Nonlinear-shaped Approximation of Fuzzy Numbers

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Abstract— *In this paper, we present a general framework for the nearest approximation of fuzzy numbers simultaneously valid for the LR or the LU representations. We suggest two families of flexible monotonic functions characterized by four or five parameters that allow multiple approximation criteria and satisfy various requirements such as preservation of value, ambiguity, support, core, level set average, etc.*

Keywords— Trapezoidal Fuzzy Numbers, Approximation of Fuzzy Numbers, LR fuzzy numbers, Expected Interval, Shape Preserving Approximation

1 Introduction

The approximation of fuzzy numbers by linear or other simple shapes (e.g. polynomial, Gaussian, logistic) is an important problem in many fields and applications where extended or massive fuzzy computations are required. The recent literature has addressed both the linear case (see [3], [9], [10], [11], [12], [13], [18] and the references therein) and the reconstruction by nonlinear functions (e.g. [1], [2], [4], [15]).

Clearly, there are at least three key points to define:

(a) the basic general model to represent fuzzy numbers such as the LR (see [6], [7]) or the LU (see [8], [16], [17]) representations,

(b) the families of parameterized monotonic functions to be used which are sufficiently flexible to cover a large set of possible curves of membership functions,

(c) the criteria for the approximation, depending usually on the application at hand, such as least squares or other distance minimization, support or core or expected interval preservation, nearest ambiguity or value approximation among others (see [12]).

2 Basic representations and tools

In this paper, we present a general framework for the nearest approximation of fuzzy numbers valid simultaneously for the LR or the LU representations. To reach this goal we'll use special families of flexible monotonic curves described by sufficiently large number of parameters to allow multiple approximation criteria and satisfy various requirements.

Let us consider the following two families of nonlinear monotonic functions to use as shape generators ([16]).

1. the (2,2)-rational standardized monotonic spline:

$$p_{R2}(t;\beta_0,\beta_1)=\frac{t^2+\beta_0 t(1-t)}{1+(\beta_0+\beta_1-2)t(1-t)}, \qquad (1)$$

where $\beta_0,\beta_1\geq 0$ and $t\in[0,1]$;

2. the mixed cubic-exponential spline

$$\begin{aligned} p_{MS}(t;\beta_0,\beta_1) &= \frac{1}{a}\left[t^2(3-2t)+\beta_0\right. \\ &\quad \left.-\beta_0(1-t)^a+\beta_1 t^a\right], \end{aligned} \qquad (2)$$

where $a=1+\beta_0+\beta_1$, $\beta_0,\beta_1\geq 0$ and $t\in[0,1]$.

It is easy to verify that for arbitrary nonnegative values of parameters β_0,β_1 the two families of functions are monotonic for all $t\in[0,1]$ and satisfy following conditions (the derivatives p' are made with respect to the first variable t):

$$p(0;\beta_0,\beta_1)=0, \qquad p(1;\beta_0,\beta_1)=1,$$
$$p\prime(0;\beta_0,\beta_1)=\beta_0, \qquad p\prime(1;\beta_0,\beta_1)=\beta_1.$$

Remark 1 All the proposed shape functions are monotonic over $[0,1]$. Note that it is not true for standard splines or other polynomials of degree greater than two.

Remark 2 Some special cases are of interest, i.e.

- if $\beta_0=\beta_1=1$ both p_{R2} and p_{MS} are linear;
- if $\beta_0+\beta_1=2$ both p_{R2} and p_{MS} are quadratic;
- if $\beta_0+\beta_1$ is integer (or zero), then p_{MS} is a polynomial function (increasing for all $t\in[0,1]$).

The properties given by following lemma are important.

Lemma 1 *For both p_{R2} and p_{MS} we have*

(i) $0<\int_0^1 p(t;\beta_0,\beta_1)dt<1 \quad \forall\beta_0,\beta_1\geq 0$

(ii) $\lim_{\beta_1\longrightarrow+\infty}\int_0^1 p(t;0,\beta_1)dt=0$

(iii) $\lim_{\beta_0\longrightarrow+\infty}\int_0^1 p(t;\beta_0,0)dt=1.$

By Lemma 1 both p_{R2} and p_{MS} are able to "cover" the whole box $[0,1]\times[0,1]$ giving an infinite number of shape functions.

Some particular cases of these functions are of interest to meet specific requirements. For example, $p_{R2}(\frac{1}{2};\beta_0,\beta_1)=\frac{1}{2}$ if and only if $\beta_0=\beta_1=\beta$ and then $\int_0^1 p_{R2}(t;\beta,\beta)dt=\frac{1}{2}$ for all $\beta\geq 0$.

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A family of curves obtained from $p_{R2}(t;\beta_0,\beta_1)$ for different β is given in Fig. 1.

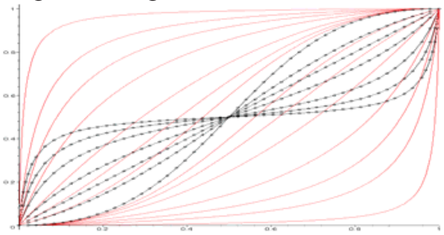

Figure 1. $p_{R2}(t;\beta_0,\beta_1)$ for different β_0,β_1; pointed lines are with $\beta_0=\beta_1$.

We can also use other single-parameter flexible forms and their combinations, like $p_{R2,1}(t,a)=p_{R2}(t;a,0)$, i.e.

$$p_{R2,1}(t,a)=\frac{t^2-at^2+at}{1+at-at^2-2t+2t^2}, \qquad (3)$$

where $t\in[0,1]$, $a\geq 0$ or $p_{R2,2}(t,b)=p_{R2}(t;0,b)$, i.e.

$$p_{R2,2}(t,b)=\frac{t^2}{1+bt-bt^2-2t+2t^2}, \qquad (4)$$

where $t\in[0,1]$, $b\geq 0$. Possible shapes obtained for different values of parameters a and b are shown in Fig. 2 and Fig. 3, respectively.

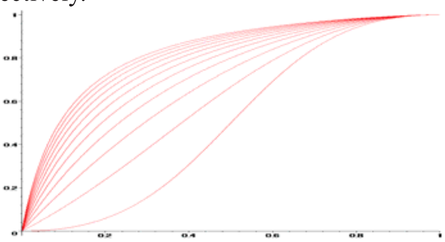

Figure 2. $p_{R2,1}(t;a)$ for different $a\geq 0$

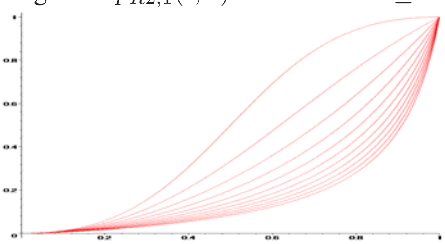

Figure 3. $p_{R2,2}(t;b)$ for different $b\geq 0$

Note that $p_{R2,1}(t;a)+p_{R2,2}(1-t;a)=1$ for all $t\in[0,1]$.

We can also combine the two using (here $\lambda\in[0,1]$)

$$P_{R2,\lambda}(t;a,b)=(1-\lambda)p_{R2,1}(t;a)+\lambda p_{R2,2}(t;b) \qquad (5)$$

and have three parameters available.

Analogous constructions can be obtained by using p_{MS}, like $p_{MS,1}(t,a)=p_{MS}(t;a,0)$, i.e.

$$p_{MS,1}(t,a)=\frac{1}{1+a}[t^2(3-2t)+a-a(1-t)^{1+a}], \qquad (6)$$

where $t\in[0,1]$, $a\geq 0$ or $p_{MS,2}(t,b)=p_{MS}(t;0,b)$, i.e.

$$p_{MS,2}(t,b)=\frac{1}{1+b}[t^2(3-2t)+bt^{1+b}], \qquad (7)$$

where $t\in[0,1]$, $b\geq 0$, and combine the two with $\lambda\in[0,1]$:

$$P_{MS,\lambda}(t;a,b)=(1-\lambda)p_{MS,1}(t;a)+\lambda p_{MS,2}(t;b). \qquad (8)$$

Examples of curves $P_{MS,0.5}(t;a,b)$ are in Fig. 4.

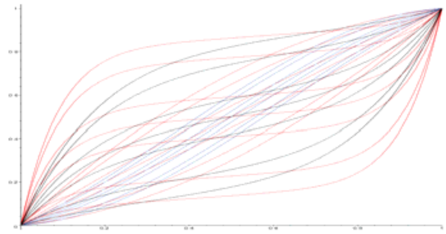

Figure 4. $P_{MS,0.5}(t;a,b)$ for different $a,b\geq 0$

The parametric functions $p_{R2}(.;\beta_0,\beta_1)$ and $p_{MS}(.;\beta_0,\beta_1)$ (or others with similar properties) can be used to construct a fuzzy number u in the LR form, by representing its membership function as

$$\mu_u(x)=\begin{cases} p(\frac{x-a}{b-a};\beta_{0,L},\beta_{1,L}) & \text{if } x\in[a,b] \\ 1 & \text{if } x\in[b,c] \\ p(\frac{d-x}{d-c};\beta_{0,R},\beta_{1,R}) & \text{if } x\in[c,d] \\ 0 & \text{otherwise} \end{cases} \qquad (9)$$

or in the LU form, by representing its alpha-cuts as the compact intervals $[u_\alpha^-,u_\alpha^+]$, for $\alpha\in[0,1]$, with

$$\begin{aligned} u_\alpha^- &= a+(b-a)p(\alpha;\beta_0^-,\beta_1^-), \\ u_\alpha^+ &= d+(c-d)p(\alpha;\beta_0^+,\beta_1^+). \end{aligned} \qquad (10)$$

Here, obviously, $a\leq b\leq c\leq d$ and $[a,d]$ is the support, while $[b,c]$ is the core of given fuzzy number u. If needed, we will denote a,b,c and d also by u_0^-,u_1^-, u_1^+ and u_0^+, respectively.

As it is possible to go from the LR to the LU representations by inverting the model functions, further on we will use the LU form only; however analogous formulations are possible for the LR form as well. The obvious relations between the derivatives of $\mu_u(x)$ and the derivatives of u_α^-, u_α^+ gives (provided they are not null)

$$\beta_{0,L}=\frac{1}{\beta_0^-},\beta_{1,L}=\frac{1}{\beta_1^-},\beta_{0,R}=\frac{1}{\beta_0^+},\beta_{1,R}=\frac{1}{\beta_1^+}. \qquad (11)$$

Let us consider the expected interval of such fuzzy number u. It is given by $[E^-u,E^+u]$, where

$$\begin{aligned} E^-u &= u_0^-+(u_1^--u_0^-)\int_0^1 p(\alpha;\beta_0^-,\beta_1^-)d\alpha, \\ E^+u &= u_0^++(u_1^+-u_0^+)\int_0^1 p(\alpha;\beta_0^+,\beta_1^+)d\alpha \end{aligned} \qquad (12)$$

and hence we have four parameters $\beta_0^-,\beta_1^-,\beta_0^+,\beta_1^+\geq 0$ free for our further purposes (or six parameters if $P_{R2,\lambda^-}(t;a^-,b^-)$ and $P_{R2,\lambda^+}(t;a^+,b^+)$ or $P_{MS,\lambda^-}(t;a^-,b^-)$ and $P_{MS,\lambda^+}(t;a^+,b^+)$ are used).

For example, if $\beta_0^-=\beta_1^-=\beta^-$ and $\beta_0^+=\beta_1^+=\beta^+$, we get $\int_0^1 p_{R2}(t;\beta^\pm,\beta^\pm)dt=\frac{1}{2}$ and the expected interval is

$$[E^-u,E^+u]=[\frac{u_0^-+u_1^-}{2},\frac{u_0^++u_1^+}{2}],$$

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for all nonnegative β^-, β^+. If a fuzzy number A with alpha cuts $[A_\alpha^-, A_\alpha^+]$ is to be approximated, we can constrain $u_0^- \leq u_1^- \leq u_1^+ \leq u_1^-$ such that

$$\begin{aligned} u_0^- + u_1^- &= 2\int_0^1 A_\alpha^- d\alpha \\ u_0^+ + u_1^+ &= 2\int_0^1 A_\alpha^+ d\alpha, \end{aligned}$$

while the two parameters $\beta^-, \beta^+ \geq 0$ still can be used for additional requirements. Thus, using our approach we obviously loose the benefits of linear shapes but the form of $p(t; \beta, \beta)$ is simple and it is easy to invert analytically by solving a quadratic equation instead of linear. It also reduces to linear if $\beta = 1$.

3 Nonlinear shape approximations

In this section, we consider the approximation problem for a fuzzy number A having membership function $\mu_A(x)$ and compact α-cuts $A_\alpha = [A_\alpha^-, A_\alpha^+]$, $\alpha \in [0, 1]$.

If we like to preserve or approximate the middle set (i.e. the alpha cut for $\alpha = 0.5$) we have

$$p_{R2}(\frac{1}{2}; \beta_0, \beta_1) = \frac{1 + \beta_0}{\beta_0 + \beta_1 + 2}$$

and we obtain the conditions

$$\begin{aligned} u_0^- + (u_1^- - u_0^-)\frac{1 + \beta_0^-}{\beta_0^- + \beta_1^- + 2} &= A_{0.5}^- \\ u_0^+ + (u_1^+ - u_0^+)\frac{1 + \beta_0^+}{\beta_0^+ + \beta_1^+ + 2} &= A_{0.5}^+ \end{aligned}$$

giving (if the $u_{0/1}^\pm$ are known) the following equations/constraints for the parameter β:

$$\begin{aligned} (u_1^- - A_{0.5}^-)\beta_0^- + (u_0^- - A_{0.5}^-)\beta_1^- &= 2A_{0.5}^- - u_0^- - u_1^- \\ (u_1^+ - A_{0.5}^+)\beta_0^+ + (u_0^+ - A_{0.5}^+)\beta_1^+ &= 2A_{0.5}^+ - u_0^+ - u_1^+. \end{aligned}$$

In the special case of $\beta_0 + \beta_1 = 2$ we obtain a parabolic shape function

$$p_{R2}(t; \beta_0, \beta_1) = Q(t; \beta) = t^2 + \beta t(1 - t)$$

where $\beta_0 = \beta, \beta_1 = 2 - \beta$ with $\beta \in [0, 2]$ and

$$\int_0^1 Q(\alpha; \beta) d\alpha = \frac{1}{3} + \frac{1}{6}\beta$$

is in the range $[\frac{1}{3}, \frac{2}{3}]$. Then the equations for expected interval preservation are as follows

$$\begin{aligned} u_0^- + (u_1^- - u_0^-)(\frac{1}{3} + \frac{1}{6}\beta^-) &= \int_0^1 A_\alpha^- d\alpha, \\ u_0^+ + (u_1^+ - u_0^+)(\frac{1}{3} + \frac{1}{6}\beta^+) &= \int_0^1 A_\alpha^+ d\alpha. \end{aligned}$$

We can also consider the value and the ambiguity of fuzzy numbers or other quantities related to ranking or defuzzification. If $s : [0, 1] \longrightarrow [0, 1]$ is a regular reducing function (i.e. $\int_0^1 s(\alpha)d\alpha = \frac{1}{2}$) the value of the fuzzy number u relative to s is defined by (see [5])

$$Val_s(u) = \int_0^1 s(\alpha)\left(u_\alpha^- + u_\alpha^+\right) d\alpha, \tag{13}$$

while the ambiguity of u relative to s is

$$Amb_s(u) = \int_0^1 s(\alpha)\left(u_\alpha^+ - u_\alpha^-\right) d\alpha. \tag{14}$$

For our parametric forms, we obtain

$$\begin{aligned} Val_s(u) = &\frac{u_0^- + u_0^+}{2} + \\ &+(u_1^- - u_0^-)\int_0^1 s(\alpha)p(\alpha; \beta_0^-, \beta_1^-)d\alpha \\ &+(u_1^+ - u_0^+)\int_0^1 s(\alpha)p(\alpha; \beta_0^+, \beta_1^+)d\alpha \end{aligned}$$

and

$$\begin{aligned} Amb_s(u) = &\frac{u_0^+ - u_0^-}{2} + \\ &+(u_0^- - u_1^-)\int_0^1 s(\alpha)p(\alpha; \beta_0^-, \beta_1^-)d\alpha \\ &+(u_1^+ - u_0^+)\int_0^1 s(\alpha)p(\alpha; \beta_0^+, \beta_1^+)d\alpha. \end{aligned}$$

The integrals above can be computed by numerical approximations (e.g. trapezoidal or Simpson formulas), but for specific cases (e.g. $s(\alpha) = 1$ or $s(\alpha) = \alpha$) we can proceed analytically.

With reference to equations (10), define the following integrals, depending on the slope parameters:

$$I(\beta_0, \beta_1) = \int_0^1 p(\alpha; \beta_0, \beta_1)d\alpha \tag{15}$$

$$J(\beta_0, \beta_1) = \int_0^1 \alpha p(\alpha; \beta_0, \beta_1)d\alpha. \tag{16}$$

For the family of parametric functions $p_{MS}(\alpha; \beta_0, \beta_1)$ we have

$$\begin{aligned} I_{MS}(\beta_0, \beta_1) &= \frac{2 + (3 + 2\beta_0)(\beta_0 + \beta_1)}{2(1 + \beta_0 + \beta_1)(2 + \beta_0 + \beta_1)} \\ J_{MS}(\beta_0, \beta_1) &= \frac{42 + (84 + 40\beta_0)(\beta_0 + \beta_1)}{20(1 + \beta_0 + \beta_1)(2 + \beta_0 + \beta_1)(3 + \beta_0 + \beta_1)} \end{aligned}$$

They are nonlinear functions of β_0, β_1 but become linear if, for example, we know or we assume a given value $m =$

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$\beta_0 + \beta_1 \geq 0$. Note that fixing the value of m (we suggest m to be integer) is equivalent to fixing the "degree" of p_{MS} as a function of α. If m is integer then p_{MS} in a polynomial function of degree one (if $\beta_0 = \beta_1 = 1$), two (if $\beta_0+\beta_1 = 2$), three (if $\beta_0 = \beta_1 = 0$) or $m + 1$ (if $m = \beta_0 + \beta_1 \geq 3$). In these cases we obtain

$$\begin{aligned} I_{MS}(\beta_0, m - \beta_0) &= \frac{2 + (3 + 2\beta_0)m}{2(1 + m)(2 + m)} \\ J_{MS}(\beta_0, m - \beta_0) &= \frac{42 + (84 + 40\beta_0)m}{20(1 + m)(2 + m)(3 + m)} \end{aligned}$$

with constraint $0 \leq \beta_0 \leq m$.

For the commonly used regular reducing functions $s(\alpha) = \frac{1}{2}$ and $s(\alpha) = \alpha$, the expressions $I_{MS}(\beta_0, \beta_1)$ and $J_{MS}(\beta_0, \beta_1)$ allow to compute $Val_s(u)$ and $Amb_s(u)$ in closed form as functions of the parameters β_0^-, β_1^- and β_0^+, β_1^+ or, for fixed m^- and m^+, of β_0^-, β_0^+.

The ordering of fuzzy numbers can be approached in many ways (see [8] or [9] for recent literature and results). In most cases, the fuzzy numbers u are transformed into real numbers by a real-valued function $D(u)$. By the use of the LU-fuzzy representation, the ranking functions can be computed, either numerically or, if possible, analytically, in terms of the parameters $u_i^{\pm}$ and $\beta_i^{\pm}$ that define u.

The first example is the so called *level set average* (see [6]), also known as the expected value of a fuzzy number, and defined by

$$u_{DP}^* = \frac{1}{2} \int_0^1 \left(u_\alpha^- + u_\alpha^+ \right) d\alpha. \quad (17)$$

Using (15) we obtain

$$\begin{aligned} u_{DP}^* &= \frac{1}{2}[u_0^- + (u_1^- - u_0^-) I_{MS}(\beta_0^-, \beta_1^-) + \\ &\quad + u_0^+ + (u_1^+ - u_0^+) I_{MS}(\beta_0^+, \beta_1^+)]. \end{aligned}$$

It is worth noting that the above integral is closely related to the *nearest interval approximation* of a fuzzy number (see [10]), expressed equivalently either in terms of the lower-upper functions or in terms of the membership function:

$$\begin{aligned} C(u) &= \left[\int_0^1 u_\alpha^- d\alpha \, , \ \int_0^1 u_\alpha^+ d\alpha \right] \quad (18) \\ &= \left[u_1^- - \int_{u_0^-}^{u_1^-} \mu(x)dx \, , \ u_1^+ + \int_{u_1^+}^{u_0^+} \mu(x)dx \right]. \end{aligned}$$

Using our parametric family p_{MS} and with reference to (10) we get

$$\begin{aligned} C(u) &= \frac{1}{2}[u_0^- + (u_1^- - u_0^-) I_{MS}(\beta_0^-, \beta_1^-), \\ &\quad u_0^+ + (u_1^+ - u_0^+) I_{MS}(\beta_0^+, \beta_1^+)]. \end{aligned}$$

Other examples are the *interval valued possibilistic mean* $M(u) = [M^-(u), M^+(u)]$ and the *level-weighted average* u_{GW}^* given by

$$M^-(u) = 2 \int_0^1 \alpha u_\alpha^- d\alpha, \quad M^+(u) = 2 \int_0^1 \alpha u_\alpha^+ d\alpha$$

and

$$u_{GW}^* = \int_0^1 \alpha \left(u_\alpha^- + u_\alpha^+ \right) d\alpha = \frac{M^-(u) + M^+(u)}{2}. \quad (19)$$

The calculations for p_{MS} family lead to

$$\begin{aligned} M^-(u) &= u_0^- + 2(u_1^- - u_0^-) J_{MS}(\beta_0^-, \beta_1^-) \\ M^+(u) &= u_0^+ - 2(u_0^+ - u_1^+) J_{MS}(\beta_0^+, \beta_1^+). \end{aligned}$$

4 Some particular approximations

We show in this section how the illustrated setting is useful to solve two specific types of approximations of a given fuzzy number A for which we know the support $[a, d]$ and the core $[b, c]$ (section 4.1) or we fix the shapes, by fixing the nonnegative parameters β_0^-, β_1^- and β_0^+, β_1^+ (section 4.2).

4.1 Approximations with known support and core

In the first case we are looking for the "best curve" under the condition that the support and the core of the approximating fuzzy number u are exactly the same of A. In other words, $u_0^- = a$, $u_0^+ = d$, $u_1^- = b$ and $u_1^+ = c$.

Now we have to estimate the shape-parameters β_0^-, β_1^- and β_0^+, β_1^+ such that a distance measure $Dist(A, u)$ is minimized, subject to the nonnegativity constraints on β_0^-, β_1^- and β_0^+, β_1^+. Here u results to be a function of the $\underline{\beta} = (\beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+)$. So our problem is

$$\begin{aligned} & Min \quad Dist(A, u(\underline{\beta})) \\ & s.t. \\ & \underline{\beta} \geq 0. \end{aligned}$$

The distance $Dist(A, u(\underline{\beta}))$ can be calculated if we have other information on A. For example, if the membership function of A is known at other points, we can approximate the distance by the least squares functional.

Suppose that $\mu_A(x_j) = \mu_j$ for given $x_j \in (a, b)$, $j = 1, 2, ..., j_L$ and for given $x_j \in (c, d)$, where $j = j_L + 1, j_L + 2, ..., j_L + j_R$ (i.e. j_L values correspond to the left arm of the fuzzy number while j_R values to the right arm). So we should minimize

$$\begin{aligned} & Dist(A, u(\underline{\beta})) = \quad (20) \\ & = \sum_{j=1}^{j_L} \left\{ x_j - \left[a + (b - a) p(\mu_j; \beta_0^-, \beta_1^-) \right] \right\}^2 \\ & + \sum_{j=j_L+1}^{j_L+j_R} \left\{ x_j - \left[d + (c - d) p(\mu_j; \beta_0^+, \beta_1^+) \right] \right\}^2. \\ & s.t. \\ & \beta_0^-, \beta_1^-, \beta_0^+, \beta_1^+ \geq 0. \end{aligned}$$

Therefore, we have obtained a nonlinear least squares problem with four variables and nonnegativity constraints which can be solved by any numerical procedure.

The minimization (20) can be split into two independent problems, one to determine β_0^-, β_1^- and the other for β_0^+, β_1^+.

A special case applies if we have available a single additional observation for the left side ($j_L = 1$), say $\mu_A(x^-) =$

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μ^- with $x^- \in (a,b)$, $\mu^- \in (0,1)$ and for the right ($j_R = 1$), say $\mu_A(x^+) = \mu^+$ with $x^+ \in (c,d)$, $\mu^+ \in (0,1)$.

An interpolating solution satisfies equations

$$\begin{aligned} a + (b-a)p(\mu^-;\beta_0^-,\beta_1^-) &= x^- \\ d + (c-d)p(\mu^+;\beta_0^+,\beta_1^+) &= x^+. \end{aligned}$$

If we use $p_{R2}(t;\beta_0,\beta_1)$ for $p(t;\beta_0,\beta_1)$, we obtain two equations and four nonnegative variables

$$(b-x^-)\beta_0^- + (a-x^-)\beta_1^- = \gamma^- \quad (21)$$
$$(c-x^+)\beta_0^+ + (d-x^+)\beta_1^+ = \gamma^+, \quad (22)$$

where

$$\gamma^- = \frac{(a-x^-)(-2\mu^{-^2} + 2\mu^- - 1) - (b-a)\mu^{-^2}}{\mu^-(1-\mu^-)}$$
$$\gamma^+ = \frac{(d-x^+)(-2\mu^{+^2} + 2\mu^+ - 1) - (c-d)\mu^{+^2}}{\mu^+(1-\mu^+)}.$$

Equations (21)-(22) represent a line in the plane (β_0,β_1) having an infinite number of nonnegative solutions (note that $b - x^- > 0$, $a - x^- < 0$ and $c - x^+ < 0$, $d - x^+ > 0$). So we have many possible choices. We suggest tree interesting solutions:

1. The unique solution having least norm of (β_0,β_1) (i.e. with minimal $\beta_0^2+\beta_1^2$), is obtained at the intersections of lines (21)-(22) with the axes. It has the following closed form:

$$\begin{aligned} \text{if } \gamma^- &= 0 \text{ then } \beta_0^- = 0, \beta_1^- = 0; \quad (23) \\ \text{if } \gamma^- &> 0 \text{ then } \beta_0^- = \frac{\gamma^-}{b-x^-}, \beta_1^- = 0; \\ \text{if } \gamma^- &< 0 \text{ then } \beta_0^- = 0, \beta_1^- = \frac{\gamma^-}{a-x^-}. \end{aligned}$$

and

$$\begin{aligned} \text{if } \gamma^+ &= 0 \text{ then } \beta_0^+ = 0, \beta_1^+ = 0; \quad (24) \\ \text{if } \gamma^+ &> 0 \text{ then } \beta_0^+ = 0, \beta_1^+ = \frac{\gamma^+}{d-x^+}; \\ \text{if } \gamma^+ &< 0 \text{ then } \beta_0^+ = \frac{\gamma^+}{c-x^+}, \beta_1^+ = 0. \end{aligned}$$

Solution (23)-(24) is illustrated in Fig. 5 with the following data: $a = 0$, $b = 2.5$, $c = 3$, $d = 5$ and the two membership functions for $\mu_A(0.5) = 0.5$, $\mu_A(3.5) = 0.9$ and $\mu_A(0.6) = 0.5$, $\mu_A(3.5) = 0.8$.

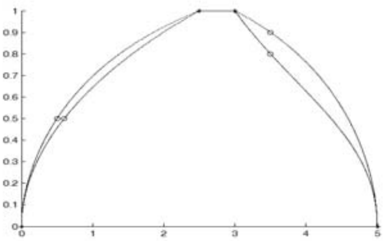

Figure 5: Solutions (23)-(24) for two examples data.

2. Recall that functions $p(t;\beta_0,\beta_1)$ represent a linear shape if and only if $\beta_0 = \beta_1 = 1$; so it is reasonable to "measure" its nonlinearity by the distance from (β_0,β_1) to $(1,1)$ and to minimize the distance $(\beta_0-1)^2 + (\beta_1-1)^2$. In this way, linear data will reproduce linear shapes. The unique solution for this criterion is obtained by the following procedure:

$$\begin{aligned} \text{if } \widehat{\beta}_0^- &\geq 0 \text{ and } \widehat{\beta}_1^- \geq 0 \text{ then } \beta_0^- = \widehat{\beta}_0^-,\ \beta_1^- = \widehat{\beta}_1^-; \quad (25) \\ \text{if } \widehat{\beta}_0^- &< 0 \text{ then } \beta_0^- = 0,\ \beta_1^- = \frac{\gamma^-}{a-x^-}; \\ \text{if } \widehat{\beta}_1^- &< 0 \text{ then } \beta_0^- = \frac{\gamma^-}{b-x^-}, \beta_1^- = 0. \end{aligned}$$

where

$$\widehat{\beta}_0^- = \frac{\gamma^-(b-x^-) + (a-x^-)^2 - (b-x^-)(a-x^-)}{(b-x^-)^2 + (a-x^-)^2}$$
$$\widehat{\beta}_1^- = 1 + \frac{(a-x^-)(\widehat{\beta}_0^- - 1)}{(b-x^-)};$$

for β_0^+, β_1^+ the procedure is analogous

$$\begin{aligned} \text{if } \widehat{\beta}_0^+ &\geq 0 \text{ and } \widehat{\beta}_1^+ \geq 0 \text{ then } \beta_0^+ = \widehat{\beta}_0^+,\ \beta_1^+ = \widehat{\beta}_1^+; \quad (26) \\ \text{if } \widehat{\beta}_0^+ &< 0 \text{ then } \beta_0^+ = 0,\ \beta_1^+ = \frac{\gamma^+}{d-x^+}; \\ \text{if } \widehat{\beta}_1^+ &< 0 \text{ then } \beta_0^+ = \frac{\gamma^+}{c-x^+}, \beta_1^+ = 0. \end{aligned}$$

where

$$\widehat{\beta}_0^+ = \frac{\gamma^+(c-x^+) + (d-x^+)^2 - (d-x^+)(c-x^+)}{(c-x^+)^2 + (d-x^+)^2}$$
$$\widehat{\beta}_1^+ = 1 + \frac{(d-x^+)(\widehat{\beta}_0^+ - 1)}{(c-x^+)}.$$

Solution (25)-(26) is illustrated in Fig. 6 with the same data as before: $a = 0$, $b = 2.5$, $c = 3$, $d = 5$, $\mu_A(0.5) = 0.5$, $\mu_A(3.5) = 0.9$ and $\mu_A(0.6) = 0.5$, $\mu_A(3.5) = 0.8$.

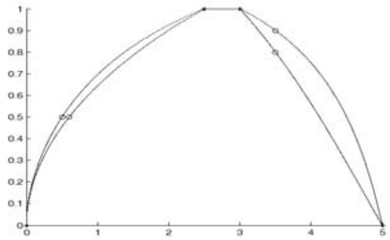

Figure 6: Solutions (25)-(26) for two examples data.

3. In cases 1. and 2. we made no assumptions on the parameters β_0^-, β_1^-, β_0^+, β_1^+, but we may know or desire to have them partially fixed. Lets consider the fitting equations in terms of the LR representation (9); if, for example, we require a differentiable membership function, this is equivalent to have $\beta_{1,L} = 0$ and $\beta_{1,R} = 0$ and we obtain the following solution

$$\beta_{0,L} = \begin{cases} \frac{\omega^-}{t^-(1-t^-)(1-\mu^-)} & \text{if } \omega^- > 0 \\ 0 & \text{if } \omega^- \leq 0, \end{cases} \quad (27)$$

where $t^- = \frac{x^- - a}{b-a}$ and $\omega^- = (1-2t^-)\mu^- - (1-2\mu^-)(t^-)^2$,

$$\beta_{0,R} = \begin{cases} \frac{\omega^+}{t^+(1-t^+)(1-\mu^+)} & \text{if } \omega^+ > 0 \\ 0 & \text{if } \omega^+ \leq 0, \end{cases} \quad (28)$$

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where $t^+ = \frac{d-x^+}{d-c}$ and $\omega^+ = (1-2t^+)\mu^+ - (1-2\mu^+)(t^+)^2$. Solution (27)-(28) is illustrated in Fig. 7 with the same data as for Fig. 5, 6. Clearly, in this case, we obtain membership functions such that $\mu(\frac{a+b}{2}) \geq \frac{1}{2}$ and $\mu(\frac{c+d}{2}) \geq \frac{1}{2}$ and interpolation is possible only for specific data.

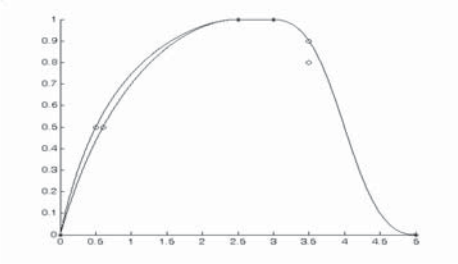

Figure 7: Solutions (27)-(28) for two examples data.

4.2 Approximations with fixed shapes

In the second case we assume that the general shape of u is fixed (by fixing the parameters $\underline{\beta}$) and we like to find the best approximation of the core and support, i.e. to find u_0^-, u_1^-, u_1^+, u_0^+ such that $d(A, u(\underline{\beta})) \to \min$.

In this case, the resulting optimization problem is easier than for Problem 1. Now, $u = u(\underline{\eta})$ is a function of $\underline{\eta} = (u_0^-, u_1^-, u_0^+, u_1^+)$ and we require $u_0^- \leq u_1^- \leq u_1^+ \leq u_0^+$.

Suppose also here that we know the membership of A at $j_L + j_R$ points, $\mu_A(x_j) = \mu_j$ for given $x_j \in [a, b]$, $j = 1, 2, ..., j_L$ and for given $x_j \in [c, d]$, $j = j_L + 1, j_L + 2, ..., j_L + j_R$.

Define $p_j = p(\mu_j; \beta_0^-, \beta_1^-)$ for $j = 1, 2, ..., j_L$ and $p_j = p(\mu_j; \beta_0^+, \beta_1^+)$ for $j = j_L + 1, j_L + 2, ..., j_L + j_R$.

Now we have to minimize

$$\begin{aligned} & Dist(A, u(\underline{\eta})) = \qquad (29) \\ = & \sum_{j=1}^{j_L} \left\{ x_j - \left[u_0^- + (u_1^- - u_0^-) p_j \right] \right\}^2 \\ & + \sum_{j=j_L+1}^{j_L+j_R} \left\{ x_j - \left[u_0^+ + (u_1^+ - u_0^+) p_j \right] \right\}^2 \\ & s.t. \\ & u_0^- - u_1^- \leq 0 \\ & u_1^- - u_1^+ \leq 0 \\ & u_1^+ - u_0^+ \leq 0. \end{aligned}$$

We obtain a linear least squares problem with four variables and three linear constraints and can be solved using standard well known procedures (see [14]).

5 Conclusions

In the present contribution we have suggested a general approach to fuzzy number approximation based on two families of parameterized functions. Our method enables a very rich set of approximating curves which are both easy to handle and useful for practical purposes.

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