Similarity based fuzzy interpolation applied to CAGD*

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Abstract— *This paper presents an approach to a system based on fuzzy logic for the* **design** *of interpolative curves and surfaces in the context of Computer Aided Geometric Design (CAGD). Some problems arising from a previous model are studied by using a similarity approach.*

Keywords— Fuzzy interpolation, indistinguishability operator, fuzzy number, Computer Aided Geometric Design,approximate reasoning, t-norm, fuzzy control.

1 Introduction

The aim of computer aided geometric design (CAGD) is to produce a curve or a surface whose shape is controlled by a set of given points. In this sense, it can be understood as an interpolative procedure where the goal object is not a function but its graphical presentation. Therefore, in the design process it is essential to control the degree on smoothness of the solution.

On the other hand, the user of a CAGD system provides the data "approximately" because, after a first trial, he usually adjusts the result to his own taste or aesthetic point of view. So, it is natural to treat the problem under the setting of approximate reasoning considering the input-output data as fuzzy points of $\mathbb{R}$ or $\mathbb{R}^2$ represented by fuzzy numbers.

In order to build such a system, we focus our attention in the fuzzification and defuzzification procedures. To define and control the shapes on the input-output fuzzy numbers, we assume that there exists a similarity relation in $\mathbb{R}$ that granulates and fuzzifies its elements. With respect to the defuzzification process, we can choose the Takagi-Sugeno model that produces a crisp output or select some of the well known defuzzification methods like the center of gravity. The paper is organized in the following way. After this introduction, section 2 recalls the preliminaries related to indistinguishability operators and fuzzy classes. Sections 3 and 4 presents the general model and a model associated to Takagi-Sugeno technique that tries to overcome some of the problems of the model proposed in [17], such as the effective construction of the fuzzy numbers and the solution for the case of sparse data. In section 5 an example is fully developed and we close with conclusions and future works.

2 Preliminaries

The large number of successful applications of fuzzy sets theory (specially in fuzzy control) has produced numerous concepts based on empirical motivations. In order to give a sound justification to the use of these applied concepts, it is necessary to provide in the setting of a clear model, a well-founded interpretation for them. The aim of this section is to introduce an appropriate background that justifies the different definitions and types of fuzzy real numbers as they are interpreted and used in the applications.

As a starting point it is assumed that the concept of fuzzy real number arises from the fact that there exists a certain equality relation in $\mathbb{R}$, in the sense that numbers laying "close" are indistinguishable to the observer [15]. When the defined equality is a T-indistinguishability operator [7,21], triangular and trapezoidal fuzzy numbers among others, appear in a natural way. The model justifies the use of different types of fuzzy numbers depending on the scale defined in $\mathbb{R}$ and also develops the way of generating indistinguishability operators on $\mathbb{R}$ associated to one or several scales.

Fuzzy numbers arise naturally from the underlying indistinguishalility as the singletons associated to the relation [2,12,13,14]. Therefore, only the fuzzy relations whose singletons fulfill a general definition of fuzzy number are considered. Reciprocally, given a "suitable" fuzzy equality in $\mathbb{R}$, only the fuzzy numbers that are the singletons of this relation will be considered as such. As a natural consequence of this setting, we also study the families of fuzzy numbers that are invariant under translations.

Throughout this paper we use the standard definitions and properties of t-norms. In the sequel Π will represent the product t-norm and $\mathcal{L}$ the Łuckasiewicz t-norm. Some definitions on T-indistiguinshability operators and fuzzy numbers are recalled. For a more detailed exposition on these topics, readers are referred to [1,7,15,16,18].

Definition 2.1. Let T be a t-norm. A *T-indistinguishability operator* E on a set X is a reflexive and symmetric fuzzy relation on X such that

$T(E(x,y),E(y,z)) \leq E(x,z)$ (T-transitive property).

If $T = \text{Min}$, then E is called a *similarity*. For $T = \mathcal{L}$, E is termed a *likeness* and when $T = \Pi$, E is a *probabilistic relation*.

T-indistinguishability operators generalize within the fuzzy framework the concept of equivalence relation and are also called equality relations [2,7]. From a semantical point of view, $E(x,y)$ can be interpreted as the degree of similarity between x and y.

Definition 2.2. Given a T-indistinguishability operator E on a set X and an element x_0 of X, the *singleton* induced by x_0 is the fuzzy set f_{x_0} defined by $f_{x_0}(x) = E(x_0, x)$.

The singleton of an element x of X can be thought as its fuzzy equivalence class with respect to E [12] and its fuzzification

*Partially supported by the project TIN2006-14311

ISBN: 978-989-95079-6-8

taking E into account [8].

Definition 2.3. A *fuzzy real number* is a map $f_a : \mathbb{R} \to [0, 1]$ such that there exists a number $a \in \mathbb{R}$ with $f_a(a) = 1$ where f_a is non-decreasing on $(-\infty, a)$ and non-increasing on $(a, +\infty)$.

Definition 2.4. A T-indistinguishability operator E on $\mathbb{R}$ is called *admissible* if and only if its singletons are fuzzy numbers.

The fuzzified of a real number via an admissible indistinguishability operator is therefore a fuzzy number.

The link between these operators and some families of fuzzy numbers will be investigated.

2.1 Admissible T-indistinguishability operators

A scale on $\mathbb{R}$ is defined as a monotonic map $f : \mathbb{R} \to \mathbb{R}$. The possibility to distinguish real numbers in a fuzzy environment is based on the use of a scale. Different scales give different degrees of accuracy and determine different fuzzy equalities on $\mathbb{R}$. Theorem 2.1.2 formalizes this idea.

Theorem 2.1.1.[14] Let m be a pseudometric on a set X and t a generator of an archimedean t-norm T. Then $E = t^{[-1]} \circ m$ is a T-indistinguishability operator on X.

N.B. Different generators of the same t-norm generate different T-indistinguishability operators.

Theorem 2.1.2.[14] Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonic map and t a generator of an archimedean t-norm T

$$E_f(x, y) = t^{[-1]} \left(|f(x) - f(y)|\right)$$

is an admissible T-indistinguishability operator.

Two monotonic maps f, g generate the same T-indistinguishability operator if and only if there exists a constant k such that $f = \pm g + k$.

In real situations where different experts produce different scales on $\mathbb{R}$ it is necessary to combine their information. The following theorem gives a way to do it.

Theorem 2.1.5. If $\{E_i\}_{i \in I}$ is a family of admissible T-indistinguishability operators on $\mathbb{R}$, then

$$E = \operatorname*{Inf}_{i \in I} E_i$$

is also an admissible T-indistinguishability operator on $\mathbb{R}$.

Therefore, we can combine admissible likeness or probabilistic relations generated by monotonic functions in order to obtain another equality on $\mathbb{R}$ of the same type.

2.2 Fuzzy numbers invariant under translations

Special families of fuzzy numbers are those who are invariant under translations. Roughly speaking, families formed by fuzzy numbers of the same “shape”. In the same sense, we can also consider T-indistinguishability operators, invariant under translations i.e.: the degree of similarity $E(x, y)$ between two numbers x, y depends only on their distance.

In this section, we study the link between the families of fuzzy numbers and T-indistinguishability operators invariant under translations.

Definition 2.2.1. A family $\{f_a\}_{a \in \mathbb{R}}$ of fuzzy numbers associated to an admissible T-indistinguishability operator is *invariant under translations* if and only if

$$f_a(x) = f_b(x - a + b).$$

A T-indistinguishability operator E is *invariant under translations* if and only if

$$E(x + a, y + a) = E(x, y).$$

Let us search for the conditions that such a family $\{f_a\}_{a \in \mathbb{R}}$ must fulfill in order to become the set of the singletons associated to a T- indistinguishability operator E, being T Archimedean.

Theorem 2.2.4.[7] Given an archimedean t-norm T with additive generator t, the families $\{f_a\}_{a \in \mathbb{R}}$ of fuzzy numbers invariant under translations corresponding to singletons of a T-indistinguishability operator on $\mathbb{R}$ can be obtained from f_0 (using definition 2.2.1) defined by

$$f_0 = t^{[-1]} \circ F$$

where F is an even and non decreasing in $\mathbb{R}^+$ subadditive function with $F(0) = 0$.

The associated T-indistinguishability operator E is defined by

$$E(x, y) = t^{[-1]} \circ F(y - x)$$

which is trivially invariant under translations.

In this section, we have introduced a general setting in order to generate fuzzy numbers associated to generalized equalities. The model allows the effective construction of families of fuzzy numbers suitable for applications since the underlying equalities can be deduced from scales proposed by experts and specially adapted to a concrete implementation.

3 Interpolation in CAGD

In CAGD, usually the user provides a set of points and the program produces a curve or surface whose shape is controlled by these points. From a classical point of view an interpolative curve means a curve passing through a given set of points. In our proposal we treat this problem in a more relaxed way in the sense that we obtain the values of the curve from a set of data points but we do not impose the condition that the resulting curve should interpolate in the classical sense. Since it is a design process, it is essential to control the degree of smoothness of the solution.

In this paper, we propose the following problems, paying special attention to the case of plane curves:

Problem 1a) *(Non parametric or functional curves).* Given $n + 1$ points of control $\overrightarrow{P_i} = (a_i, b_i) \in \mathbb{R}^2$, $i = 0, 1, \ldots, n$, with $a_0 < a_1 < ... < a_n$, we want to construct the graphic of a function $f : [a_0, a_n] \to \mathbb{R}$ such that $\{a_i\}$ are values of the independent variable and $\{b_i\}$ are values or approximations of $\{f(a_i)\}$.

Problem 1b) *(Parametric curves).* Given $n + 1$ points of control $\overrightarrow{P_i} = (a_i, b_i) \in \mathbb{R}^2$, or $\overrightarrow{P_i} = (a_i, b_i, c_i) \in \mathbb{R}^3$, $i = 0, 1, \ldots, n$ we want to construct $n + 1$ values $t_0 < t_1 < \ldots <$

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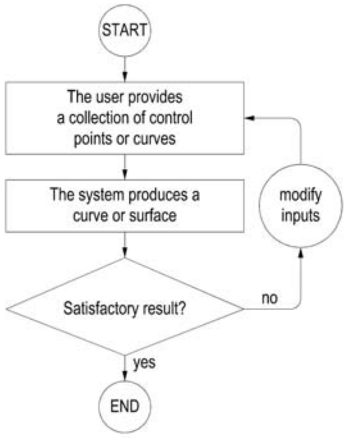


Figure 3.1

t_n of a parameter and a parametric curve $\overrightarrow{f} : [t_0, t_n] \rightarrow \mathbb{R}^2$ or $\mathbb{R}^3$ in such a way that the points $\{\overrightarrow{P}_i\}$ are the values or approximations of $\{\overrightarrow{f}(t_i)\}$.

Problem 2a) *(Non parametric or functional surfaces)*. Given $n+1$ arbitrarily distributed points of control $\overrightarrow{P_i} = (a_i, b_i, c_i) \in \mathbb{R}^3$, $i = 0, 1, \dots, n$, we want to construct a surface passing through or controlled by these points as the graphic of a map $f : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ (for example, $A = [a_{\min}, a_{\max}] \times [b_{\min}, b_{\max}]$) such that $\{(a_i, b_i)\}$ are the values of the independent variables and $\{c_i\}$ values or approximations of $\{f(a_i, b_i)\}$.

Problem 2b) *(Parametric surfaces)*. Given a net of $(n+1) \times (m+1)$ points $\overrightarrow{P}_{ij} = (a_{ij}, b_{ij}, c_{ij}) \in \mathbb{R}^3 \quad i = 0, \dots, n$, $j = 0, \dots, m$, we want to construct $n+1$ values $s_0 < s_1 < ... < s_n$ of a parameter s and $m+1$ values $t_0 < t_1 < ... < t_m$ of a parameter t and a parametric surface $\overrightarrow{f} : [s_0, s_n] \times [t_0, t_m] \rightarrow \mathbb{R}^3$ in such a way that the points $\{\overrightarrow{P}_{ij}\}$ are the values or approximations of $\{\overrightarrow{f}(s_i, t_j)\}$.

There are many standard techniques to treat the preceding problems. For a more complete information, also about methods to construct values of the parameter in parametric cases, the reader is refered to [9,10,11].

The common algorithm of all design methods that interacts between the procedure and the user is shown in the Fig. 3.1.

The algorithm shows the existing vagueness of the data points and its relation with the goal curves. Therefore, it is natural to focus the problem in the setting of approximate reasoning. The model proposed in section 4 uses fuzzy numbers and techniques of Takagi-Sugeno in fuzzy control to design curves and surfaces.

4 The Takagi-Sugeno Model

We propose a system based on fuzzy linguistic rules used in fuzzy control [5,18]. We have adopted the point of view of Takagi and Sugeno [23], in order to avoid the defuzzification process.

The main idea is to use rules of the following type:

If $\overrightarrow{x}$ is **close** to $\overrightarrow{x}_i$ in the domain, then $\overrightarrow{y}$ is **close** to $\overrightarrow{y}_i$ in the image. This idea is developed in the following **model**:

- Use $n+1$ rules

  R_i : If $\overrightarrow{x} \simeq \overrightarrow{x}_i$, then $\overrightarrow{y} \simeq \overrightarrow{y}_i$; $i = 0, 1, \dots, n$.

- Model the degree of fulfillment of the antecedent of the i^{th} rule with a fuzzy set α_i such that $\alpha_i(\overrightarrow{x}_i) = 1$ and $\alpha_i(\overrightarrow{x})$ is a decreasing function of the distance to $\overrightarrow{x}_i$ [6].

- The consequence function of each rule, $f_i(\overrightarrow{x})$, is the constant $\overrightarrow{y}_i$. (Therefore, the system coincides with the **height defuzzification method**).

- The output of the system is obtained as a mean of the outputs of each rule weighted by the fuzzy sets α_i:

$$\overrightarrow{y}(\overrightarrow{x}) = \frac{\sum_{i=0}^{n} \alpha_i(\overrightarrow{x}) \cdot \overrightarrow{y}_i}{\sum_{i=0}^{n} \alpha_i(\overrightarrow{x})} = \sum_{i=0}^{n} F_i(\overrightarrow{x}) \cdot \overrightarrow{y}_i \quad (2.1.)$$

where $F_i(\overrightarrow{x}_i) = \frac{\alpha_i(\overrightarrow{x})}{\sum_{j=1}^{n} \alpha_j(\overrightarrow{x})}$.

So, the result is a convex linear combination of the outputs of the rules where the coefficients depend on the data points and the fuzzy sets α_i. This is the general description of our proposal already outlined in [17] that contained the following remarks

4.1 Remarks

4.1.1. If the domain is in $\mathbb{R}$ (the case of a curve), then $\alpha_i = \mu_i$ are **fuzzy numbers**.

4.1.2. If the domain is in $\mathbb{R}^2$ (the case of a surface), then the closeness α_i of a point $\overrightarrow{x} = (x, y)$ to $\overrightarrow{x}_i = (x_i, y_i)$ can be modeled in the following way:
$\overrightarrow{x} = (x, y)$ is **close** to $\overrightarrow{x}_i = (x_i, y_i)$ if and only if x is **close** to x_i **and** y is **close** to y_i. That is, if μ_i and ν_i are fuzzy numbers modeling the closeness of x and y to x_i and y_i respectively, then the fuzzy set α_i can be defined as $\alpha_i(\overrightarrow{x}) = T(\mu_i(x), \nu_i(y))$, where T is a suitable t-norm that plays the semantic role of a conjunction. In order to maintain the degree of smoothness of the resulting surface inherited from the smoothness of the fuzzy numbers, it is convenient to choose a C^∞ t-norm. In our model, we use the **product**.

4.1.3. The fuzzy sets $\{\mu_i\}$ must cover the domain in the sense that for every point $\overrightarrow{x}$ in the domain, there must exists a fuzzy set μ_i with $\mu_i(\overrightarrow{x}) \neq 0$. This is a technical requirement in order to avoid dividing by zero in (2.1) and has the semantical meaning that the image of every point of the domain must be controlled by at least one of the given points $\{\overrightarrow{x}_i\}$.

4.1.4. Curves of type 1b) generated by this model cannot be, simultaneously, interpolative and smooth (i.e. tangent continuous). For interpolative curves $\overrightarrow{y}(t_i) = \overrightarrow{P}_i$, so $\mu_i(t_i) = 1$ and $\mu_j(t_i) = 0 \quad j \neq i$. Due to the monotonicity of the μ_i, $\mu_i(t) \equiv 0 \quad t \notin (t_{i-1}, t_{i+1})$. That means that for every interval $[t_{i-1}, t_i]$ there are only two fuzzy numbers different from zero. Consequently, in this interval the curve is a convex combination of the points $\overrightarrow{P}_{i-1}$ and $\overrightarrow{P}_i$ and we obtain a polygonal curve.

ISBN: 978-989-95079-6-8

4.1.5. It is worth noticing that, whereas in the classical methods used in CAGD only the selection of points is important, here we also have the possibility of choosing the type and size of the fuzzy number associated to each point as a consequence of the underlying indistinguisability, even within the same set of nodes. This aspect gives great versatility to our model.

From these remarks we can detect some problems in order to obtain a general procedure. The first one deals with the effective construction of the fuzzy numbers associated to the crisp data; that is, the fuzzification problem. If we consider that all the given points have the same weight in the construction of the curve or surface, it is natural to accept that the fuzzy points are invariant under translations and we can apply the ideas contained in the subsection 2.2. Otherwise, we can use the scales in order to graduate the importance of the points contained in different intervals of the domain. The fuzzification process of $\mathbb{R}$ is completely meaningful using the underlying equality relations. A second and important problem is mentioned in the remark 4.1.3., since our initial proposal is not applicable when dealing with sparse data. This problem has been extensively treated in the literature[8,10,19,20] from various points of view. Also, we can find many papers where different methods are compared [3,4,6,22]. In order to overcome this problem, we distinguish two different situations:

- a) The data set consists of equidistant x-values. To solve this case, we propose to build a family of parametrized fuzzy equalities in $\mathbb{R}$ whose associated fuzzy numbers cover all the domain for values of the parameter and for every $\vec{x_i}$ of the data set there exists a fuzzy number μ_i such that $\mu_i(\vec{x_j}) = 1$ if $i = j$ and $\mu_i(\vec{x_j}) = 0$ otherwise. Under this hypothesis we can obtain a curve that interpolates the points in the classical sense (see the example in section 5).

- b) The data has sparse non equidistant values. Our proposal consists in constructing a monotonic function f between a subset of the natural numbers and our numerable (finite) data set and using the fuzzy equality $E(x_i, x_j) = t^{-1} \circ F(f^{-1}(x_i) - f^{-1}(x_j))$ in order to define the fuzzy numbers associated to data.

With all this in mind, the solutions proposed to the problems of the introduction are the following:

4.2 Solutions.

Problem 1a) *(Non parametric curves).* In this case, $\overrightarrow{P}_i = (a_i, b_i)$, the domain is some interval in $\mathbb{R}$, $\overrightarrow{x}_i = a_i$, $\overrightarrow{y}_i = b_i$ and $\alpha_i = \mu_i$ are normalized fuzzy numbers. The solution curve is

$$y = f(x) = \frac{\sum_{i=0}^{n} \mu_i(x) \cdot b_i}{\sum_{i=0}^{n} \mu_i(x)} = \sum_{i=0}^{n} F_i(x) \cdot b_i. \tag{2.2}$$

Problem 1b) *(Parametric curves).* In this case, $\overrightarrow{P}_i = (a_i, b_i) \in \mathbb{R}^2$ or $\overrightarrow{P}_i = (a_i, b_i, c_i) \in \mathbb{R}^3$. The domain is an interval in $\mathbb{R}$, the variable is a parameter t, $\overrightarrow{x}_i = t_i$, $\overrightarrow{y}_i = \overrightarrow{P}_i$ and $\alpha_i = \mu_i$ are normalized fuzzy numbers. The solution curve is

$$\overrightarrow{y} = \overrightarrow{f}(t) = \frac{\sum_{i=0}^{n} \mu_i(t) \cdot \overrightarrow{P}_i}{\sum_{i=0}^{n} \mu_i(t)} = \sum_{i=0}^{n} F_i(t) \cdot \overrightarrow{P}_i. \tag{2.3}$$

Problem 2a). *(Non parametric surfaces).* In this case, $\overrightarrow{P}_i = (a_i, b_i, c_i) \in \mathbb{R}^3$, the domain is some region on $\mathbb{R}^2$, $\overrightarrow{x}_i = (a_i, b_i)$, $\overrightarrow{y}_i = c_i$ and $\alpha_i(x, y) = \mu_i(x) \cdot \nu_i(y)$ where μ_i, ν_i are normalized fuzzy numbers.

The solution surface is

$$z = f(x, y) = \frac{\sum_{i=0}^{n} \alpha_i(x, y) \cdot c_i}{\sum_{i=0}^{n} \alpha_i(x, y)} = \sum_{i=0}^{n} F_i(x, y) \cdot c_i. \tag{2.4}$$

Problem 2b). *(Parametric surfaces).*

In this case, $\overrightarrow{P}_{ij} = (a_{ij}, b_{ij}, c_{ij})$, the domain is a rectangle in $\mathbb{R}^2$, the variables are the parameters $s \in [s_0, s_n]$, $t \in [t_0, t_m]$, $\overrightarrow{x}_{ij} = (s_i, t_j)$, $\overrightarrow{y}_{ij} = \overrightarrow{P}_{ij}$ and $\alpha_{ij}(s, t) = \mu_i(s) \cdot \nu_j(t)$ where μ_i, ν_j are normalized fuzzy numbers.

The solution surface is

$$\overrightarrow{P}(s, t) = \frac{\sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{ij}(s, t) \cdot \overrightarrow{P}_{ij}}{\sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{ij}(s, t)} = \sum_{i=0}^{n} \sum_{j=0}^{m} F_{ij}(s, t) \overrightarrow{P}_{ij} \tag{2.5}$$

where

$$F_{ij}(s, t) = \frac{\alpha_{ij}(s, t)}{\sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{ij}(s, t)}$$

Note: Since in this case, the set of points determines a net, we have used a double index for convenience. If we want to use a single index k, we can, for instance, reassign indices in the following way: $\{i, j\} \to \{k\}$ with $k = i \cdot (m + 1) + j$.

5 An example

In this section we develop an example for fuzzy interpolation of curves and surfaces in its functional and parametric forms. In order to generate the underlying fuzzy equality in $\mathbb{R}$ invariant under translations, we need to select a subadditive, even and non decreasing function in $\mathbb{R}^+$. We have choosen $F = \text{Arctan}\,|x|$. This function fulfills all the conditions of theorem 2.2.1. On the other hand we have selected a parametrized family of strictly decreasing functions $f(x, k) : [0, 1] \to \mathbb{R}^+$ that, jointly with its inverses (Fig. 5.1 and Fig. 5.2), allows us to build a parametrized family of t-norms represented in Fig. 5.3 for $k = 5$.

Fig. 5.4 contains a representation of the associated T-indistinguishability that produces the fuzzification of $\mathbb{R}$ depicted in Fig. 5.5 for a discrete sub-family. By using this fuzzification and the procedure explained in section 4, we produce an interpolating curve for the case of equidistant values in functional form (Fig. 5.6).

Fig. 5.7 and Fig. 5.8 present two solutions of a curve constructed by means of its parametrized representation. As it is

ISBN: 978-989-95079-6-8

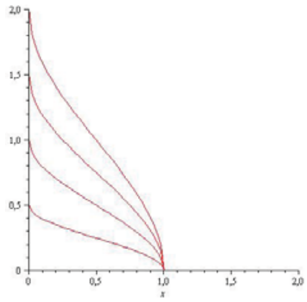


Figure 5.1

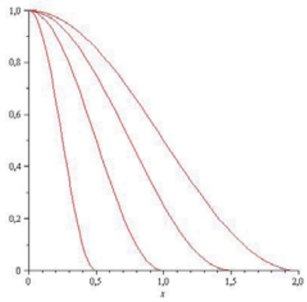


Figure 5.2

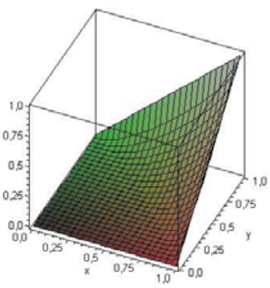
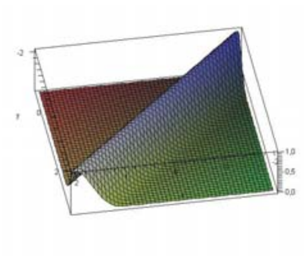


Figure 5.3



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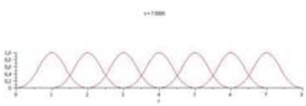

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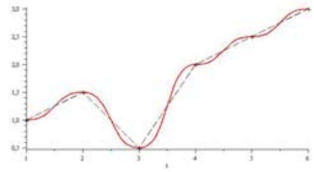

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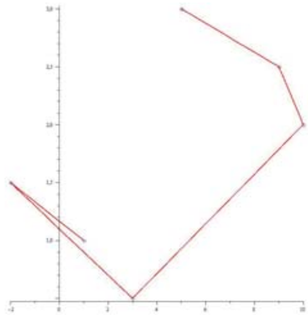

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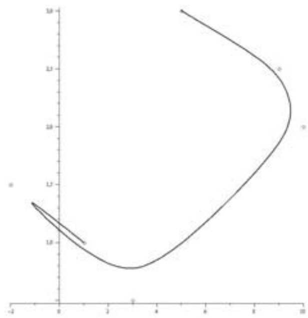

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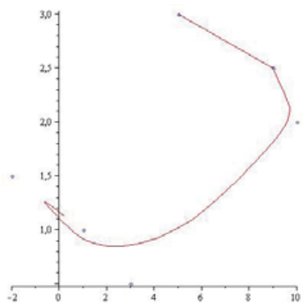

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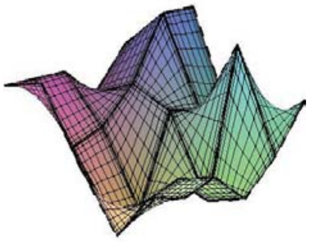

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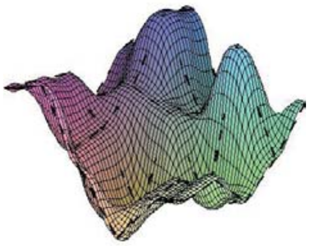

Figure 5.11

pointed out in the remarks, this representation can not be interpolative, in the classical sense, and, at the same time smooth.

Finally, in Fig. 5.9 a parametric curve is represented using a fuzzification of $\mathbb{R}$ defined by a scaled indistinguishability.

In a similar way we can construct surfaces.In this setting,the same incompatibility between interpolation and smoothnes, in the case on parametric representation, applies. Fig. 5.10 shows the case of interpolation for parametric representation. Fig. 5.11 shows the interpolative surface for the functional case.

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ISBN: 978-989-95079-6-8