Systems of Fuzzy Relation Equations in a Space with Fuzzy Preorder

Irina Perfilieva

Institute for Research and Applications of Fuzzy Modeling, University of Ostrava
Ostrava, Czech Republic
Email: Irina.Perfilieva@osu.cz

Abstract— In this paper, we consider the problem of solving systems of fuzzy relation equations in a space with fuzzy preorder. Two types of these systems with different compositions are considered. New solvability criteria are proposed for systems of both types. The new criteria are weaker than all the known ones that are based on the assumption that fuzzy sets on the left-hand side of a system establish a fuzzy partition of a respective universe.

Keywords— system of fuzzy relation equations, fuzzy preorder, criterion of solvability

1 Preliminaries

Let throughout this contribution $\mathscr{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be an integral, residuated, commutative l-monoid (*a residuated lattice*), X a non-empty set and L^X a set of L-valued functions on X. Fuzzy subsets of X are identified with L-valued functions on X (membership functions).

Let X and Y be two universes, not necessary different, $A_i \in L^X$, $B_i \in L^Y$ arbitrarily chosen fuzzy subsets of respective universes, and $R \in L^{X\times Y}$ a fuzzy subset of $X \times Y$. The latter is called a fuzzy relation. Lattice operations $\vee$ and $\wedge$ induce respective union and intersection of fuzzy sets. Two other binary operations $*, \rightarrow$ of $\mathscr{L}$ are used for compositions - binary operations on $L^{X\times Y}$. We will consider two of them: sup-*-composition that is usually denoted by $\circ$, and $\inf$-$\rightarrow$-composition that is denoted by $\lhd$. The first one has been introduced by L. Zadeh [16] and the second one by W. Bandler and L. Kohout [1]. We will demonstrate definitions of both compositions on particular examples of set-relation compositions $A \circ R$ and $A \lhd R$ where $A \in L^X$ and $R \in L^{X\times Y}$:

$$(A \circ R)(y) = \bigvee_{x\in X} (A(x) * R(x, y)),$$

$$(A \lhd R)(y) = \bigvee_{x\in X} (A(x) \rightarrow R(x, y)).$$

Remark 1
Let us remark that both compositions can be considered as set-set compositions where R is assumed to be replaced by a fuzzy set. In this reduced form they are used in instances of systems of fuzzy relation equations below.

By a *system of fuzzy relation equations with sup-*-composition* (SFRE$*$), we mean the following system of equations

$$A_i \circ R = B_i, \quad 1 \le i \le n, \tag{1}$$

that is considered with respect to unknown fuzzy relation $R \in L^{X\times Y}$. Its counterpart is a *system of fuzzy relation equations with* $\inf$-$\rightarrow$*composition* (SFRE$\rightarrow$)

$$A_i \lhd R = B_i, \quad 1 \le i \le n, \tag{2}$$

that is considered with respect to unknown $R \in L^{X\times Y}$ also. System (1) and its potential solutions are well investigated in the literature (see e.g. [3, 2, 4, 5, 12, 8, 13, 15]). On the other hand, investigation of solvability of (2) is not so intensive (see [2, 10]).

Both systems of fuzzy relation equations arise when a system of fuzzy IF-THEN rules is modeled by a fuzzy relation (below in (3) it is denoted by R), and continuity of the model [9] is requested. In order to explain this request, we recall that in relation models, a computation of an output value (B) which relates to a given input $A \in L^X$ is performed with the help of sup-*, respectively $\inf$-$\rightarrow$ composition:

$$B = A \circ R \quad \text{or} \quad B = A \lhd R. \tag{3}$$

(3) is a computational realization of the *Generalized Modus Ponens* inference scheme in fuzzy logic (in a broader sense). It is often welcome if thus constructed model is *continuous* in the sense that when (input) fuzzy sets $A', A'' \in L^X$ are close to each other (in some space) so do output fuzzy sets $A' \circ R$ and $A'' \circ R$ (respectively, $A' \lhd R$ and $A'' \lhd R$). We proved in [9] that this is possible if and only if R solves the respective system (1) or (2) of fuzzy relation equations. This fact gives additional importance to the problem of solvability of systems of fuzzy relation equations.

In general, solutions of (1) or (2) may not exist. Therefore, investigation of necessary and sufficient conditions for solvability (or at least, sufficient conditions) is needed. This problem has been widely studied in the literature, and some nice theoretical results have been obtained in the cited above papers.

If the universes of discourse X and Y are infinite then the complexity of verification of necessary and sufficient conditions is comparable with the direct checking of solvability. Therefore, the problem of discovering easy-to-check solvability conditions or criteria is still open (see [11, 14] for some results). Besides other, this paper is a contribution to this topic.

We recall basic facts concerning solvability of system (1) of fuzzy relation equations (similar conditions are known for system (2), so that we will not recall theme (see e.g. [2, 10])).

Theorem 1
(a) [13] If system (1) with respect to unknown fuzzy relation R is solvable then relation

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \rightarrow B_i(y)) \tag{4}$$

is the greatest solution to (1).

ISBN: 978-989-95079-6-8

(b) *[5] Let fuzzy sets $A_i \in L^X$ and $B_i \in L^Y$, $1 \le i \le n$, be normal. Then the fuzzy relation*

$$\check{R}(x,y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y)) \quad (5)$$

is a solution to (1) if and only if

$$\bigvee_{x\in X} (A_i(x) * A_j(x)) \le \bigwedge_{y\in Y} (B_i(y) \leftrightarrow B_j(y)) \quad (6)$$

holds for all $i, j = 1, \dots, n$.

(c) *[7] Let fuzzy sets $A_1, \dots, A_n \in L^X$ be normal and $x_1, \dots, x_n \in X$ be pairwise different elements such that for all $i = 1, \dots, n$, $A_i(x_i) = 1$. Let moreover, for all $i, j = 1, \dots, n$,*

$$\bigvee_{x\in X} (A_j(x) * A_i(x)) \le \bigwedge_{x\in X} (A_j(x) \leftrightarrow A_i(x)) \quad (7)$$

holds true. Then (1) is solvable if and only if

$$\bigvee_{x\in X} (A_j(x) * A_i(x)) \le \bigwedge_{y\in Y} (B_i(y) \leftrightarrow B_j(y)).$$

In this contribution, we will prove weaker criteria of solvability for systems (1) and (2) than those ones, consider above in cases (b) and (c) (see Section 5 with the Discussion). Moreover, an algorithm of verifying solvability that is based on each proposed here criterion has a polynomial complexity.

2 Fuzzy Preorders and Their Normal Upper Sets

This section is auxiliary with respect to the problem of solvability discussed above. In this section, we will first recall basic facts that relate to spaces with fuzzy preorder [6]. Then we will prove new results that are used in Sections below where we consider the problem of solvability.

We will fix $\mathscr{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ and a non-empty universe X. Recall that a binary fuzzy relation is a $*$-*fuzzy preorder* if it is reflexive and $*$-transitive. Fuzzy preorder $Q : X \times X \longrightarrow L$ on X can be generated by an arbitrary family of fuzzy subsets $(A_i)_{i\in I}$ of X:

$$Q(x,y) = \bigwedge_{i\in I} (A_i(x) \rightarrow A_i(y)).$$

If Q is a fuzzy preorder on X then fuzzy set $A \in L^X$ is called [6] an *upper set* of Q if

$$A(x) * Q(x,y) \le A(y), \quad x, y \in X.$$

A necessary and sufficient condition that a family of fuzzy subsets of X constitutes a family of upper sets of Q has been proved in [6]. A new result will be proved below in Theorem 2 for a family of normal fuzzy subsets of X. Let us remark that our assumptions are different from those in [6].

Theorem 2
Let $(A_i)_{i\in I} \subseteq L^X$ be a family of normal fuzzy subsets of X and $(x_i)_{i\in I} \subseteq X$ be a family of pairwise different elements such that for all $i \in I$, $A_i(x_i) = 1$. Then the following two statements are equivalent:

(i) *there exists fuzzy preorder Q on X such that for each $i \in I$, $A_i(x) = Q(x_i, x)$, $x \in X$.*

(ii) *For all $i, j \in I$,*

$$A_i(x_j) \le \bigwedge_{x\in X} (A_j(x) \rightarrow A_i(x)). \quad (8)$$

PROOF: It is easy to see that (i) $\Rightarrow$ (ii) so that we will prove the reverse implication. Assume that $(A_i)_{i\in I} \subseteq L^X$ is a family of normal fuzzy subsets of X, and (8) holds true. If $Q(x,y) = \bigwedge_{i\in I}(A_i(x) \rightarrow A_i(y))$ then by the assertion above, Q is a fuzzy preorder on X. We will show that statement (i) is valid for Q.

Let us choose and fix i, $i \in I$. For all $x \in X$, and arbitrary $j \in I$, such that $i \ne j$, we have:

$$Q(x_i, x) = \bigwedge_{j\in I} (A_j(x_i) \rightarrow A_j(x)) \le$$
$$\le A_i(x_i) \rightarrow A_i(x) = A_i(x).$$

On the other hand, by (8), $A_j(x_i) \le A_i(x) \rightarrow A_j(x)$ or, equivalently, $A_i(x) \le A_j(x_i) \rightarrow A_j(x)$. Therefore,

$$A_i(x) \le \bigwedge_{j\in I} (A_j(x_i) \rightarrow A_j(x)) = Q(x_i, x).$$

Hence, for all $x \in X$, $A_i(x) = Q(x_i, x)$. □

Corollary 1
Let $(A_i)_{i\in I} \subseteq L^X$ be a family of normal fuzzy subsets of X and $(x_i)_{i\in I} \subseteq X$ be a family of pairwise different elements such that for all $i \in I$, $A_i(x_i) = 1$. Then $Q(x,y) = \bigwedge_{i\in I}(A_i(x) \rightarrow A_i(y))$ is the coarsest fuzzy preorder on X such that (8) holds true.

The following lemma gives another necessary and sufficient condition that a family of normal fuzzy subsets of X constitutes a family of upper sets of Q. We will use that condition in our new criteria of solvability.

Lemma 1
Let A_i, $i \in I$, be a family of normal fuzzy subsets of L^X, such that $A_i(x_i) = \mathbf{1}$ for the respective $x_i \in X$, $i \in I$. Moreover, let for all $i, j \in I$, inequality (8) hold true. Then inequality (8) turns to the equality

$$A_i(x_j) = \bigwedge_{x\in X} (A_j(x) \rightarrow A_i(x)). \quad (9)$$

PROOF: Assume that for all $i, j \in I$, (8) holds true, i.e. $A_i(x_j) \le \bigwedge_{x\in X}(A_j(x) \rightarrow A_i(x))$. On the other hand,

$$\bigwedge_{x\in X} (A_j(x) \rightarrow A_i(x)) \le A_j(x_j) \rightarrow A_i(x_j) = A_i(x_j)$$

so that (9) follows.

Assume that for all $i, j \in I$, (9) holds true. Then (8) follows immediately. □

ISBN: 978-989-95079-6-8

3 SFRE$*$ in a Space with a Fuzzy Preorder and Their Solvability

Assume that $\mathscr{L}$, X, Y are as above, and we are given fuzzy sets $A_1, \ldots, A_n \in L^X$ and $B_1, \ldots, B_n \in L^Y$. In this section we will investigate the problem of solvability of system (1) and propose a new criterion that is weaker than all criteria, based on the assumption that fuzzy sets $A_1, \ldots, A_n$, establish a fuzzy partition of X, i.e. that they are classes of a respective similarity on X.

In order to simplify denotation we will choose and fix $y \in Y$ and work with the following instance of the system:

$$A_i \circ r = b_i, \quad 1 \leq i \leq n, \tag{10}$$

where $b_i = B_i(y)$, $1 \leq i \leq n$. In this particular case, system (10) is considered with respect to unknown fuzzy set $r \in L^X$.

Theorem 3
Let fuzzy sets $A_1, \ldots, A_n \in L^X$ be normal and $x_1, \ldots, x_n \in X$ be pairwise different elements such that for all $i = 1, \ldots, n$, $A_i(x_i) = 1$. Let moreover, for all $i, j = 1, \ldots, n$,

$$A_i(x_j) = \bigwedge_{x \in X} (A_j(x) \to A_i(x)) \tag{11}$$

holds true. Then system (10) is solvable if and only if

$$A_i(x_j) \leq (b_j \to b_i). \tag{12}$$

PROOF: By Lemma 1, (11) is equivalent to

$$(\forall i, j)(\forall x \in X) \quad (A_j(x) \leq (A_i(x_j) \to A_i(x))). \tag{13}$$

Assume that system (10) is solvable. Then

$$\hat{r}(x) = \bigwedge_{j=1}^{n} (A_j(x) \to b_j)$$

is a solution so that

$$(\forall j) \quad \bigvee_{x \in X} (A_j(x) * \hat{r}(x)) \geq b_j.$$

Let us fix j, $j = 1, \ldots, n$. By (13),

$$(\forall k) \quad \bigvee_{x \in X} ((A_k(x_j) \to A_k(x)) * \hat{r}(x)) \geq b_j.$$

Then

$$\bigvee_{x \in X} ((A_k(x_j) \to A_k(x)) * (A_k(x) \to b_k)) \geq b_j$$

so that

$$A_k(x_j) \to b_k \geq b_j.$$

The last inequality is equivalent to

$$A_k(x_j) \leq b_j \to b_k,$$

and by arbitrariness of k, j, it is equivalent to (12).

On the other hand, assume that (12) holds true. We will prove that $\hat{r}$ is a solution of system (10). Let us fix i, $i = 1, \ldots, n$ and prove that $\hat{r}$ solves the i-th equation of (10). Indeed,

$$\bigvee_{x \in X} (A_i(x) * \hat{r}(x)) \leq \bigvee_{x \in X} (A_i(x) * (A_i(x) \to b_i)) \leq b_i.$$

Let us prove the opposite inequality.

$$\bigvee_{x \in X} (A_i(x) * \hat{r}(x)) \geq A_i(x_i) * \hat{r}(x_i) = \bigwedge_{j=1}^{n} (A_j(x_i) \to b_j).$$

By (12), $A_j(x_i) \to b_j \geq b_i$, so that the opposite inequality easily follows:

$$\bigvee_{x \in X} (A_i(x) * \hat{r}(x)) \geq b_i.$$

Thus, $\hat{r}$ is a solution of (10), and the system is solvable. □

4 SFRE$\to$ in a Space with a Fuzzy Preorder and Their Solvability

Assume that $\mathscr{L}$, X, Y and fuzzy sets $A_1, \ldots, A_n \in L^X$ and $B_1, \ldots, B_n \in L^Y$ are as above. In this section we will investigate the problem of solvability of system (2) and propose a new criterion. As above we will work with an instance of system (2), i.e. with the following equation:

$$A_i \triangleright r = b_i, \quad 1 \leq i \leq n, \tag{14}$$

where $b_i = B_i(y)$, $1 \leq i \leq n$. System (14) will be considered with respect to an unknown fuzzy set $r \in L^X$.

Theorem 4
Let fuzzy sets $A_1, \ldots, A_n \in L^X$ be normal and $x_1, \ldots, x_n \in X$ be pairwise different elements such that for all $i = 1, \ldots, n$, $A_i(x_i) = 1$. Let moreover, for all $i, j = 1, \ldots, n$, (11) holds true. Then system (14) is solvable if and only if

$$(\forall i, j) \quad A_i(x_j) \leq (b_i \to b_j). \tag{15}$$

PROOF: By Lemma 1, (11) is equivalent to

$$(\forall i, j)(\forall x \in X) \quad A_j(x) \leq (A_i(x_j) \to A_i(x)),$$

and can be equivalently transformed to

$$(\forall i, j)(\forall x \in X) \quad b_i * A_j(x_j) \leq (A_j(x) \to b_i * A_i(x)). \tag{16}$$

Assume that system (14) is solvable. Then

$$\check{r}(x) = \bigvee_{j=1}^{n} (A_j(x) * b_j)$$

is a solution so that

$$(\forall j) \quad \bigwedge_{x \in X} (A_j(x) \to \check{r}(x)) \leq b_j,$$

and hence

$$(\forall j) \quad \bigwedge_{x \in X} (A_j(x) \to A_i(x) * b_i) \leq b_j,$$

ISBN: 978-989-95079-6-8

where $i = 1, \ldots, n$. By (16),

$$(\forall i, j) \quad b_i * A_j(x_j) \le b_j$$

that is equivalent to (15).

On the other hand, assume that (15) holds true. We will prove that $\check{r}$ is a solution of system (14). Let us fix i, $i = 1, \ldots, n$ and prove that $\check{r}$ solves the i-th equation of (14). First we observe that

$$\bigwedge_{x \in X} (A_i(x) \to \check{r}(x)) \ge \bigwedge_{x \in X} (A_i(x) \to A_i(x) * b_i) \ge b_i.$$

On the other hand,

$$\bigwedge_{x \in X} (A_i(x) \to \check{r}(x)) \le A_i(x_i) \to \check{r}(x) = \bigvee_{j=1}^{n} (A_j(x_i) * b_j).$$

By (15), for all i, j, $A_j(x_i) * b_j \le b_i$, so that

$$\bigwedge_{x \in X} (A_i(x) \to \check{r}(x)) \le b_i.$$

Thus, $\check{r}$ is a solution of (14), and the system is solvable. □

5 Discussion

In this section, we will show how the discovered criteria relate to those existed in literature. Moreover, we will justify our claim in the Abstract that the new criteria are weaker than all known ones that are based on the assumption that fuzzy sets on the left-hand side of a system establish a fuzzy partition of a respective universe. The following criterion is among those to which we have referred to as known ones (it has been recalled in Theorem 1). The formulation below is adapted to the instance (10).

Let fuzzy sets $A_1, \ldots, A_n \in L^X$ be normal and $x_1, \ldots, x_n \in X$ be pairwise different elements such that for all $i = 1, \ldots, n$, $A_i(x_i) = 1$. Let moreover, for all $i, j = 1, \ldots, n$,

$$\bigvee_{x \in X} (A_j(x) * A_i(x)) \le \bigwedge_{x \in X} (A_j(x) \leftrightarrow A_i(x)) \qquad (17)$$

holds true. Then (10) is solvable if and only if

$$\bigwedge_{x \in X} (A_j(x) \to A_i(x)) \le (b_i \leftrightarrow b_j).$$

Let us examine condition (7) and compare it with (11). For simplicity, assume that $n = 2$ and sets A_1, A_2 are ordinary (not fuzzy). It is easy to see that in this case, membership functions A_1, A_2 are characteristic functions of the respective sets and $A_1(x) * A_2(x)$ is a characteristic function of the intersection $A_1 \cap A_2$. For this particular case, (7) is fulfilled if and only if either $A_1 = A_2$ or $A_1 \cap A_2 = \emptyset$. Assume that $A_1 \subseteq A_2$ and $A_1 \neq A_2$. Then (7) fails while condition (11) is valid provided that $x_2 \notin A_1$.

Therefore, in Theorems 3, 4 we have obtained new criteria of solvability that have weaker assumptions than all those that explicitly or implicitly use the condition that fuzzy sets $A_1, \ldots, A_n \in L^X$ establish a fuzzy partition of X.

Conclusion

The problem of solvability of systems of fuzzy relation equations with two different compositions has been considered. We established new solvability criteria for systems of both types. The new criteria are based on the assumption that fuzzy sets on the left-hand side of a system are upper sets of a respective $*$-fuzzy preorder. This assumption is weaker than the assumption that fuzzy sets on the left-hand side of a system establish a fuzzy partition of a respective universe.

Acknowledgment

The paper has been supported partially by the grant IAA108270902 of GA AV ČR and partially by the project 1M0572 of the MŠMT ČR.

References

[1] W. Bandler, L. Kohout, Semantics of implication operators and fuzzy relational products, Int. J. Man-Machine Studies 12 (1980) 89–116.

[2] B. De Baets, Analytical solution methods for fuzzy relation equations, in: D. Dubois, H. Prade (eds.), The Handbooks of Fuzzy Sets Series, Vol.1, Kluwer, Dordrecht, 2000, pp. 291–340.

[3] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez, Fuzzy Relation Equations and Their Applications to Knowledge Engineering, Kluwer, Boston, 1989.

[4] S. Gottwald, On the existence of solutions of systems of fuzzy equations, Fuzzy Sets and Systems 12 (1984) 301–302.

[5] F. Klawonn, Fuzzy points, fuzzy relations and fuzzy functions, in: V. Novák, I. Perfilieva (eds.), Discovering the World with Fuzzy Logic, Springer, Berlin, 2000, pp. 431–453.

[6] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, Fuzzy Sets and Systems 157 (2006) 1865–1885.

[7] I. Perfilieva, Fuzzy function as an approximate solution to a system of fuzzy relation equations, Fuzzy Sets and Systems 147 (2004) 363–383.

[8] I. Perfilieva, S. Gottwald, Solvability and approximate solvability of fuzzy relation equations, Int. J. of General Systems 32 (2003) 361–372.

[9] I. Perfilieva, S. Lemhke, Correct models of fuzzy if–then rules are continuous, Fuzzy Sets and Systems 157 (2006) 3188–3197.

[10] I. Perfilieva, L. Nosková, System of fuzzy relation equations with inf $\to$ composition: complete set of solutions, Fuzzy Sets and Systems 159 (2008) 2256–2271.

[11] I. Perfilieva, V. Novák, System of fuzzy relation equations as a continuous model of if–then rules, Information Sciences 177 (2007) 3218–3227.

[12] I. Perfilieva, A. Tonis, Compatibility of systems of fuzzy relation equations, Int. J. of General Systems 29 (2000) 511–528.

[13] E. Sanchez, Resolution of composite fuzzy relation equations, Information and Control 30 (1976) 38–48.

[14] M. Štěpnička, B. De Baets, L. Nosková, On additive and multiplicative fuzzy models, in: M. Štěpnička, V. Novák, U. Bodenhofer (eds.), New Dimensions in Fuzzy Logic and Related Technologies. Int. Conference EUSFLAT, Ostrava, 2007, pp. 95–102.

ISBN: 978-989-95079-6-8

[15] X. Wang, Infinite fuzzy relational equations on a complete Brouwerian lattice, Fuzzy Sets and Systems 138 (2003) 657–666.

[16] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, II, III, Information Sciences 8-9 (1975) 199–257, 301–357, 43–80.

ISBN: 978-989-95079-6-8