On $\mathbb{M}$-Approximative Operators and $\mathbb{M}$-Approximative Systems

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Abstract— *The concept of an $\mathbb{M}$-approximative system is introduced. Basic properties of the category of $\mathbb{M}$-approximative systems and in a natural way defined morphisms between them are studied. It is shown that categories related to fuzzy topology as well as categories related to rough sets can be described as special subcategories of the category of $\mathbb{M}$-approximative systems.*

Keywords— Approximative operator, approximative system, *cl*-monoid, fuzzy topology, L-topology, rough set

1 Introduction and motivation

In 1968, that is only 3 years after L. Zadeh has published his famous work *"Fuzzy Sets"*, thus laying down the principles of what can be called *Mathematics of Fuzzy Sets*, his student C.L. Chang [3] introduced the concept of a fuzzy topological space thus marking the beginning of Fuzzy Topology, the conterpart of General Topology in the context of fuzzy sets. Now Fuzzy Topology is one of the most well developed fields of Mathematics of Fuzzy Sets, and there are published dozens of fundamental works on this subject.

In 1982 Z. Pawlak [14] has introduced the concept of a rough set which can be viewed as a certain alternative for the concept of a fuzzy set for the study of mathematical problems of applied nature. Pawlak's work was followed by many other publications where rough sets and mathematical structures on the basis of rough sets were introduced, studied, and applied.

Although at the first glance it may seem that the concepts of a fuzzy set, of a (fuzzy) topological space and of a rough set are of an essentially different nature and "have nothing in common", this is not the case. Probably, the first one to start studying the intermediate relations between topologies, fuzzy sets and rough sets was J. Kortelainen [11], see also [12], etc. Further a detailed analysis of different relations between fuzzy sets, rough sets and some other related concepts was done in a series of papers by Y. Yao (e.g. [20]), and other researchers.

The aim of this work is to present an alternative view on the relations between fuzzy sets, fuzzy topological spaces and rough sets and to develop a framework allowing to generalize these concepts and corresponding theories. In order to realize this aim we introduce the concept of an $\mathbb{M}$-approximative system (cf [19]) and thus come to the category $\mathbf{AS}^{\mathbb{M}}$ of $\mathbb{M}$-approximative systems. Properties of this category are studied and connections between $\mathbf{AS}^{\mathbb{M}}$ and its subcategories related to fuzzy topology, fuzzy sets and rough sets are described.

2 The context

In our work two lattices will play the fundamental role. The first one is a complete infinitely distributive lattice

$$\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee),$$

whose top and bottom elements are $1_{\mathbb{L}}$ and $0_{\mathbb{L}}$ respectively. Besides sometimes we will assume that the lattice $\mathbb{L}$ is equipped with one of the following operations: a monotone mapping ${}^c : \mathbb{L} \to \mathbb{L}$ or a binary operation $* : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$.

A lattice $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, {}^c)$ will be called *adjunctive* if the pair $({}^c, {}^c)$ is an adjunction

$$({}^c, {}^c) : \mathbb{L} \vdash \mathbb{L}^{op},$$

that is $a \leq b^c \iff b \leq a^c \quad \forall\, a, b \in \mathbb{L}$, cf e.g. [4]. A lattice $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, {}^c)$ will be called *involutive* if ${}^c : \mathbb{L} \to \mathbb{L}$ is an involution, that is if $(a^c)^c = a \quad \forall a \in \mathbb{L}$. One can easily see that in an adjunctive involutive lattice involution ${}^c : \mathbb{L} \to \mathbb{L}$ is order reversing:

$$a \leq b \implies b^c \leq a^c \, \forall a, b \in \mathbb{L},$$

and conversely, if ${}^c : \mathbb{L} \to \mathbb{L}$ is order reversing involution, then $({}^c, {}^c) : \mathbb{L} \vdash \mathbb{L}^{op}$ is an adjunction.

Concerning the second, binary operation $* : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ (conjunction) it will be assumed that $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, *)$ is a commutative cl-monoid (see e.g. [2]), that is
$*$ is commutative: $a * b = b * a$ for all $a, b \in \mathbb{L}$;
$*$ distributes over arbitrary joins:
$a * \left(\bigvee_{i \in \mathcal{I}} b_i\right) = \bigvee_{i \in \mathcal{I}} (a * b_i) \, \forall\, a \in \mathbb{L}, \, \forall\, \{b_i \mid i \in \mathcal{I}\} \subseteq \mathbb{L}$
and $a * 1_{\mathbb{L}} = a, \quad a * 0_{\mathbb{L}} = 0_{\mathbb{L}}$.

It is well-known (see e.g. [2]) that in a cl-monoid a further binary operation $\mapsto : \mathbb{L} \to \mathbb{L}$ (residuation) is defined related to conjunction $*$ by Galois connection:

$$a * b \leq c \iff a \leq b \mapsto c \, \forall a, b, c \in \mathbb{L}.$$

One can easily see that residuation is nonincreasing by the first argument and nondecreasing by the second argument, and that $b * (b \mapsto a) \leq a \, \forall a, b \in \mathbb{L}$. In particular $b * (b \mapsto 0) \leq 0$, and hence

$$b \leq (b \mapsto 0) \mapsto 0.$$

This allows to conclude, that by setting $a^c = a \mapsto 0$ we obtain an adjunction $({}^c, {}^c) : \mathbb{L} \vdash \mathbb{L}^{op}$. Indeed, if $a \leq b \mapsto 0$, then

$$b \leq (b \mapsto 0) \mapsto 0 \leq a \mapsto 0.$$

A cl-monoid is called a Girard monoid [10] if $(a \mapsto 0) \mapsto 0 = a \, \forall a \in \mathbb{L}$. Hence in case $\mathbb{L}$ is a Girard monoid, residuation $\mapsto$ induces an order reversing involution ${}^c : \mathbb{L} \to \mathbb{L}$.

An important situation in our research will be the following. Let $\mathrm{L} = (\mathrm{L}, \leq, \wedge, \vee)$ be a a lattice and X be a set. Then the L-powerset $\mathrm{L}^X =: \mathbb{L}$ becomes a lattice $(\mathbb{L}, \leq, \wedge, \vee)$ by pointwise extending the lattice structure from L to $\mathbb{L}$. Besides $\mathbb{L}$ is

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infinitely distributive whenever L was infinitely distributive. Moreover, if $\mathrm{L} = (\mathrm{L}, \leq, \wedge, \vee, ^c)$ is an adjunctive (involutive) lattice then by pointwise extending operation ^{c} from L to $\mathbb{L}$, an adjunctive (resp. involutive) lattice $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, ^c)$ is obtained. In case $\mathrm{L} = (\mathrm{L}, \leq, \wedge, \vee, *)$ is a cl-monoid, by pointwise extension of $* : \mathrm{L} \times \mathrm{L} \to \mathrm{L}$ to $* : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ we obtain a cl-monoid $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, *)$

The second lattice belonging to the context of our work is denoted by $\mathbb{M}$. At the moment we assume only its completeness, however sometimes it will be requested that $\mathbb{M}$ is completely distributive. The bottom and the top elements of $\mathbb{M}$ are $0_\mathbb{M}$ and $1_\mathbb{M}$ resp. As different from $\mathbb{L}$ we do not exclude the case when $\mathbb{M}$ is a one-point lattice and hence in this case $0_\mathbb{M} = 1_\mathbb{M}$.

3 Basic definitions

Definition 3.1 *An upper $\mathbb{M}$-approximative operator on* L *is a mapping $u : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ such that*

1. $u(0, \alpha) = 0\ \forall \alpha \in \mathbb{M}$;
2. $a \leq u(a, \alpha)\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
3. $u(a \vee b, \alpha) = u(a, \alpha) \vee u(b, \alpha)$
4. $u(u(a, \alpha), \alpha) = u(a, \alpha)$;
5. $\alpha \leq \beta, \alpha, \beta \in \mathbb{M} \Longrightarrow u(a, \alpha) \leq u(a, \beta)$.

Operator u is called (upper) semicontinuous (usc) if

- (usc) $u(a, \bigvee_{i\in\mathcal{I}} \alpha_i) = \bigwedge_{i\in\mathcal{I}} u(a, \alpha_i)$;

u is called (upper) weakly semicontinuous (uwsc) if

- (uwsc) *If* $u(a, \alpha_i) = \bar{a}\ \forall \alpha_i, i \in \mathcal{I}$ *and* $\alpha = \bigwedge_{i\in\mathcal{I}} \alpha_i$, *then* $u(a, \alpha) = \bar{a}$

Definition 3.2 *A lower $\mathbb{M}$-approximative operator on $\mathbb{L}$ is a mapping $l : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ such that*

1. $l(1, \alpha) = 1\ \forall \alpha \in \mathbb{M}$;
2. $a \geq l(a, \alpha)\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
3. $l(a \wedge b, \alpha) = l(a, \alpha) \wedge l(b, \alpha)$
4. $l(l(a, \alpha), \alpha) = l(a, \alpha)$;
5. $\alpha \leq \beta, \alpha, \beta \in \mathbb{M} \Longrightarrow l(a, \alpha) \geq l(a, \beta)$.

Operator l is called (lower) semicontinuous (lsc) if

- (lsc) $l(a, \bigvee_{i\in\mathcal{I}} \alpha_i) = \bigvee_{i\in\mathcal{I}} l(a, \alpha_i)$;

l is called (lower) weakly semicontinuous (lwsc) if

- (lwsc) *If* $l(a, \alpha_i) = a^0\ \forall \alpha_i, i \in \mathcal{I}$ *and* $\alpha = \bigvee_{i\in\mathcal{I}} \alpha_i$, *then* $l(a, \alpha) = a^0$

Definition 3.3 *A triple $(\mathbb{L}, u, l)$, where $u : \mathbb{L}\times\mathbb{M} \to \mathbb{L}$ and $l : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ are upper and lower $\mathbb{M}$-approximative operators on $\mathbb{L}$, is called an $\mathbb{M}$-approximative system. In case when X is a set* L *is a lattice, $\mathbb{L} = \mathrm{L}^X$ and $(\mathbb{L}, u, l)$ is an approximative system, the quadruple (X, L, u, l) is called an $\mathbb{M}$-approximative space.*

Definition 3.4 *An $\mathbb{M}$-approximative system $(\mathbb{L}, u, l)$ is called semicontinuous (s.c) if u is u.s.c. and l is l.s.c. An $\mathbb{M}$-approximative system $(\mathbb{L}, u, l)$ is called weakly semicontinuous (w.s.c) if u is u.w.s.c. and l is l.w.s.c.*

Definition 3.5 *In case $\mathbb{L}$ is equipped with unary operation $^c : \mathbb{L} \to \mathbb{L}$, an $\mathbb{M}$-approximative system $(\mathbb{L}, u, l)$ is called self dual if*

$$u(a^c, \alpha) = (l(a, \alpha))^c \text{ and}$$

$$l(a^c, \alpha) = (u(a, \alpha))^c\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$$

Note that in case when $(\mathbb{L}, \leq, \wedge, \vee, ^c)$ is involutive, the system is self-dual iff $(u(a^c, \alpha))^c = l(a, \alpha)$, and $(l(a^c, \alpha))^c = u(a, \alpha),\ \forall a \in \mathbb{L}, \forall \alpha \in \mathbb{M}$;

Remark 3.6 Sometimes we consider $\mathbb{M}$-approximative systems in case of a one-point lattice $\mathbb{M} = \{\cdot\}$. Obviously, in this case the use of the second argument in the notation of approximative systems is redundant and we write just $u(a)$ and $l(a)$ instead of $u(a, \cdot)$ and $l(a, \cdot)$ respectively. Besides, in this case we use the terms *upper and lower approximative operator*, *approximative system*, etc., omitting the prefix $\mathbb{M}$.

4 Lattice of $\mathbb{M}$-approximative systems on a lattice $\mathbb{L}$

Let $\mathcal{AS}^\mathbb{M}(\mathbb{L})$ stand for the family of $\mathbb{M}$-approximative systems $(\mathbb{L}, u, l)$ where $\mathbb{L}$ and $\mathbb{M}$ are fixed. We introduce an order $\preceq$ on $\mathcal{AS}^\mathbb{M}(\mathbb{L})$ by setting $(\mathbb{L}, u_1, l_1) \preceq (\mathbb{L}, u_2, l_2)$ iff $u_1 \geq u_2$ and $l_1 \leq l_2$.

Theorem 4.1 *$(\mathcal{AS}^\mathbb{M}(\mathbb{L}), \preceq)$ is a complete lattice. Its top and bottom elements are given respectively by*

$$u_\top(a, \alpha) = l_\top(a, \alpha) = a\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M};$$

$$u_\perp(a, \alpha) = \begin{cases} 1_\mathbb{L} & \text{if } a \neq 0_\mathbb{L} \\ 0_\mathbb{L}, & \text{if } a = 0_\mathbb{L} \end{cases}$$

$$l_\perp(a, \alpha) = \begin{cases} 0_\mathbb{L} & \text{if } a \neq 1_\mathbb{L} \\ 1_\mathbb{L}, & \text{if } a = 1_\mathbb{L} \end{cases}$$

The infimum of a family $\{(\mathbb{L}, u_i, l_i) \mid i \in \mathcal{I}\} \subseteq \mathcal{AS}^\mathbb{M}(\mathbb{L})$ is given by $(u_0, l_0) = (\bigvee_i u_i, \bigwedge_i l_i)$ and its supremum $\mathbb{L}$ is given by $(u^0, l^0) = (\bigwedge_i u_i, \bigvee_i l_i)$.

Proof Let $\mathcal{S} = \{(\mathbb{L}, u_i, l_i) \mid i \in \mathcal{I}\} \subseteq \mathcal{AS}^\mathbb{M}(\mathbb{L})$. Let

$$\bigwedge \mathcal{S} := (\mathbb{L}, u_0, v_0) \text{ where } u_0 = \bigvee_i u_i,\ l_0 = \bigwedge_i l_i.$$

Since $u_0 \geq u_i$ and $l_0 \leq l_i$ for all $i \in \mathcal{I}$, to show the completeness of $\mathcal{AS}^\mathbb{M}(\mathbb{L})$, it is sufficient to show that u_0 and l_0 are respectively the upper and lower $\mathbb{M}$-approximative operators on $\mathbb{L}$. However this can be established by the direct verification of the conditions (1)-(5) in definitions 3.1, 3.2.

Further, let

$$\bigvee \mathcal{S} = (\mathbb{L}, u^0, l^0) \text{ where } u^0 = \bigwedge_i u_i, l^0 = \bigvee_i l_i.$$

To show that (u^0, l^0) is the supremum of $\mathcal{S}$ it is sufficient to notice that u^0 and l^0 are resp. the upper and lower $\mathbb{M}$-approximative operators. However the validity of properties

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(1),(2),(4),(5) in definitions 3.2, 3.1 is obvious while the validity of property (3) can be established referring to the distributivity of $\mathbb{L}$.

One can easily establish also the following

Theorem 4.2 *1. The family $(SC\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$ of semicontinuous $\mathbb{M}$-approximative systems is a complete sublattice of $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$.*

2. The family $(WSC\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$ of weakly semicontinuous $\mathbb{M}$-approximative systems is a complete sublattice of $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$.

3. The family $(SD\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$ of self-dual $\mathbb{M}$-approximative systems is a complete sublattice of $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$.

5 Category $\mathrm{AS}^{\mathbb{M}}$ of $\mathbb{M}$-approximative systems

Let $\mathbb{M}$ be fixed and let $\mathbf{AS}^{\mathbb{M}}$ be the family of all $\mathbb{M}$-approximative systems $(\mathbb{L}, u, v)$. To consider $\mathbf{AS}^{\mathbb{M}}$ as a category whose class of objects are all $\mathbb{M}$-approximative systems we have to specify its morphisms. Given $(\mathbb{L}_1, u_1, l_1), (\mathbb{L}_2, u_2, l_2) \in \mathcal{O}b(\mathbf{AS}^{\mathbb{M}})$ by a morphism

$$f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$$

we call a mapping $f : \mathbb{L}_2 \to \mathbb{L}_1$ such that

1. $f : \mathbb{L}_1 \to \mathbb{L}_2$ is a morphism in the category $\mathbf{LAT^{op}}$ where $\mathbf{LAT}$ is the category of complete infinitely distributive lattices;
2. $u_1(f(b), \alpha) \leq f(u_2(b, \alpha))\ \forall b \in \mathbb{L}_2, \forall \alpha \in \mathbb{M}$;
3. $f(l_2(b, \alpha)) \leq l_1(f(b), \alpha)\ \forall b \in \mathbb{L}_2, \forall \alpha \in \mathbb{M}$

A morphsim $f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$ is aslo referred to as a continuous mapping between the corresponding $\mathbb{M}$-approximative systems

Theorem 5.1 $\mathbf{AS}^{\mathbb{M}}$ *thus obtain is indeed a category.*

Proof Let $f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$ and $g : (\mathbb{L}_2, u_2, l_2) \to (\mathbb{L}_3, u_3, l_3)$ be continuous mappings and let $g \circ f : \mathbb{L}_1 \to \mathbb{L}_3$ be their composition in $\mathbf{LAT^{op}}$. We have to verify that $g \circ f$ satisfies conditions (2) and (3) above. Since it is sufficient to verify these conditions for a fixed $\alpha \in \mathbb{M}$, to simplify the reasonings we omit the second argument in the notation of the approximative operators. Let $c \in \mathbb{L}_3$. Then

$$u_1(f(g(c))) \leq f(u_2(g(c))) \leq f(g(u_3(c))),$$

In a similar way we can show that $f(g(l_3(c))) \leq l_1(g(f(c)))$. Thus the composition $g \circ f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_3, u_3, l_3)$ is continuous whenever f and g are continuous. We conclude the proof noticing that the identity mapping $f : (\mathbb{L}, u, l) \to (\mathbb{L}, u, l)$ is continuous.
$\square$

In the sequel, when discussing categorical properties of $\mathbf{AS}^{\mathbb{M}}$ and other categories we refer to the monograph [1].

Theorem 5.2 *Every source $f_i : \mathbb{L}_1 \to (\mathbb{L}_i, u_i, l_i), i \in \mathcal{I}$ has a unique initial lift $f_i : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$.*

Proof Taking into account Theorem 4.1 it is sufficient to consider the case when the source contains only one morphism $f : \mathbb{L}_1 \to (\mathbb{L}_2, u_2, l_2)$ in $\mathbf{LAT}^{op}$.

Define upper approximative operator $u_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ by

$$u_1(a, \alpha) = \bigwedge \{f(u_2(b, \alpha)) \mid f(b) \geq a\}\ \forall a \in \mathbb{L}_1, \alpha \in \mathbb{M}.$$

Note first that the condition

$$u_1(f(b), \alpha) \leq f(u_2(b, \alpha))\ \forall b \in \mathbb{L}_2\ \forall \alpha \in \mathbb{M}$$

is obviously fulfilled. We verify that u_1 thus defined is indeed an upper approximative operator. As in the previous theorem in our reasoning we fix $\alpha \in \mathbb{M}$ and omit it in notation of approximative operators when verifying the properties (1) - (4).

The first two properties are obvious: $u_1(0_{\mathbb{L}_1}) = 0_{\mathbb{L}_1}$; $u_1(a) \geq a \forall a \in \mathbb{L}_1$.
To verify property (3) let $a_1, a_2 \in \mathbb{L}_1$, then

$$u_1(a_1 \vee a_2) = \bigwedge \{f(u_2(b)) \mid f(b) \geq a_1 \vee a_2\} \leq$$
$$\bigwedge \{f(u_2(b_1 \vee b_2)) f(b) \geq a_1, f(b_2) \geq a_2\} =$$
$$\bigwedge \{f(u_2(b_1)) \vee f(u_2(b_2)) \mid f(b_1) \geq a_1, f(b_2) \geq a_2\} =$$
$$\bigvee_{i=1,2} (\bigwedge \{f(u_2(b_i)) \mid f(b_i) \geq a_i\}) = u_1(a_1) \vee u_1(a_2).$$

The converse inequality is obvious.

To verify the fourth condition notice that $u_1(u_1(a)) = u_1 (\bigwedge \{f(u_2(b)) \mid f(b) \geq a\}) \leq \bigwedge \{u_1(f(u_2(b))) \mid f(b) \geq a\} \leq \bigwedge \{f(u_2(u_2(b))) \mid f(b) \geq a\} = u_1(a)$. The converse inequality is obvious

To verify property (5) for u_1 note that

$$\alpha \leq \beta, \alpha, \beta \in \mathbb{M} \Longrightarrow u_1(a, \alpha) \leq u_1(a, \beta)$$

is guaranted by the analogous property of the operator $u_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ and the definition of u_1.

Define lower $\mathbb{M}$-approximative operator $l_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ by

$$l_1(a, \alpha) = \bigvee \{f(l_2(b), \alpha) \mid f(b) \leq a\}\ \forall a \in \mathbb{L}_1 \forall \alpha \in \mathbb{M}.$$

Notice first that

$$f(l_2(b, \alpha)) \leq l_1(f(b), \alpha)\ \forall b \in \mathbb{L}_2, \alpha \in \mathbb{M}.$$

We show that $l_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ thus defined is an lower $\mathbb{M}$-approximative operator. Again, we omit in notation α when it is fixed. The first two conditions from Definition 3.2 are obvious. To verify the third condition let $a_1, a_2 \in \mathbb{L}_1$. Then $l_1(a) \wedge l_1(a_2) = \bigvee \{f(l_1(b_1)) \wedge f(l_1(b_2)) \mid f(l_1(b_i)) \leq a_i, i = 1, 2\} \leq \bigvee \{f(l_1(b_1 \wedge b_2) \mid f(b_1) \wedge f(b_2) \leq a_1 \wedge a_2\} = \bigvee \{f(b) \mid f(b) \leq a_1 \wedge a_2\} = l_1(a_1 \wedge a_2)$,
The converse inequality is obvious, The idempotence of the operator $l_1 : \mathbb{L}_1 \to \mathbb{L}_1$ is establihed as follows:
$l_1(l_1(a)) = l_1 (\bigvee \{f(l_2(b)) \mid f(b) \leq a\}) \geq \bigvee \{l_1 f(l_2(b)) \mid f(b) \leq a\} \geq \bigvee \{f(l_2(b)) \mid f(b) \leq a\} = l_1(a)$.
The opposite inequality is obvious.

Finaly, the condition $\alpha \leq \beta, \alpha, \beta \in \mathbb{M} \Longrightarrow l_1(a, \alpha) \geq l_1(a, \beta)$ is guaranted by the analogous property of the operator $l_2 : \mathbb{L}_2 \to \mathbb{L}_2$ and the definition of l_1.

Let $g : (\mathbb{L}_3, u_3, l_3) \to (L_2, u_2, l_2)$ be a morphism in $\mathbf{AS}^{\mathbb{M}}$ and $h : \mathbb{L}_3 \to \mathbb{L}_1$ be a morphism in $\mathbf{LAT}^{op}$ such that $f \circ h = g$.

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Then from the construction it is clear that $h : (\mathbb{L}_3, u_3, l_3) \to (L_1, u_1, l_1)$ is a morphism in $\mathbf{AS}^{\mathbb{M}}$. Thus $f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$ is indeed the initial lift of $f : \mathbb{L}_1 \to (\mathbb{L}_2, u_2, l_2)$. The uniqueness of the lift is obvious.

Let $\mathcal{AI}\mathbf{LAT}$ denote the category of adjunctive involutive complete infinitely distributive systems.

Theorem 5.3 *Let $\mathbb{L}_1, \mathbb{L}_2$ be adjunctive involutive lattices and $f : \mathbb{L}_1 \to \mathbb{L}_2$ be a morphism in $\mathcal{AI}\mathbf{LAT}^{op}$. If $\mathbb{M}$-approximative operators $l_2 : u_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ are self-dual, then $\mathbb{M}$-approximative operators $l_1, u_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ constructed above are self-dual as well.*

Indeed, let $a \in \mathbb{L}$. Then
$l_1(a^c) = \bigvee\{f(l_2(b)) \mid f(b) \le a^c\} = (\bigwedge\{(f(l_2(b)))^c \mid f(b) \le a^c\})^c = (\bigwedge\{f(u_2(b^c)) \mid f(b) \le a^c\})^c = (\bigwedge\{f(u_2(b^c)) \mid f(b^c) \ge a\})^c = (\bigwedge\{f(u_2(d)) \mid f(d) \ge a\})^c = (u_1(a))^c$.

Theorem 5.4 *If approximative system (u_2, l_2) on $\mathbb{L}_2$ is semicontinuous (weakly semicontinuous), then the approximation system (u_1, l_1) constructed in the previous theorem is semicontinuous (resp. weakly semicontinuous), too.*

Indeed, if (u_2, l_2) is semicontinuous, then $u_1(a, \bigvee_i \alpha_i) = \bigwedge f(u_2(b, \bigvee_i \alpha_i) \mid f(b) \ge a\} = \bigwedge_i \bigwedge\{f(u_2(b, \alpha_i) \mid f(b) \ge a\} = \bigwedge_i u_1(a, \alpha_i)$, and $l_1(a, \bigvee_i \alpha_i) = \bigvee\{f(l_2(b, \bigvee_i \alpha_i) \mid f(b) \le a\} = \bigvee_i \bigvee\{f(l_2(b, \alpha_i) \mid f(b) \le a\} = \bigvee_i l_1(b, \alpha_i)$.
In a similar way one can establish weak semicontinuity of (u_1, l_1) in case (u_2, l_2) was weakly semicontinuous.

Theorem 5.5 *Every sink $f_i : (\mathbb{L}_i, u_i, l_i) \to \mathbb{L}_2,\ i \in \mathcal{I}$ has a unique final lift: $f_i : (\mathbb{L}_i, u_i, l_i) \to (\mathbb{L}_2, u_2, l_2)\ i \in \mathcal{I}$.*

Proof . Taking into account Theorem 4.1 it is sufficient to consider the case of the sink consisting of a single morphism $f : (\mathbb{L}_1, u_1, l_1) \to \mathbb{L}_2$. We define an upper $\mathbb{M}$-approximative operator $u_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ by:

$$u_2(b, \alpha) = \bigwedge\{c \in \mathbb{L}_2 \mid c \ge b, f(c) \ge u_1(f(b), \alpha)\}.$$

It is obvious that $u_1(f(b), \alpha) \le f(u_2(b, \alpha))\ \forall\ b \in \mathbb{L}_2$. We show that $u_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ is an upper $\mathbb{M}$-approximative operator. We omit notation α when it can be fixed.

It is obvious that $u_2(0) = 0$ and $b \le u_2(b)$ for every $b \in \mathbb{L}_2$. Further, let $b_1, b_2 \in \mathbb{L}_2$. Then $u_2(b_1) \vee u_2(b_2) = (\bigwedge\{c_1 \in \mathbb{L}_2 \mid c_1 \ge b_1, f(c_1) \ge u_1(f(b_1))\}) \vee (\bigwedge\{c_2 \in \mathbb{L}_2 \mid c_2 \ge b_2, f(c_2) \ge u_1(f(b_2))\}) = \bigwedge\{c_1 \vee c_2 \mid c_i \ge b_i, f(c_i) \ge u_1(f(b_i)), i = 1, 2\} \ge \bigwedge\{c \mid c \le b_1 \vee b_2, f(c) \ge b_1 \vee b_2, f(c) \ge u_1(f(b_1)) \vee u_1(f(b_2))\} = \bigwedge\{c \mid c \le b_1 \vee b_2, f(c) \ge b_1 \vee b_2, f(c) \ge u_1(f(b_1 \vee b_2))\} = u_2(b_1 \vee b_2)$.
The opposite inequality is obvious.

$u_2(u_2(b)) = \bigwedge\{c \mid c \ge u_2(b), u_1(f(u_2(b))) \le f(c)\}$. Noticing that $u_2(b)$ is among the elements c satisfying the above conditions, we conclude that $u_2(u_2(b)) \le u_2(b)$. The opposite inequality is obvious and hence $u_2(u_2(b)) = u_2(b)$.

Property (5) for u_2 is guaranted by the analogous property of the operator $u_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ and the definition of u_2.

Define lower approximation operator $l_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ by

$$l_2(b, \alpha) = \bigvee\{c \in \mathbb{L}_2 \mid c \le b, l_1(f(c), \alpha) \le f(b)\}.$$

The validity of the first two conditions for $l_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ is obvious. To verify the third property let $b_1, b_2 \in \mathbb{L}_2$. Then $l_2(b_1) \wedge l_2(b_2) = \bigvee\{c_1 \wedge c_2 \mid c_1 \le b_1, c_2 \le b_2, l_1(f(c_1) \le f(b_1), l_1(f(c_2)) \le f(b_2)\} \le \bigvee\{c_1 \wedge c_2 \mid c_1 \wedge c_2 \le b_1 \wedge b_2,\ l_1(f(c_1)) \wedge l_1(f(c_2)) \le f(b_1) \wedge f(b_2)\} \le \bigvee\{c \mid c \le b_1 \wedge b_2, l_1(f(c)) \le f(b_1 \wedge b_2)\} = l_2(b_1 \wedge b_2)$.
The opposite inequality is obvious.

To show the fourth axiom note that

$$l_2(l_2(b)) = \bigvee\{c \mid c \le l_2(b), l_1(f(c)) \le f(l_2(b))\}$$

and since $l_2(b)$ is one of c appearing in the above formula, it holds $l_2(l_2(b)) \ge l_2(b)$. The converse inequality is obvious.

Property (5) for l_2 is guaranted by the analogous property of the operator $l_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_2$ and the definition of l_2.

Theorem 5.6 *If approximation system (u_1, l_1) on $\mathbb{L}_1$ is self-dual, and $f : \mathbb{L}_1 \to \mathbb{L}_2$ is a morphism in the category of adjunkctive involutive lattices, then the approximation system (u_2, l_2) constructed in the previous theorem is self-dual.*

Indeeed, given $b \in \mathbb{L}_2$, we have
$(l_2(b))^c = (\bigvee\{d \mid d \le b, l_1(f(d)) \le f(b)\})^c = \bigwedge\{d^c \mid d \le b,\ l_1(f(d)) \le f(b)\} = \bigwedge\{d^c \mid d^c \ge b^c, (l_1(f(d)))^c \ge f(b^c)\} = \bigwedge\{d^c \mid d^c \ge b^c,\ u_1((f(d))^c) \ge f(b^c)\} = \bigwedge\{d^c \mid d^c \ge b^c,\ u_1((f(d^c))) \ge f(b^c)\} = \bigwedge\{e \mid e \ge b^c,\ u_1(f(e)) \ge f(b^c)\} = u_2(b^c)$.

One can easily establish also the following

Theorem 5.7 *If $\mathbb{M}$-approximation system (u_1, l_1) on $\mathbb{L}_1$ is semicontinuous (weakly semicontinuous), then the approximation system (u_2, l_2) constructed in the previous theorem is semicontinuous (resp. weakly semicontinuous).*

From theorems 5.2, 5.5, we obtain the following important

Corollary 5.8 *Category $\mathbf{AS}^{\mathbb{M}}$ is topological over the category $\mathbf{LAT}^{op}$ lattices with respect to the forgetful functor $\mathfrak{F} : \mathbf{AS}^{\mathbb{M}} \to \mathbf{LAT}^{op}$.*

Besides, taking into account theorems 5.3, 5.6, 5.4, 5.7 we have

Corollary 5.9 *The category $SD\mathbf{AS}^{\mathbb{M}}$ of self-dual $\mathbb{M}$-approximative systems is topological over the category $\mathcal{AI}\mathbf{LAT}^{op}$ with respect to the forgetful functor $\mathfrak{F} : SD\mathbf{AS}^{\mathbb{M}} \to \mathcal{AI}\mathbf{LAT}^{op}$.*

Corollary 5.10 *The categories $SC\mathbf{AS}^{\mathbb{M}}$ and $WSC\mathbf{AS}^{\mathbb{M}}$ of semicontinuous and weakly semicontinuous $\mathbb{M}$-approximative systems are topological over the category $\mathbf{LAT}^{op}$ with respect to the forgetful functor $\mathfrak{F} : (W)SC\mathbf{AS}^{\mathbb{M}} \to \mathbf{LAT}^{op}$.*

6 Categories $\mathrm{AS}^{\mathbb{M}}(\mathrm{L})$ of $\mathbb{M}$-approximative L-spaces

An important subcategory of the category $\mathbf{AS}^{\mathbb{M}}$ is the category whose ojects are L-powersets L^X of arbitrary sets (where L is a fixed lattice) and whose morphisms are induced by mappings of the corresponding sets X. Here are the details:

Let L be a fixed complete infinitely distributive lattice and let the objects of $\mathbf{AS}^{\mathbb{M}}(\mathrm{L})$ be approximation systems of the

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form (L^X, u, l) where L^X are L-powersets of arbitrary sets X. Sometimes it is more convenient to interpret objects of this category as the corresponding quadruples (X, L, u, l). To define a morphism $F : (\mathrm{L}^{X_1}, u_1, l_1) \to (\mathrm{L}^{X_2}, u_2, l_2)$ consider a mapping $f : X_1 \to X_2$ and let $f^{\leftarrow} : \mathrm{L}^{X_2} \to \mathrm{L}^{X_1}$ be the backward powerset operator induced by f [16]. Now as morphisms in $\mathbf{AS}^{\mathbb{M}}(\mathrm{L})$ we take $F := f^{\leftarrow^{op}} : (\mathrm{L}^{X_1}, u_1, l_1) \to (\mathrm{L}^{X_2}, u_2, l_2)$ in case it is a morphism in the category $\mathbf{AS}^{\mathbb{M}}$.

An important special case is a two point lattice $\mathrm{L} = \mathbf{2}$: in this case we come to the category of $\mathbb{M}$-approximative structures on ordinary sets (of course, for this one has to interpret a subset A of a set X as the characteristic function $\chi_A : X \to \mathbf{2}$). In particular, if $\mathbb{M}$ is a one-point lattice we come to the concept of an approximation system as it was considered by some authors, see e.g. [20].

7 Categories of fuzzy topologies as subcategories of $\mathrm{AS}^{\mathbb{M}}$

7.1 *Category of* $(\mathrm{L}, \mathbb{M})$*-fuzzy topological spaces*

We start with interpreting the category **FTOP**$(\mathrm{L}, \mathbb{M})$ of $(\mathrm{L}, \mathbb{M})$-fuzzy topological spaces see, e.g. [17], [13], [18], [7] as a subcategory of $\mathbf{AS}^{\mathbb{M}}$. In this section $\mathbb{M}$ is assumed to be completely distributive.

Definition 7.1 *A mapping* $\mathcal{T} : \mathrm{L}^X \to \mathbb{M}$ *is an* $(\mathrm{L}, \mathbb{M})$*-fuzzy topology on* X *if*

1. $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1$;
2. $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V) \forall U, V \in \mathrm{L}^X$;
3. $\mathcal{T}\left(\bigvee_{i \in \mathcal{I}} U_i\right) \geq \bigwedge_{i \in \mathcal{I}} \mathcal{T}(U_i) \ \forall \{U_i \mid i \in \mathcal{I}\} \subseteq \mathrm{L}^X$

A pair $(X, \mathcal{T})$ *is called an* $(\mathrm{L}, \mathbb{M})$*-fuzzy topological space and the value* $\mathcal{T}(U), U \in \mathrm{L}^X$ *is interpreted as the degree of openess of a fuzzy set* U. *A mapping* $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ *is called continuous if* $\mathcal{T}_X\left(f^{-1}(V)\right) \geq \mathcal{T}_Y(V)) \ \forall V \in \mathrm{L}^Y$.

Let $(X, \mathcal{T})$ be an $(\mathrm{L}, \mathbb{M})$-fuzzy topological space. By setting

$$\mathrm{int}_{\mathcal{T}}(A, \alpha) = \bigvee \{U \in \mathrm{L}^X \mid U \leq A, \ \mathcal{T}(U) \geq \alpha\},$$

we define the s.c. interior operator $\mathrm{int}_{\mathcal{T}} : \mathrm{L}^X \times \mathbb{M} \to \mathrm{L}^X$.

The relations between (L, M)-fuzzy topologies and lower $\mathbb{M}$-approximative operators are described in the theorem;

Theorem 7.2 *The interior operator* int *is a weakly s.c. lower* $\mathbb{M}$*-approximative operator on* $\mathbb{L} = \mathrm{L}^X$. *Conversely, if* $l : \mathrm{L}^X \times \mathbb{M} \to \mathrm{L}^X$ *is a weakly s.c. lower* $\mathbb{M}$*-approximative operator, then by setting*

$$T_l(U) = \bigvee \{\alpha \mid l(U, \alpha) \geq U\}$$

we obtain a mapping $\mathcal{T}_l : \mathrm{L}^X \to \mathrm{L}^X$ *satisfying conditions (1) and (3) of Definition 7.1. Besides* $\mathcal{T}_{\mathrm{int}_{\mathcal{T}}} = \mathcal{T}$ *and* $l_{\mathcal{T}_l} = l$.

Further, assume that L is an adjunctive involutive lattice and let $^c : \mathrm{L} \to \mathrm{L}$ be the corresponding involution. Then by setting

$$cl_{\mathcal{T}}(A, \alpha) = \bigwedge \{B \mid B \geq A, \mathcal{T}(B^c) \geq \alpha\}$$

a closure operator $cl_{\mathcal{T}} : \mathrm{L}^X \times \mathbb{M} \to \mathrm{L}^X$ is defined. One can easily show that $cl_{\mathcal{T}}$ is a weakly s.c. upper $\mathbb{M}$-approximative operator and prove a theorem establishing relations between weakly s.c. upper $\mathbb{M}$-approximative operators and $(\mathrm{L}, \mathbb{M})$-fuzzy topologies via closure operators, analogous to Theorem 7.2. Besides the $\mathbb{M}$-approximation system $(\mathrm{L}^X, cl_{\mathcal{T}}, int_{\mathcal{T}})$ is self-dual.

Thus in case of an involutive adjunctive lattice L an $(\mathrm{L}, \mathbb{M})$-fuzzy topological space $(X, \mathcal{T})$ can be interpreted as a weakly s.c. $\mathbb{M}$-approximative self-dual system $(\mathbb{L}, \mathrm{cl}, \mathrm{int})$ where $\mathbb{L} = \mathrm{L}^X$.

This allows us to identify the category $\mathbf{FTOP}(\mathrm{L}, \mathbb{M})$ with the subcategory $\mathbf{TopAS}^{\mathbb{M}}(\mathrm{L})$ of the category $\mathbf{AS}^{\mathbb{M}}$ whose objects are self dual weakly s.c. $\mathbb{M}$-approximative systems of the form $(\mathrm{L}^X, \mathrm{int}, \mathrm{cl})$ and the morphisms are $F = f^{\leftarrow^{op}} : (\mathrm{L}^X, \mathrm{int}_X, \mathrm{cl}_X) \to (\mathrm{L}^Y, \mathrm{int}_Y, \mathrm{cl}_Y)$, where $f^{\leftarrow} : \mathrm{L}^Y \to \mathrm{L}^X$ are powerset operators induced by (see e.g. [9]) by coninuous mappings $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ (cf also the previous section).

7.2 *Category of Chang-Goguen* L*-topological space*

To obtain characterization of L-topological spaces (see [3], [5]) by means of approximative systems we can restrict the theory developed in 7.1 by taking the two-point lattice $\mathbf{2} = \{0, 1\}$ in the role of $\mathbb{M}$. Then the category of Chang-Goguen L-topological spaces can be identified with the subcategory $\mathbf{TopAS}(\mathrm{L})$ of the category $\mathbf{TopAS}^{\mathbf{2}}(\mathrm{L})$. In particular $\mathbf{TopAS}^{\mathbf{2}}(\mathbf{2})$ can be identified with the classical category **TOP** of ordinary topological spaces and continuous mappings.

7.3 *Category of* L*-fuzzifying topological space*

To obtain characterization of L-fuzzifying topological spaces, originally defined by U.Hohle [6] and then independently discovered by Mingsheng Ying [21], by means of approximative systems we restrict the theory developed in 7.1 by taking the two-point lattice $\mathbf{2}$ in the role of L (thus $\mathbb{L} = 2^X$) and the lattice L in the role of $\mathbb{M}$. Then the category of L-fuzzifying topological spaces can be identified with the category $\mathbf{TopAS}^{\mathrm{L}}(\mathbf{2})$.

7.4 *Category of Hutton fuzzy topological spaces*

According to B. Hutton [8], a fuzzy topological space is a pair (L, τ) where L is a completely distributive lattice and $\tau \subseteq \mathrm{L}$ such that $0, 1 \in \tau$; $\ a, b \in \tau \Longrightarrow a \wedge b \in \tau$: $\ a_i \in \tau \forall i \in \mathcal{I} \Longrightarrow \bigvee_{i \in \mathcal{I}} a_i \in \tau$. The morphisms $f : (L_1, \tau_1) \to (L_2, \tau_2)$ in the category **H-TOP** of Hutton fuzzy topological spaces are mappings $f : L_2 \to L_1$ such that $f(\tau_2) \subseteq \tau_1$. One can show that the category **H-TOP** can be identified with the subcategory $\mathbf{HAS}$ of the category $\mathbf{AS}^{\mathbb{M}}$. whose objects are self dual approximative systems $(\mathbb{L}, l, u)$ where $\mathbb{L}$ is a completely distributive involutive adjunctive lattice and $\mathbb{M} = \{\cdot\}$

7.5 *Category of variable basis fuzzy topological spaces*

In [15] S.E. Rodabaugh has introduced the notion of a variable-basis fuzzy topological space and defined the corresponding category **R-TOP**. Further the theory of variable basis fuzzy topological spaces and some related categories was developed in a series of papers by S.E. Rodabaugh, P. Eklund and other authors. The category of variable-basis fuzzy topological spaces also can be obtained as a subcategory of the category $\mathbf{AS}^{\mathbb{M}}$. However to describe it in this way and to give an

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explicite characterization by means of $\mathbb{M}$-approximative systems we need more space than it is allowed here.

8 Categories related to rough sets

8.1 Rough sets

Let $\rho \subseteq X \times X$ be a binary relation on a set X and let $R(x) = \{x' \mid x\rho x'\}$ be the right ρ-class of $x \in X$. Given $A \in 2^X$ let $l(A) = A^{\blacktriangledown} = \{x \mid R(x) \subseteq A\}$, $u(A) = A^{\blacktriangle} = \{x \mid R(x) \cap A \neq \emptyset\}$. In case ρ is reflexive and transitive $u : 2^X \to 2^X$ and $l : 2^X \to 2^X$ are, respectively, upper and lower approximative operators on $2^X = \mathcal{P}(X)$ and $(X, 2,^{\blacktriangledown},^{\blacktriangle})$ is an approximative space. Besides, one can easily see that the system $(\mathbf{2}^X,^{\blacktriangledown},^{\blacktriangle})$ is self dual: $A^{c\blacktriangle} = A^{\blacktriangledown c})$ for any $A \subseteq X$. Such operators and corresponding approximative spaces in case when ρ is an equivalence relation were introduced by Pawlak [14] under the name "rough set". Further approximative operators operators induced by binary relations, either general or satisfying special properties, were studied by different authors, see e.g. [11], [12], [20], etc. Note however, that in case ρ is not reflexive or transitive, this operators may fail to be approximative operators in our sense.

In case when ρ is only reflexive, J. Järvinen and J. Kortelainen [12] along with operators $A^{\blacktriangledown}$ and $A^{\blacktriangle}$ consider also operators $u'(A) = A^{\triangle} = \{x \mid R^{-1}(x) \cap A \neq \emptyset\}$, $l'(A) = A^{\triangledown} = \{x \mid R^{-1}(x) \subseteq A\}$ and show that (u, l') and (u', l) form Galois connection: $u(a) \leq b \Longleftrightarrow a \leq l'(b)$; $\quad l(a) \leq b \Longleftrightarrow a \leq u'(b)$. Thus in case ρ is also transitive, we obtain "Galoi-connected" approximative systems $(\mathbf{2}^X,^{\blacktriangledown},^{\triangle})$ and $(\mathbf{2}^X,^{\triangledown},^{\blacktriangle})$.

8.2 L*-rough sets*

Generalizing the previous situation let L be a cl-monoid $(\mathrm{L}, \wedge, \vee, *)$, X be a set and $\rho : X \times X \to \mathrm{L}$ be an L-relation on X. Further, assume that ρ is reflexive (that is $\rho(x,x) = 1\ \forall x \in X$) and transitive (that is $\rho(x,y) * \rho(y,z) \leq \rho(x,z)\ \forall x,y,z \in X$.) For every $x \in X$ we define $\mathcal{R}(x) : X \to \mathrm{L}$ by $\mathcal{R}(x)(x') = \rho(x,x')$ Further, given $A \in \mathrm{L}^X$ let lower and upper approximative operators $l(A) \in \mathrm{L}^X$, and $u(A) \in \mathrm{L}^X$ be defined by $l(A)(x) = \inf_{x' \in X}(\mathcal{R}(x)(x') \mapsto A(x'))$ and $u(A)(x) = \sup_{x' \in X}(\mathcal{R}(x)(x') * A(x'))$ respectively. One can show that (L^X, u, l) is an L-approximative system. We refer to such kind of an approximative system as an L-rough system induced by the L-relation ρ. In case $(\mathrm{L}, \wedge, \vee, *)$ is a Girard monoid, the system (L^X, u, l) is self dual. Further, if $\mathrm{L} = \mathbf{2}$ is a two-point lattice we come to the situation described in the previous subsection. In an natural way we define morphisms for the category $\mathbf{Rgh}(\mathbb{L})$ of L- rough systems and characterize it as a category of approximative systems.

9 Defuzzification approximation operators

Finally we sketch how the concept of an approximative systems can be applied for fuzzy sets themselves.

Let $\mathrm{L} = (\mathrm{L}, \wedge, \vee, \leq)$ be a complete lattice, X be a set and $\mathbb{L} = \mathrm{L}^X$ Define $u : \mathbb{L} \times \mathrm{L} \to \mathbb{L}$ and $l : \mathbb{L} \times \mathrm{L} \to \mathbb{L}$ as follows: Given $A \in \mathrm{L}^X$ let

$$u(A, \alpha) = A \vee 1_{A_\alpha},\ l(A, \alpha) = \alpha \cdot 1_{A_\alpha} \wedge A,$$

where $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$ In this way we obtain L-approximation operators on the L-powerset of a set X which can be interpreted as resp. upper and lower level-defuzzification operators .

References

[1] J. Adámek, H. Herrlich and G.E. Strecker, **Abstract and Concrete Categories**, John Wiley & Sons, New York 1990.

[2] G. Birkhoff, **Lattice Theory**, AMS Providence, RI, 1995.

[3] C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.

[4] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, D.S. Scott, **Continuous Lattices and Domains**, Cambridge University Press, Cambridge, 2003.

[5] J.A. Goguen, *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl., **18** (1973), 734-742.

[6] U. Höhle, *Upper semicontinuous fuzzy sets and applications*, J. Math. Anal. Appl., **78** (1980) 659-673.

[7] U. Höhle, A. Šostak, *Axiomatics for fixed-based fuzzy topologies*, Chapter 3 in [8]

[8] B. Hutton, *Normality in fuzzy topologicl spaces*, J. Math. Anal. Appl., **50** (1975), 74-79.

[9] **Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory** , U. Höhle, S.E. Rodabaugh eds. - Handbook Series, vol.3, Kluwer Acad. Publ., - 1999.

[10] S. Jenei, *Structure of Girad Monoids on [0,1]* Chapter 10 In: **Topological and Algebraic Structures in Fuzzy Sets** S.E. Rodabaugh, E.P. Klement eds., Kluwer Acad. Publ. 2003.

[11] J. Korteleinen, On relationship between modified sets, topological spaces and rough sets, Fuzzy Sets and Syst. **61** (1994), 91-95.

[12] J. Järvinen, J. Kortelainen, **A unified study between modal-like operators, Topologies and Fuzzy Sets**, Turku Centre for Computer Sciences, Turku, Finland, Technical Reports **642**, 2004.

[13] T. Kubiak, *On Fuzzy Topologies*, PhD Thesis, Adama Mickiewicz University Poznan, Poland, 1985.

[14] Z. Pawlak, *Rough sets*, International J. of Computer and Information Sciences, **11** (1982), 341-356.

[15] S.E. Rodabaugh, A categorical accomodation of various notions of fuzzy topology, *Fuzzy Sets and Syst.*, **9** (1983), 241-265.

[16] S.E. Rodabaugh, *Powerset Operator Foundations For Poslat Fuzzy Set Theories and Topologies*, Chapter 2 in [8].

[17] A. Šostak, *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Palermo, Ser II **11** (1985), 125-186.

[18] A. Šostak, *Basic structures of fuzzy topology*, J. Math. Sci. **78** (1996), 662-701.

[19] A. Šostak, *Approximative fuzzy systems and approximative fuzzy spaces*, In: 9^{th} International Conference on Fuzzy Set Theory and Applications: FSTA2008. Abstracts, p. 118.

[20] Y.Y. Yao, *A comparative study of fuzzy sets and rough sets*, Inf. Sci., **109** (1998), 227 - 242.

[21] Ying Mingsheng, *A new approach to fuzzy topology, Part* I, Fuzzy Sets and Syst., **39** (1991), 303-321.

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