Determining OWA weights by maximizing consensus

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Abstract— *In this paper we propose a method for generating OWA weighting vectors from the individual assessments on a set of alternatives in such a way that these weights maximize the consensus among individual assessments with respect to the outcome provided by the OWA operator.*

Keywords— OWA operators; consensus; distances; mathematical programming.

1 Introduction

In 1988 Yager [9] introduced OWA operators as a tool for aggregating numerical values in multi-criteria decision making. An OWA operator is similar to a weighted mean, but with the values of the variables previously ordered in a decreasing way. Thus, contrary to the weighted means, the weights are not associated with specific variables and, therefore, they are anonymous. Moreover, they satisfy other interesting properties, such as monotonicity, unanimity, continuity and compensativeness.

Initially, the weights of an OWA operator may be fixed taking into account the importance we want to give to the assessments. So, the outcome of an OWA operator may be the maximum, the minimum, the average or a median of the individual assessments, among a large number of possibilities.

It is important to note that the determination of the weights of OWA operators is a relevant issue since the origins of the theory of OWA operators. In this way, Yager [9] proposes to use linguistic quantifiers for generating the OWA weights; O'Hagan [6] generates the OWA weights by maximizing their entropy whenever a degree of orness has been fixed; Filev and Yager [3] consider an exponential smoothing approach for generating the OWA weights. After these seminal papers, a large variety of techniques have been proposed in the literature (see, for instance, Wang and Parkan [7] and Xu [8]).

When a group of individuals provides assessments on an alternative and these values are aggregated, it is relevant to know the degree of agreement or consensus among the individual assessments with respect to the aggregated value. In fact, it is desirable that the aggregation function used to obtain the collective value reflects the opinions of as many agents as possible. Using a specific OWA operator for aggregating individual assessments does not necessarily ensure such consensus for every opinion situation.

In our proposal, we do not fix the OWA weighting vector, but we generate an OWA operator for each profile of individual assessments, just one that maximizes the consensus (or equivalently, minimizes the disagreement) in the group with respect to the outcome provided by the OWA operator. More concretely, once the agents opinions are known, we first calculate the distances among individual assessments on the alternatives and the collective assessments generated by an arbitrary OWA operator. Secondly, we use an aggregation operator for obtaining a representative measure of disagreement from the individual assessments to the collective one. By solving a mathematical program, we obtain the weighting vector(s) that maximize(s) the consensus among individual and collective opinions.

The paper is organized as follows. Section 2 is devoted to introduce notation, basic notions and our proposal for generating an OWA operator for each profile of individual assessments. Section 3 contains some illustrative examples. Finally, Section 4 shows some open problems and further research.

2 A model for generating OWA weights

Consider a set of agents (experts or voters) $V = \{1, \ldots, m\}$ ($m \geq 2$) who show their opinions on a set of alternatives $A = \{a_1, \ldots, a_n\}$ ($n \geq 2$) through numbers in the interval $[0, 1]$.

A *profile* is a $m \times n$ matrix

$$P = \begin{pmatrix} a_1^1 & \cdots & a_j^1 & \cdots & a_n^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1^i & \cdots & a_j^i & \cdots & a_n^i \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1^m & \cdots & a_j^m & \cdots & a_n^m \end{pmatrix}$$

where $a_j^i \in [0, 1]$ is the assessment that agent i assigns to alternative a_j. The set of profiles is denoted by $\mathcal{P}$.

Given a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_m) \in [0, 1]^m$ such that $\sum_{i=1}^m w_i = 1$, the *OWA operator associated with* $\boldsymbol{w}$ is the mapping $F_{\boldsymbol{w}} : [0, 1]^m \longrightarrow [0, 1]$ defined by

$$F_{\boldsymbol{w}}(x_1, \ldots, x_m) = \sum_{i=1}^m w_i \cdot y_i$$

where y_i is the i-th greatest number of $\{x_1, \ldots, x_m\}$. We use the following notation:

$$\mathcal{W} = \left\{ \boldsymbol{w} \in [0, 1]^m \;\middle|\; \sum_{i=1}^m w_i = 1 \right\}.$$

Given $\boldsymbol{w} \in \mathcal{W}$, from the opinions given by the agents on an alternative a_j, $\{a_j^1, \ldots, a_j^m\}$, we generate a collective assessment on a_j:

$$v_j(\boldsymbol{w}) = F_{\boldsymbol{w}}\left(a_j^1, \ldots, a_j^m\right).$$

We denote $\boldsymbol{v}(\boldsymbol{w}) = (v_1(\boldsymbol{w}), \ldots, v_n(\boldsymbol{w}))$ the vector that contains the collective assessments on the alternatives of A generated by the OWA operator $F_{\boldsymbol{w}}$.

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By $\boldsymbol{a}^i = \left(a_1^i, \ldots, a_n^i\right)$ we denote the vector that contains the assessments of individual $i \in \{1, \ldots, m\}$ on the alternatives of A.

In our context, an *aggregation operator* is a continuous mapping $A : [0,1]^m \longrightarrow [0,1]$ that satisfies the following conditions:

1. *Monotonicity*, i.e., $A(x_1, \ldots, x_m) \leq A(y_1, \ldots, y_m)$ for all $(x_1, \ldots, x_m), (y_1, \ldots, y_m) \in [0,1]^m$ such that $x_i \leq y_i$ for every $i \in \{i, \ldots, m\}$.
2. *Unanimity*, i.e., $A(x, \ldots, x) = x$ for every $x \in [0,1]$.

It is easy to see that every aggregation operator is *compensative*, i.e.,

$$\min\{x_1, \ldots, x_m\} \leq A(x_1, \ldots, x_m) \leq \max\{x_1, \ldots, x_m\},$$

for every $(x_1, \ldots, x_m) \in [0,1]^m$.

On aggregation operators, see Fodor and Roubens [4], Grabisch, Orlovski and Yager [5], Calvo, Kolesárova, Komorníková and Mesiar [2] and Beliakov, Pradera and Calvo [1], among others.

2.1 The general model

In order to present our general model, it is necessary to fix two ingredients:

- A distance $d : [0,1]^n \times [0,1]^n \longrightarrow [0,1]$.
- An aggregation operator $A : [0,1]^m \longrightarrow [0,1]$.

Given a profile $P \in \mathcal{P}$, we propose to find weighting vector(s) $\boldsymbol{w} \in \mathcal{W}$ being solution(s) of the following mathematical program

$$\begin{array}{ll} \text{Min} & A\Big(d(\boldsymbol{a}^1, \boldsymbol{v}(\boldsymbol{w})), \ldots, d(\boldsymbol{a}^m, \boldsymbol{v}(\boldsymbol{w}))\Big) \\ \text{s. t. :} & \boldsymbol{w} \in \mathcal{W} \end{array} \tag{1}$$

Notice that from continuity of A and compactness of $\mathcal{W}$, the existence of solution(s) in (1) is always guaranteed.

Among the large variety of distances and aggregation operators that we may use in (1), we present with more detail those cases where Manhattan and Chebyshev distances are used, and the aggregation operators are the arithmetic mean and the maximum.

The Manhattan distance is defined by

$$d_1\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) = \sum_{i=1}^{n} |x_i - y_i|.$$

The Chebyshev distance is defined by

$$\begin{aligned} d_\infty\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) &= \\ = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}&. \end{aligned}$$

2.2 Using the arithmetic mean as aggregation operator

If we consider the arithmetic mean as aggregation operator, then (1) is equivalent to find the weighting vector that minimizes the sum of distances between the individual assessments and the collective assessments generated by the OWA operator associated with that weighting vector, i.e.

$$\begin{array}{ll} \text{Min} & \displaystyle\sum_{i=1}^{m} d\Big((a_1^i, \ldots, a_n^i), (v_1(\boldsymbol{w}), \ldots, v_n(\boldsymbol{w}))\Big) \\ \text{s. t. :} & \boldsymbol{w} \in \mathcal{W} \end{array} \tag{2}$$

1. If we use the Manhattan distance, then it is necessary to solve the following mathematical program:

$$\text{Min} \sum_{i=1}^{m} \Big(|a_1^i - v_1(\boldsymbol{w})| + \cdots + |a_n^i - v_n(\boldsymbol{w})|\Big)$$
$$\text{s. t. :}\ w_1 \geq 0, \ldots, w_m \geq 0,\ w_1 + \cdots + w_m = 1$$

2. If we use the Chebyshev distance, then it is necessary to solve the following mathematical program:

$$\text{Min} \sum_{i=1}^{m} \max\Big\{\, |a_1^i - v_1(\boldsymbol{w})|, \ldots, |a_n^i - v_n(\boldsymbol{w})| \,\Big\}$$
$$\text{s. t. :}\ w_1 \geq 0, \ldots, w_m \geq 0,\ w_1 + \cdots + w_m = 1$$

In the first case, i.e., when the Manhattan distance is used, it is possible to give the analytical solution of the problem.

Proposition 1 *The solution of Problem (2) for the Manhattan distance is the median operator.*

Proof

When the Manhattan distance is used in Problem (2), we obtain the following mathematical program:

$$\begin{aligned} &\min_{\boldsymbol{w} \in \mathcal{W}} \sum_{i=1}^{m} \sum_{j=1}^{n} |a_j^i - v_j(\boldsymbol{w})| = \\ &\min_{\boldsymbol{w} \in \mathcal{W}} \sum_{j=1}^{n} \sum_{i=1}^{m} |a_j^i - v_j(\boldsymbol{w})|. \end{aligned} \tag{3}$$

On the other hand, it is known that given $x_1, \ldots, x_m \in \mathbb{R}$, the median operator is the solution of the following problem:

$$\min_{x \in \mathbb{R}} \frac{1}{m} \sum_{i=1}^{m} |x_i - x|.$$

Therefore, it is also the solution of

$$\min_{\boldsymbol{w} \in \mathcal{W}} \sum_{i=1}^{m} |a_j^i - v_j(\boldsymbol{w})|.$$

for all $j \in \{1, \ldots, n\}$, and, consequently it is the solution of Problem (3). □

2.3 Using the maximum as aggregation operator

We now consider the maximum as aggregation operator. Thus, with (4) we look for the weighting vector that minimizes the maximum distance between the individual assessments and the collective assessments generated by the OWA operator associated with that weighting vector, i.e.

$$\begin{array}{ll} \text{Min} & \displaystyle\max_{i=1,\ldots,m} \Big\{d\Big((a_1^i, \ldots, a_n^i), (v_1(\boldsymbol{w}), \ldots, v_n(\boldsymbol{w}))\Big)\Big\} \\ \text{s. t. :} & \boldsymbol{w} \in \mathcal{W} \end{array} \tag{4}$$

1. If we use the Manhattan distance, then it is necessary to solve the following mathematical program:

$$\text{Min} \max_{i=1,\ldots,m} \Big\{|a_1^i - v_1(\boldsymbol{w})| + \cdots + |a_n^i - v_n(\boldsymbol{w})|\Big\}$$
$$\text{s. t. :}\ w_1 \geq 0, \ldots, w_m \geq 0,\ w_1 + \cdots + w_m = 1$$

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2. If we use the Chebyshev distance, then it is necessary to solve the following mathematical program:

$$\text{Min} \max_{i=1,\dots,m} \max \left\{ |a_1^i - v_1(\boldsymbol{w})|, \dots, |a_n^i - v_n(\boldsymbol{w})| \right\}$$

$$\text{s. t.}: \ w_1 \geq 0, \dots, w_m \geq 0, \ w_1 + \cdots + w_m = 1$$

In this last case, i.e., when the Chebyshev distance is used, the solution of the mathematical program generates the mid-range OWA operator, as we show in the following proposition.

Proposition 2 *The solution of Problem (4) for the Chebyshev distance is the weighting vector* $\boldsymbol{w}^*$ *given by*

$$w_i^* = \begin{cases} 0.5, & \textit{if } i \in \{1, m\}, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof

When the Chebyshev distance is used in Problem (4), we obtain the following mathematical program:

$$\min_{\boldsymbol{w} \in \mathcal{W}} \max_{i=1,\dots,m} \max_{j=1,\dots,n} \left\{ |a_j^i - v_j(\boldsymbol{w})| \right\} = \min_{\boldsymbol{w} \in \mathcal{W}} \max_{j=1,\dots,n} \max_{i=1,\dots,m} \left\{ |a_j^i - v_j(\boldsymbol{w})| \right\}. \quad (5)$$

On the other hand, it is obvious that for all $j \in \{1, \dots, n\}$, the weighting vector given by

$$w_i^* = \begin{cases} 0.5, & \text{if } i \in \{1, m\}, \\ 0, & \text{otherwise,} \end{cases}$$

is the solution of the following problem:

$$\min_{\boldsymbol{w} \in \mathcal{W}} \max_{i=1,\dots,m} \left\{ |a_j^i - v_j(\boldsymbol{w})| \right\}.$$

Therefore, $\boldsymbol{w}^*$ is also the solution of Problem (5). □

2.4 Restricted models

The general model (1) does not impose any restriction to the OWA weighting vectors that maximize the consensus among the individual assessments and the collective one. However, in some cases it could be interesting to consider some requirements for these weights by following a desired pattern, through a subclass $\mathcal{W}^* \subset \mathcal{W}$. For instance, we may restrict the search of the weighting vectors within one of the following cases.

1. Weighting vectors with a fixed *orness* or *attitudinal character* $\alpha \in (0, 1)$ (Yager [9]):

$$\mathcal{W}_\alpha^1 = \left\{ \boldsymbol{w} \in \mathcal{W} \;\middle|\; \frac{1}{m-1} \sum_{i=1}^{m} (m-i) w_i = \alpha \right\}.$$

2. Symmetric weights:

$$\mathcal{W}^2 = \left\{ \boldsymbol{w} \in \mathcal{W} \mid w_i = w_{m+1-i} \ \forall i \in \left\{1, \dots, \left[\frac{m}{2}\right]\right\} \right\}.$$

3. Centered OWAs (after Yager [10]):

$$\mathcal{W}^3 = \left\{ \boldsymbol{w} \in \mathcal{W}^2 \mid w_1 \leq w_2 \leq \cdots \leq w_{\left[\frac{m+1}{2}\right]} \right\}.$$

4. Trimmed OWAs:

$$\mathcal{W}_1^4 = \{ \boldsymbol{w} \in \mathcal{W} \mid w_1 = w_m = 0 \},$$
$$\mathcal{W}_2^4 = \{ \boldsymbol{w} \in \mathcal{W} \mid w_1 = w_2 = w_{m-1} = w_m = 0 \},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

It is worth noting that $\mathcal{W}^3 \subseteq \mathcal{W}^2 \subseteq \mathcal{W}_{0.5}^1 \subseteq \mathcal{W}$.

Analogously to the general model, consider a distance $d : [0,1]^n \times [0,1]^n \longrightarrow [0,1]$ and an aggregation operator $A : [0,1]^m \longrightarrow [0,1]$. Given a profile $P \in \mathcal{P}$, we now propose to find weighting vector(s) $\boldsymbol{w} \in \mathcal{W}^*$ being solution(s) of the following mathematical program

$$\begin{aligned} &\text{Min} \quad A\Big(d(\boldsymbol{a}^1, \boldsymbol{v}(\boldsymbol{w})), \dots, d(\boldsymbol{a}^m, \boldsymbol{v}(\boldsymbol{w}))\Big) \\ &\text{s. t.}: \ \boldsymbol{w} \in \mathcal{W}^* \end{aligned} \quad (6)$$

Notice that from continuity of A and compactness of $\mathcal{W}^*$ in the previous cases, the existence of solution(s) in (6) is always guaranteed.

3 Some illustrative examples

The following matrix shows the opinions of four experts on three alternatives.

$$P = \begin{pmatrix} 0.7 & 0.6 & 0.1 \\ 0 & 0.5 & 0.8 \\ 0.6 & 0.1 & 1 \\ 0.6 & 0.7 & 0 \end{pmatrix}$$

Case 1. If we use Model (2) with the Chebyshev distance, the following mathematical programming must be solved:

Min

$$\begin{aligned} &\max\{|0.7 - v_1(\boldsymbol{w})|, |0.6 - v_2(\boldsymbol{w})|, |0.1 - v_3(\boldsymbol{w})|\} + \\ &\max\{|0 - v_1(\boldsymbol{w})|, |0.5 - v_2(\boldsymbol{w})|, |0.8 - v_3(\boldsymbol{w})|\} + \\ &\max\{|0.6 - v_1(\boldsymbol{w})|, |0.1 - v_2(\boldsymbol{w})|, |1 - v_3(\boldsymbol{w})|\} + \\ &\max\{|0.6 - v_1(\boldsymbol{w})|, |0.7 - v_2(\boldsymbol{w})|, |0 - v_3(\boldsymbol{w})|\} \end{aligned}$$

$$\begin{aligned} \text{s. t.}: \quad & w_1 \geq 0, \ w_2 \geq 0, \ w_3 \geq 0, \ w_4 \geq 0, \\ & w_1 + w_2 + w_3 + w_4 = 1, \end{aligned}$$

where

$$\begin{aligned} v_1(\boldsymbol{w}) &= 0.7w_1 + 0.6w_2 + 0.6w_3, \\ v_2(\boldsymbol{w}) &= 0.7w_1 + 0.6w_2 + 0.5w_3 + 0.1w_4, \\ v_3(\boldsymbol{w}) &= 1w_1 + 0.8w_2 + 0.1w_3. \end{aligned}$$

The previous problem is a non-smooth optimization problem. However, it can be easily replaced by an equivalent smooth linear problem by using some auxiliary variables. We have used LINGO software to solve this linear problem. The solution obtained is showed in Table 1. Likewise, that table summarizes the solutions obtained when some restrictions are imposed to the OWA weighting vectors in the problem.

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Table 1: Solutions for Case 1.

	w_1	w_2	w_3	w_4
$\mathcal{W}$	0.4706	0	0	0.5294
$\mathcal{W}^1_{0.75}$	0.75	0	0	0.25
$\mathcal{W}^1_{0.5}$	0.5	0	0	0.5
$\mathcal{W}^1_{0.25}$	0	0.375	0	0.625
$\mathcal{W}^2$	0.5	0	0	0.5
$\mathcal{W}^3$	0.25	0.25	0.25	0.25
$\mathcal{W}^4_1$	0	0.1429	0.8571	0

Case 2. If we consider Model (4) with the Manhattan distance, the mathematical programming to solve is:

$$\begin{aligned} \text{Min} \quad \max \Big\{ & \\ & |0.7 - v_1(\boldsymbol{w})| + |0.6 - v_2(\boldsymbol{w})| + |0.1 - v_3(\boldsymbol{w})|, \\ & |0 - v_1(\boldsymbol{w})| + |0.5 - v_2(\boldsymbol{w})| + |0.8 - v_3(\boldsymbol{w})|, \\ & |0.6 - v_1(\boldsymbol{w})| + |0.1 - v_2(\boldsymbol{w})| + |1 - v_3(\boldsymbol{w})|, \\ & |0.6 - v_1(\boldsymbol{w})| + |0.7 - v_2(\boldsymbol{w})| + |0 - v_3(\boldsymbol{w})| \Big\} \\ \text{s. t. :} \quad & w_1 \geq 0,\, w_2 \geq 0,\, w_3 \geq 0,\, w_4 \geq 0, \\ & w_1 + w_2 + w_3 + w_4 = 1, \end{aligned}$$

where

$$\begin{aligned} v_1(\boldsymbol{w}) &= 0.7w_1 + 0.6w_2 + 0.6w_3, \\ v_2(\boldsymbol{w}) &= 0.7w_1 + 0.6w_2 + 0.5w_3 + 0.1w_4, \\ v_3(\boldsymbol{w}) &= 1w_1 + 0.8w_2 + 0.1w_3. \end{aligned}$$

As in Case 1, this problem can be also substituted by an equivalent smooth linear problem by introducing some auxiliar variables. Table 2 shows the solution obtained by using LINGO software for different requirements on the weights.

Table 2: Solutions for Case 2.

	w_1	w_2	w_3	w_4
$\mathcal{W}$	0.667	0	0.222	0.111
$\mathcal{W}^1_{0.75}$	0.673	0	0.2307	0.0963
$\mathcal{W}^1_{0.5}$	0	0.722	0.056	0.222
$\mathcal{W}^1_{0.25}$	0	0.375	0	0.625
$\mathcal{W}^2$	0	0.5	0.5	0
$\mathcal{W}^3$	0	0.5	0.5	0
$\mathcal{W}^4_1$	0	0.834	0.166	0

4 Further research

It is worth noting that the solutions obtained in the above described models might not be unique. Even more, depending the weighting vector we choose, the outcomes provided by the corresponding OWA operator could be different. In this way, it would be necessary to provide an appropriate procedure for choosing a single weighting vector among the set of multiple solutions. This open problem constitutes part of our further research.

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