the tolerance thresholds of some goals are unattainable. The first step (M.I) is to identify one minimum-cardinality subset of {1, 2,…, k}, that will be denoted *CS*, so that the goals {1, 2,…, k}−*CS* provide a feasible (B-Z) problem. This procedure is related to the minimum cardinality irreducible infeasible subsystem (IIS) set-covering problem (MIN IIS COVER) (see [3, 4]). Hence the Chinneck's algorithm [4] will be applied to the following sets of linear constraints

$$z_i(x) \le g_i + t_i \qquad i = 1,2,...,k$$
$$x \in X$$

where only the k-first constraints (the goals) are elasticized. So we will work with the model

$$\text{(M.I)} \quad \text{Min} \sum_{i=1}^{k} p_i$$
$$\text{S.t.} \quad z_i - p_i \le g_i + t_i \quad i = 1,\ldots,k$$
$$x \in X$$

Next, in order to repair infeasibility, we have to shift the unattainable goals, i.e. the *CS*-goals, as low as possible without penalizing excessively the achievement degree of other goals. Taking this into account we consider the following two positions: 1) we allow the feasible goals to deteriorate to their corresponding tolerance threshold 2) we aim to attain a satisfactory solution for the feasible goals. Obviously the relaxation of the unattainable goals will be smaller in the first position than in the second. Therefore we assign a satisfaction degree equal to 1 to the relaxed values obtained in the first position and a satisfaction degree equal to 0 to that obtained in the second position, decreasing monotonously between these values. This way, we obtain the membership functions $\mu_i^*(z_i(x))$ that rebuild the new fuzzy goals corresponding to the, previously, unattainable goals.

Assuming, for the sake of simplicity, that the unattainable goals are the first *r*. Consider the aforementioned position 1, that is to say, we allow the feasible goals to deteriorate until their tolerance threshold and, being an inclusive condition, we look for the lowest relaxation levels for the unattainable goals by conventional goal programming approach [7]. Thus we propose to solve the following model

$$\text{(M.II)} \quad \text{Min} \sum_{i=1}^{r} w_i p_i$$
$$\text{S.t.} \quad z_i(x) - p_i \le g_i + t_i, \quad i = 1,...,r$$
$$z_i(x) \le g_i + t_i, \qquad i = r+1,...,k$$
$$x \in X, p_i \ge 0$$

where $\sum_{i=1}^{r} w_i = 1$ and $w_i \ge 0$ for all i=1,…,r. The variables p_i have the same significance as the positive deviation in goal programming (GP). (M.II) searches for a solution with the smallest weighted average relaxation levels in order to achieve feasibility, considering that the other goals could reach their corresponding tolerance threshold. Weight w_i represents the priority or importance of the relaxation levels of tolerance threshold g_i+t_i. It is well known that when $w_i>0$ for all i=1,…,r, any optimal solution of (M.II) is efficient [8], which give us a large set of possible goal shifts that provide a feasible solution.

Let x^{II} be an optimal solution of (M.II). In line with what we have said before, the values $z_i^{II} = z_i(x^{II}), i = 1,...,r$, have membership degree equal to 1 in the new (repaired) fuzzy goals that we are constructing.

Now let us consider the aforementioned position 2, that is to say we are looking for a good degree of satisfaction of the goals that are attainable in (M.I). In order to achieve this we solve the following max-min model in which only the feasible goals are included

$$\text{(M.III)} \quad \text{Max } \lambda$$
$$\text{S. t. } z_i(x) \ge z_i^{II}, \ i = 1,...,r$$
$$0 \le \lambda \le \mu_i(z_i(x)) \quad i = r+1,\ldots,k$$
$$x \in X \ , \lambda \le 1$$

Let x^{III} be an optimal solution of (M.III) and $z_i^{III} = z_i(x^{III})$, i=r+1,…,k. We can consider the values $\max\{z_i^{III}, g_i\}$, i=r+1,…,k as sufficiently satisfactory. In the process of searching for the lowest relaxation levels of the infeasible goals, maintaining the values of the feasible goals sufficiently satisfactory, we solve the following GP model

$$\text{(M.IV)} \quad \text{Min} \sum_{i=1}^{r} w_i p_i$$
$$\text{S.t.} \quad z_i(x) - p_i \le g_i + t_i, \quad i = 1,...,r$$
$$z_i(x) \ge z_i^{II}, \qquad i = 1,...,r$$
$$z_i(x) \le \max\{z_i^{III}, g_i\}, \qquad i = r+1,...,k$$
$$x \in X, p_i \ge 0$$

Let x^{IV} be an optimal solution of (M.IV). The values $z_i^{IV} = z_i(x^{IV}), i = 1,...r$, achieved by the unattainable goals, have membership degree equal to 0 in the new (repaired) fuzzy goals that we are constructing.

We write $z_i^{II} \equiv g_i^*$ and $z_i^{IV} \equiv g_i^* + t_i^*$.

This way we have obtained the new fuzzy aspiration levels $\mu_i^*(z_i(x))$ which restore feasibility with the lowest relaxation levels (see figure 1).

Finally we solve the following max-min model, by replacing the unattainable aspiration levels substituted with the new aspiration levels, which restores feasibility.

$$\text{(B-Z*)} \quad \text{Max } \lambda$$
$$\text{S.t.} \quad \mu_i^*(z_i(x)) \ge \lambda \ge 0, i = 1,...,r$$
$$\mu_i(z_i(x)) \ge \lambda \ge 0, i = r+1,...,k$$
$$x \in X \ , \quad \lambda \le 1$$

ISBN: 978-989-95079-6-8

The optimal solutions of (B-Z*) can be considered as solutions of (B-Z) well balanced between attainable fuzzy goals an unattainable fuzzy goals. Moreover, if the solution of this problem is not fuzzy-efficient of (FGP) and/or not Pareto optimal of (MOP), the solution can be improved following a procedure showed in [9].

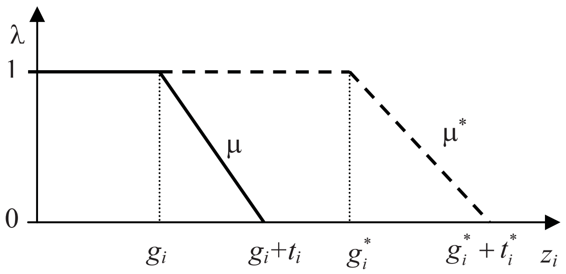

Figure1: Relaxation of an unattainable fuzzy aspiration level.

3 Example

To check the usefulness of our approach we consider the following numerical example.
Let a MOLP problem be

$$\begin{aligned} &\text{Min } z_1=3x_1+3x_2+3x_3+2x_4+3x_5\\ &\text{Min } z_2=2x_1+x_2+2x_3+x_4+3x_5\\ &\text{Min } z_3=4x_1+4x_2+2x_3+3x_4+x_5\\ &\text{Max } z_4=x_1+6x_2+5x_3+7x_4+6x_5\\ .\ &\text{Min } z_5= 5x_1+5x_2+4x_3+7x_4+5x_5\\ &\text{S.t. } 4x_1+2x_2+4x_3+2x_4+3x_5 \geq 18\\ &\quad x_1 \geq 1,\ 0 \leq x_3 \leq 3\\ &\quad x_2 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0 \end{aligned}$$

Assume the DM proposes the following imprecise aspiration levels: "z_1 should be essentially less than or equal to 18", "z_2 should be essentially less than or equal to 6", "z_3 should be essentially less than or equal to 13", "z_4 should be essentially greater than or equal to 24" and "z_5 should be essentially less than or equal to 17", the respective tolerance threshold being 21, 8, 15, 21 and 20.

Step 1. We solve the (B-Z) problem

$$\begin{aligned} &\text{Max } \lambda\\ &\text{S.t.}\\ &\lambda \leq \frac{1}{3}\left(21-\left(3x_1+3x_2+3x_3+2x_4+3x_5\right)\right)\\ &\lambda \leq \frac{1}{2}\left(8-\left(2x_1+x_2+2x_3+x_4+3x_5\right)\right)\\ &\lambda \leq \frac{1}{2}\left(15-\left(4x_1+4x_2+2x_3+3x_4+x_5\right)\right)\\ &\lambda \leq \frac{1}{3}\left(\left(x_1+6x_2+5x_3+7x_4+6x_5\right)-21\right)\\ &\lambda \leq \frac{1}{3}\left(20-\left(5x_1+5x_2+4x_3+7x_4+5x_5\right)\right)\\ &4x_1+2x_2+4x_3+2x_4+3x_5 \geq 18\\ &x_1 \geq 1,\ 0 \leq x_3 \leq 3\\ &x_2 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ 0 \leq \lambda \leq 1 \end{aligned}$$

This problem is infeasible because the tolerance thresholds of some goals are unattainable.

Step 2. We will identify one minimum-cardinality irreducible infeasible subsystem of goals by applying the Chinneck's algorithm to the following system of linear constraints where only the five first constraints (the goals) are elasticized:

$$\begin{aligned} &3x_1+3x_2+3x_3+2x_4+3x_5 \leq 21\\ &2x_1+x_2+2x_3+x_4+3x_5 \leq 8\\ &4x_1+4x_2+2x_3+3x_4+x_5 \leq 15\\ &x_1+6x_2+5x_3+7x_4+6x_5 \geq 21\\ .\ &5x_1+5x_2+4x_3+7x_4+5x_5 \leq 20\\ &4x_1+2x_2+4x_3+2x_4+3x_5 \geq 18\\ &x_1 \geq 1,\ 0 \leq x_3 \leq 3\\ &x_2 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0 \end{aligned}$$

We obtain $CS = \{2, 5\}$.

Step 3. In order to repair infeasibility we propose to solve the following model (see M.II)

$$\begin{aligned} &\text{Min } p_2+p_5\\ &\text{S.t.}\\ &2x_1+x_2+2x_3+x_4+3x_5-p_2 \leq 8\\ &5x_1+5x_2+4x_3+7x_4+5x_5-p_5 \leq 20\\ &3x_1+3x_2+3x_3+2x_4+3x_5 \leq 21\\ &4x_1+4x_2+2x_3+3x_4+x_5 \leq 15\\ &x_1+6x_2+5x_3+7x_4+6x_5 \geq 21\\ &4x_1+2x_2+4x_3+2x_4+3x_5 \geq 18\\ &x_1 \geq 1,\ 0 \leq x_3 \leq 3\\ &x_2 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0\\ &p_2 \geq 0,\ p_5 \geq 0 \end{aligned}$$

We get $z_2^{\text{II}} = 9$ and $z_5^{\text{II}} = 21.5$. These values have membership degree equal to 1 in the new (repaired) fuzzy goals that we are constructing.
In order to look for the balanced relaxation levels of the infeasible goal we need to find the values sufficiently satisfactory for the feasible goals.

Step 4. We solve the following model (M. III)

$$\begin{aligned} &\text{Max } \lambda\\ &\text{S.t.}\\ &\lambda \leq \frac{1}{3}\left(21-\left(3x_1+3x_2+3x_3+2x_4+3x_5\right)\right)\\ &\lambda \leq \frac{1}{2}\left(15-\left(4x_1+4x_2+2x_3+3x_4+x_5\right)\right)\\ &\lambda \leq \frac{1}{3}\left(\left(x_1+6x_2+5x_3+7x_4+6x_5\right)-21\right)\\ &2x_1+x_2+2x_3+x_4+3x_5 \geq 9\\ &5x_1+5x_2+4x_3+7x_4+5x_5 \geq 21.5\\ &4x_1+2x_2+4x_3+2x_4+3x_5 \geq 18\\ &x_1 \geq 1,\ 0 \leq x_3 \leq 3\\ &x_2 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0\\ &0 \leq \lambda \leq 1 \end{aligned}$$

As $z_i^{\text{III}} = 15.48$, $z_3^{\text{III}} = 11.32$, $z_4^{\text{III}} = 26.52$ then in the process of searching for the lowest relaxation levels of the infeasible

ISBN: 978-989-95079-6-8

goals, maintaining the values of the feasible goals sufficiently satisfactory, we solve the following GP model:

Step 5. (see M.IV)

$$
\begin{aligned}
&\text{Min } p_2 + p_5\\
&\text{S.t.}\\
&2x_1+x_2+2x_3+x_4+3x_5 - p_2 \le 8\\
&5x_1+5x_2+4x_3+7x_4+5x_5 - p_5 \le 20\\
&3x_1+3x_2+3x_3+2x_4+3x_5 \le 18\\
&2x_1+x_2+2x_3+x_4+3x_5 \ge 9\\
&4x_1+4x_2+2x_3+3x_4+x_5 \le 13\\
&x_1+6x_2+5x_3+7x_4+6x_5 \ge 24\\
&5x_1+5x_2+4x_3+7x_4+5x_5 \ge 21.5\\
&4x_1+2x_2+4x_3+2x_4+3x_5 \ge 18\\
&x_1 \ge 1,\ 0 \le x_3 \le 3\\
&x_2 \ge 0,\ x_4 \ge 0,\ x_5 \ge 0\\
&p_2 \ge 0,\ p_5 \ge 0
\end{aligned}
$$

We get $z_2^{IV} = 9.5,\ z_5^{IV} = 24.96$. These values achieved by the unattainable goals, have membership degree equal to 0 in the new (repaired) fuzzy goals that we are constructing.

This way we have obtained the new fuzzy aspiration levels which restore the feasibility with the lowest relaxation levels: "z_2 should be essentially greater than 9" and "z_5 should be essentially greater than 21.5" with tolerance threshold equal to 9.5 and 25, respectively. Finally we solve the following (B-Z*) model

$$
\begin{aligned}
&\text{Max } \lambda\\
&\text{s.t.}\\
&\lambda \le \frac{1}{3}\left(21-\left(3x_1+3x_2+3x_3+2x_4+3x_5\right)\right)\\
&\lambda \le \frac{1}{0.5}\left(9.5-\left(2x_1+x_2+2x_3+x_4+3x_5\right)\right)\\
&\lambda \le \frac{1}{2}\left(15-\left(4x_1+4x_2+2x_3+3x_4+x_5\right)\right)\\
&\lambda \le \frac{1}{3}\left(\left(x_1+6x_2+5x_3+7x_4+6x_5\right)-21\right)\\
&\lambda \le \frac{1}{3.5}\left(25-\left(5x_1+5x_2+4x_3+7x_4+5x_5\right)\right)\\
&4x_1+2x_2+4x_3+2x_4+3x_5 \ge 18\\
&x_1 \ge 1,\ 0 \le x_3 \le 3\\
&x_2 \ge 0,\ x_4 \ge 0,\ x_5 \ge 0\\
&0 \le \lambda \le 1
\end{aligned}
$$

We get $x_1^* = 1,\ x_2^* = 0.79,\ x_3^* = 2.86,\ x_4^* = 0.25,\ x_5^* = 0.13$; $z_1(x^*) = 14.92$, $z_2(x^*) = 9.2$, $z_3(x^*) = 13.83$, $z_4(x^*) = 22.75$ and $z_5(x^*) = 22.95$. This can be considered as a solution of model (B-Z) well balanced between attainable fuzzy goals and unattainable fuzzy goals.

Acknowledgment

This work was supported by the *Ministerio Español de Educación y Ciencia* under grant no. MTM2007-67634. and by the *Universidad del País Vasco* under grant no. EHU08/28.

4 Conclusions

This paper describes a method to deal with infeasibility in fuzzy goal programming problems in case of the DM proposes unattainable tolerance thresholds for some goals. Considering the inherent flexibility of fuzzy approach we restore the feasibility by shifting the unattainable goals as low as possible but, at the same time, taking care not to penalize excessively the degree of achievement of the others goals. This way we rebuilt the unattainable goals obtaining a new feasible fuzzy goal programming problem whose solution is well balanced between attainable and unattainable fuzzy goals.

References

[1] R.E. Bellman and L.A. Zadeh, Decision making in a fuzzy environment, *Management Science,* 17: 141-164, 1970.

[2] H.J. Zimmermann, Fuzzy programming and linear programming with several objective functions, *Fuzzy Sets and Systems*, 1 (1): 45-55, 1978.

[3] J.W. Chinnek, *Feasibility and Infeasibility in Optimization. Algorithms and Computational Methods.* New York: Springer, 2008.

[4] J. W. Chinnek, Fast Heuristics for the Maximum Feasible subsystem Problem, *INFORMS Journal on Computing,* Vol. 13, Nº 3, Suumer: 210-223, 2001.

[5] T. León and V. Liern, A fuzzy method to repair infeasibility in linearly constrained problem, *Fuzzy Sets and Systems,* 122: 237-243, 2001.

[6] P. Gupta, M. Vlach and D. Bhatia, Fuzzy approximation to an infesible generalized linear complementary problem, *Fuzzy Sets and Systems,* 146: 221-233, 2004.

[7] J. Jang, Infeasibility resolution based on goal programming, Computers & Operations Research, 35:1483-1493, 2008.

[8] Y. Sagarawi, H. Nakayama, T. Tanino. Theory of Multiobjective Optimization. London: Academia Press, 1985.

[9] M. Jiménez and A. Bilbao, Pareto-optimal solutions in fuzzy multi-objective linear programming, *Fuzzy Set and Systems*, 2008, doi: 10.1016/j.fss.2008.12.005.