Weighted Trapezoidal Approximations of Fuzzy Numbers

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Abstract— *Fuzzy number approximation by trapezoidal fuzzy numbers which preserves the expected interval is discussed. New algorithms for calculating the proper trapezoidal approximation of fuzzy numbers with respect to the distance based on bi-symmetrical weighted functions are proposed. It is shown that the adequate approximation operator is chosen with respect both to the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of a fuzzy number.*

Keywords— fuzzy number, expected interval, trapezoidal approximation, width, weighted distance.

1 Introduction

Trapezoidal approximation of fuzzy numbers was considered by many authors (see, e.g. [1, 2, 3, 5, 11, 12, 9, 10, 15]). In [11] a list of criteria which trapezoidal approximation operators should possess was formulated. Then [18] considered the trapezoidal approximation under weighted distance. Some gaps in that paper were removed in [16] and [17] where so-called extended trapezoidal fuzzy numbers were introduced and applied.

The most common weighing applied to fuzzy numbers is a linearly increasing one (see, e.g. [5, 18]). However, it seems that a slightly modified weighing would be more natural and convenient in many situations. Such weighted functions, called regular bi-symmetrical weighted functions, were introduced and the nearest trapezoidal approximation operator preserving the expected interval, with respect to the distance based on such weighted functions, was suggested.

The paper is organized as follows. Firstly we recall some basic notions related to fuzzy numbers and present a few principle ideas connected with trapezoidal approximation. Then in Sec. 3 we introduce the notion of bi-symmetrical weighted functions. Finally, in Sec 4, we show trapezoidal approximation operators based on the weighted distance utilizing bi-symmetrical weighted functions and discuss some properties of the above mentioned operators.

2 Basic concepts

Let A denote a fuzzy number, i.e. such fuzzy subset A of the real line $\mathbb{R}$ with membership function $\mu_A : \mathbb{R} \to [0,1]$ which is (see [6]): normal (i.e. there exist an element x_0 such that $\mu_A(x_0) = 1$), fuzzy convex (i.e. $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$, $\forall x_1, x_2 \in \mathbb{R}$, $\forall \lambda \in [0,1]$), μ_A is upper semicontinuous, supp$A$ is bounded, where supp$A = cl(\{x \in \mathbb{R} : \mu_A(x) > 0\})$, and cl is the closure operator. A space of all fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$.

Moreover, let $A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$, $\alpha \in (0,1]$, denote an α-cut of a fuzzy number A. As it is known, every α-cut of a fuzzy number is a closed interval, i.e. $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where $A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$.

The expected interval $EI(A)$ of a fuzzy number A is given by (see [7, 13])

$$\begin{aligned} EI(A) &= [EI_L(A), EI_U(A)] \\ &= \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right]. \end{aligned} \tag{1}$$

The middle point of the expected interval given by

$$EV(A) = \frac{1}{2}\left(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha\right) \tag{2}$$

is called the *expected value* of a fuzzy number and it represents the typical value of the fuzzy number A (see [7, 13]). Sometimes its generalization, called *weighted expected value*, might be interesting. It is defined as

$$EV_q(A) = (1-q)\int_0^1 A_L(\alpha)d\alpha + q\int_0^1 A_U(\alpha)d\alpha, \tag{3}$$

where $q \in [0,1]$ (see [8]).

Another useful parameter characterizing a fuzzy number is called the *width* of a fuzzy number (see [4]) and is defined by

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x)dx = \int_0^1 (A_U(\alpha) - A_L(\alpha))d\alpha. \tag{4}$$

Suppose that for a certain reason or just for simplicity we want to find a suitable approximation of a fuzzy number under study. It seems that sufficiently effective simplification of the fuzzy number shape can be reached by the piecewise linear curves leading to triangle, trapezoidal or orthogonal membership functions. Since these three mentioned shapes are particular cases of the trapezoidal one, further on we will consider just the trapezoidal approximation of fuzzy numbers. It means that we want to substitute given fuzzy number A by the trapezoidal fuzzy number $T(A)$, i.e. by a fuzzy number with a following membership function

$$\mu_{T(A)}(x) = \begin{cases} 0 & \text{if } x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if } t_1 \leq x < t_2, \\ 1 & \text{if } t_2 \leq x \leq t_3, \\ \frac{t_4-x}{t_4-t_3} & \text{if } t_3 < x \leq t_4, \\ 0 & \text{if } t_4 < x. \end{cases} \tag{5}$$

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Since we can do this in many ways we need some additional constraints which guarantee that our approximation would be reasonable. One of the most natural idea is to construct $T(A)$ that is the closest to the original fuzzy number A with respect to given distance d. Sometimes we add additional requirements which warrant that our approximation would possess some desired properties, like preservation fixed parameters or relations, continuity, etc. This problem was considered by many authors (see, e.g. [1, 2, 3, 5, 11, 12, 9, 10, 15, 16]). For example, it was suggested in [11] to consider *the nearest trapezoidal approximation operator preserving the expected interval*, i.e. the approximation operator T which produces a trapezoidal fuzzy number $T(A)$ that is the closest with respect to distance

$$d(A,T(A)) = \left(\int_0^1 [A_L(\alpha) - T(A)_L(\alpha)]^2 d\alpha + \int_0^1 [A_U(\alpha) - T(A)_U(\alpha)]^2 d\alpha\right)^{1/2} \quad (6)$$

to given original fuzzy number A among all trapezoidal fuzzy numbers having identical expected interval as the original one, i.e. satisfying a following condition

$$EI(T(A)) = EI(A). \quad (7)$$

It is worth noting the invariance of the expected interval assures many other properties (for more details we refer the reader to [11] where the broad list of desired requirements that the approximation operator should possess is also given). The research on this operator was continued in [3, 12, 9, 10].

Whatever trapezoidal approximation is considered the goal reduces to finding such real numbers $t_1 \le t_2 \le t_3 \le t_4$ that characterize $T(A) = T(t_1, t_2, t_3, t_4)$. It is so because any trapezoidal fuzzy number is completely described by four real numbers that are borders of its support and core. Let us mention that mathematical formulae for these points corresponding to the nearest trapezoidal approximation operator preserving the expected interval are given in [9].

In some applications other distances than (6) are more suitable. It is easily seen that all α-cuts in (6) are treated evenly. This feature is sometimes criticized by authors who claim that elements belonging to α_1-cut should be treated with the higher attention that those from α_2-cut if $\alpha_1 > \alpha_2$ because the membership degree for the first group is higher and so they are less uncertain. Such reasoning in trapezoidal approximation can be found in [18] devoted to weighted trapezoidal approximation. More precisely, the authors consider there a trapezoidal approximation with respect to the weighted distance

$$d_{ZL}(A,T(A)) = \left(\int_0^1 \alpha[A_L(\alpha) - T(A)_L(\alpha)]^2 d\alpha + \int_0^1 \alpha[A_U(\alpha) - T(A)_U(\alpha)]^2 d\alpha\right)^{1/2} \quad (8)$$

with increasing weighting function.

3 A bi-symmetrical weighted distance

Although such increasing weighting might be useful in some occasions, another weighted distances would be more interesting in general. This is a straightforward conclusion from the fact that the least informative α-cut is not zero but 0.5. Actually, situation $\mu_A(x) = 1$ leads to perfect information that x surely belongs to A. If $\mu_A(x)$ is close to 1 we'll say that x rather belongs to A. And conversely, $\mu_A(x) = 0$ shows that x surely does not belong to A (and belongs to $\neg A$) which is also a perfect information. Similarly, x such that $\mu_A(x)$ is close to 0 is interpreted as a point that rather does not belong to A. However, if $\mu_A(x) = 0.5$ we dot know how to classify x because it belong to A and to it's completion $\neg A$ with the same degree. The same happens if $\mu_A(x)$ is close to 0.5. Thus, to sum up, degrees of membership both high (close to 1) and low (close to 0) are much more informative than those close to 0.5. Hence, if we try to incorporate this quite obvious remark into practice we have to consider a so-called bi-symmetrical weighted distance suggested below. Before defining this distance we will introduce the notion of the bi-symmetrical weighted function.

Definition 1
A function $\lambda : [0,1] \to [0,1]$ symmetrical around $\frac{1}{2}$, i.e. $\lambda(\frac{1}{2} - \alpha) = \lambda(\frac{1}{2} + \alpha)$ for all $\alpha \in [0, \frac{1}{2}]$, which reaches its minimum in $\frac{1}{2}$, is called the bi-symmetrical weighted function. Moreover, the bi-symmetrical weighted function is called regular if

(a) $\lambda(\frac{1}{2}) = 0,$

(b) $\lambda(0) = \lambda(1) = 1,$

(c) $\int_0^1 \lambda(\alpha) d\alpha = \frac{1}{2}.$

Definition 2
For two arbitrary fuzzy numbers A and B with α-cuts $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, respectively, the quantity

$$d_\lambda(A,T(A)) = \left(\int_0^1 \lambda(\alpha)[A_L(\alpha) - B_L(\alpha)]^2 d\alpha + \int_0^1 \lambda(\alpha)[A_U(\alpha) - B_U(\alpha)]^2 d\alpha\right)^{1/2} \quad (9)$$

where $\lambda : [0,1] \to [0,1]$ is a bi-symmetrical (regular) weighted function is called the bi-symmetrical (regular) weighted distance between A and B based on λ.

One can, of course, propose many regular bi-symmetrical weighted functions and hence obtain different bi-symmetrical weighted distances. Further on we will consider mainly a following function

$$\lambda(\alpha) = \begin{cases} 1 - 2\alpha & \text{if } \alpha \in [0, \frac{1}{2}], \\ 2\alpha - 1 & \text{if } \alpha \in [\frac{1}{2}, 1], \end{cases} \quad (10)$$

which is in some sense a bi-symmetrical counterpart of the increasing weighted function such as applied in [18].

4 Trapezoidal approximations based on bi-symmetrical weighted distances

Let us go back to the trapezoidal approximation operators $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which produce a trapezoidal fuzzy number $T(A)$ that is the closest to given original fuzzy number

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A among all trapezoidal fuzzy numbers having identical expected interval as the original one, i.e. satisfying (7). However, now we will look for the operators which minimize the bi-symmetrical weighted distance based on bi-symmetrical function (10).

It is easily seen that the α-cut of $T(A)$ is equal to $[t_1 + (t_2 - t_1)\,\alpha, t_4 - (t_4 - t_3)\alpha]$. Since a trapezoidal fuzzy number is completely described by four real numbers that are borders of its support and core, our goal reduces to finding such real numbers $t_1 \le t_2 \le t_3 \le t_4$ that characterize $T(A) = T(t_1, t_2, t_3, t_4)$. Substituting it into (9) and (7) our problem might be expressed as follows: find t_1, t_2, t_3, t_4 which minimize

$$d(A, T(A)) = \tag{11}$$
$$= \left(\int_0^1 \lambda(\alpha)\,[A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)]^2\, d\alpha + \int_0^1 \lambda(\alpha)\,[A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)]^2\, d\alpha \right)^{1/2}$$

with respect to conditions

$$\frac{t_1 + t_2}{2} = \int_0^1 A_L(\alpha) d\alpha, \tag{12}$$
$$\frac{t_3 + t_4}{2} = \int_0^1 A_U(\alpha) d\alpha \tag{13}$$
$$t_1 \le t_2 \le t_3 \le t_4. \tag{14}$$

Before showing final results let us introduce some notation. Firstly let us notice that by Def. 1 the centroid of a bi-symmetrical weighted function λ is $\frac{1}{2}$. Therefore, the dispersion of the bi-symmetrical weighted function λ is given by

$$\eta = \int_0^1 (\alpha - \frac{1}{2})^2 \lambda(\alpha)\ d\alpha. \tag{15}$$

Easy computation shows that the dispersion of our bi-symmetrical weighted function (10) is $\frac{1}{16}$.

We also introduce another two parameters characterizing the dispersion of the left side and of the right side of given fuzzy number A with respect to considered bi-symmetrical weighted function λ.

Definition 3
The left (lower) spread of a fuzzy number A with respect to considered bi-symmetrical weighted function λ is a number $LSP_\lambda(A)$ given by

$$LSP_\lambda(A) = \frac{1}{2\eta} \int_0^1 (\alpha - \frac{1}{2})\lambda(\alpha) A_L(\alpha)\ d\alpha, \tag{16}$$

while the right (upper) spread of a fuzzy number A with respect to considered bi-symmetrical weighted function λ is a following number $USP_\lambda(A)$

$$USP_\lambda(A) = \frac{1}{2\eta} \int_0^1 (\frac{1}{2} - \alpha)\lambda(\alpha) A_U(\alpha)\ d\alpha. \tag{17}$$

It can be shown that $LSP_\lambda(A) \ge 0$ and $USP_\lambda(A) \ge 0$.

Let us also denote the total spread of given fuzzy number A with respect to considered bi-symmetrical weighted function λ by $TSP_\lambda(A)$, i.e.

$$TSP_\lambda(A) = LSP_\lambda(A) + USP_\lambda(A), \tag{18}$$

while the difference between the right and the left spread of a fuzzy number will be denoted by $\Delta SP_\lambda(A)$, i.e.

$$\Delta SP_\lambda(A) = USP_\lambda(A) - LSP_\lambda(A). \tag{19}$$

It is easily seen that as $TSP_\lambda(A)$ is always nonnegative, $\Delta SP_\lambda(A)$ might be positive or negative as A is more asymmetrical to the right or to the left.

We can now formulate our main result.

Theorem 4
The nearest trapezoidal approximation operator preserving expected interval with respect to distance (11) for bi-symmetrical weighted function λ given by (10) is such operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$, which assigns a following trapezoidal fuzzy number $T(A) = T(t_1, t_2, t_3, t_4)$ for any fuzzy number A with α-cuts $[A_L(\alpha), A_U(\alpha)]$

(a) *if* $w(A) \ge TSP_\lambda(A)$ *then*

$$t_1 = EI_L(A) - LSP_\lambda(A), \tag{20}$$
$$t_2 = EI_L(A) + LSP_\lambda(A), \tag{21}$$
$$t_3 = EI_U(A) - USP_\lambda(A), \tag{22}$$
$$t_4 = EI_U(A) + USP_\lambda(A); \tag{23}$$

(b) *if* $|\Delta SP_\lambda(A)| \le w(A) < TSP_\lambda(A)$ *then*

$$t_1 = EI_L(A) - \frac{1}{2}w(A) + \frac{1}{2}\Delta SP_\lambda(A), \tag{24}$$
$$t_2 = t_3 = EV(A) - \frac{1}{2}\Delta SP_\lambda(A), \tag{25}$$
$$t_4 = EI_U(A) + \frac{1}{2}w(A) + \frac{1}{2}\Delta SP_\lambda(A); \tag{26}$$

(c) *if* $w(A) < \Delta SP_\lambda(A)$ *then*

$$t_1 = t_2 = t_3 = EI_L(A), \tag{27}$$
$$t_4 = 2EI_U(A) - EI_L(A); \tag{28}$$

(d) *if* $w(A) < -\Delta SP_\lambda(A)$ *then*

$$t_1 = 2EI_L(A) - EI_U(A), \tag{29}$$
$$t_2 = t_3 = t_4 = EI_U(A). \tag{30}$$

The prove this result we apply the Karush-Kuhn-Tucker theorem for the local minimizer under given constraints.

Actually, we have received four different operators providing the nearest trapezoidal fuzzy number that preserves the expected value of the original fuzzy number, where T_1 leads to trapezoidal fuzzy number, T_2 stands for the operator that leads to triangular fuzzy number with two sides, while T_3 and T_4 produce triangular fuzzy numbers with the right side only or with the left side only, respectively.

Roughly speaking, we approximate a fuzzy number A by the trapezoidal approximation operator T_1 provided the total dispersion of given fuzzy number with respect to considered bi-symmetrical weighted function, measured by the sum of the lower and upper spread, is large enough. Otherwise, we will approximate A by a triangular number. It means that for less dispersed fuzzy numbers the solution is always a triangular fuzzy number.

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However, here we have also three possible situations: to approximate a fuzzy number A we apply operator T_2 provided the asymmetry of A is not too big (i.e.there is no big difference between the lower and upper spread). If A reveals high right asymmetry (i.e. the right spread is significantly larger than the lower spread) it would be approximated by a triangular fuzzy number with the right side only, produced by operator T_3. Otherwise, a fuzzy number with high left asymmetry would be approximated by a triangular fuzzy number with the left side only, produced by operator T_4.

It is worth noticing that we have obtained four possible solutions like in the problem with non-weighted distance considered in [3] or [9]. Moreover, operators T_3 and T_4 are identical as in [3] or [9]. It means that for very asymmetrical fuzzy numbers its nearest trapezoidal approximation preserving the expected interval remains independent whether we use weighted or non-weighted distance.

Our operators possess many desired properties, like invariance to translations and scale, monotonicity, continuity, etc. Some of those properties are guaranteed by the expected interval invariance. For the detailed list of requirements related to trapezoidal approximations we refer the reader to [11].

One may ask what happen if we use a different bi-symmetrical weighted function. Thus now, contrary to previous continuous weighted function (10) let us consider a non-continuous one

$$\lambda_1(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1], \\ 0 & \text{if} \quad \alpha \in (\frac{1}{4}, \frac{3}{4}), \end{cases} \tag{31}$$

which appreciates only elements with high or low degree of membership and does not take into account the other. Thus λ_1 corresponds to Pedrycz's viewpoint expressed in his shadowed sets (see [14]) where we consider only these points which rather belong to a set under study or those that rather do not belong to it. The other elements with intermediate membership degree form the so-called shadow. Easy computation shows that the dispersion of our bi-symmetrical weighted function (31) is $\frac{7}{96}$. It is interesting and worth stressing that the final solution both for λ_1 is identical with that obtained before, i.e. a following theorem holds.

Theorem 5
The nearest trapezoidal approximation operator preserving expected interval with respect to distance (11) for bi-symmetrical weighted function λ given by (31) is such operator $T : \mathbb{F}(\mathbb{R}) \rightarrow \mathbb{F}^T(\mathbb{R})$, which assigns a following trapezoidal fuzzy number $T(A) = T(t_1, t_2, t_3, t_4)$ for any fuzzy number A with α-cuts $[A_L(\alpha), A_U(\alpha)]$, has the same form as for for bi-symmetrical weighted function (10), i.e. given in Theorem 4.

5 Conclusion

In the present contribution we have considered the problem of trapezoidal approximation of fuzzy numbers for bi-symmetrical weighted functions. It was shown that the choice of adequate approximation operator depends mainly on the global spread of a fuzzy number and possible asymmetry between the spread of the left-hand and right-hand part of the original fuzzy number. It is interesting that identical operators for two different bi-symmetrical weighted functions were obtained. Thus one may ask whether the solution remains valid for other weighted function too. This would be the topic of our further research.

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