KM-Fuzzy Approach Space

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Abstract— *Using the idea of approach space due to Robert Lowen [Approach spaces. The missing link in the topology-uniformity-metric triad], we define the notion of fuzzy approach spaces as a natural generalization of fuzzy metric spaces due to Kramosil and Michalek [Kybernetika , 11, 1975, 336–344] and prove some properties of fuzzy approach spaces.*

Keywords— Approach spaces, Fuzzy approach spaces, Fuzzy metric spaces.

1 Introduction

The notion of fuzzy sets was introduced by Zadeh [1]. Since then, several concepts of fuzzy metric spaces were considered in [2, 3, 4]. In the sequel, we shall adopt the usual terminology, notation and conventions of the theory of fuzzy metric spaces introduced by Kramosil and Michalek [5]. In this section we recall the definitions of a fuzzy metric space, approach space and some elementary properties.

Definition 1.1 *A binary operation* $*\colon [0,1]\times[0,1]\to[0,1]$ *is a continuous* t*-norm if* $([0,1],*)$ *is a topological monoid with unit 1 such that* $a*b\le c*d$ *whenever* $a\le c$ *and* $b\le d$ *for* $a,b,c,d\in[0,1]$.

Definition 1.2 *A* KM-fuzzy quasi-metric space *(briefly qFM-space) is a triple* $(X,M,*)$ *where* X *is an arbitrary set,* $*$ *is a continuous* t*-norm and* $M\colon X\times X\times[0,+\infty]\to[0,1]$ *is a (fuzzy) mapping such that for all* $x,y,z\in X$*:*

FM1. $M(x,y,0)=0$*;*

FM2. $x=y$ *if and only if* $M(x,y,t)=1$ *for all* $t>0$*;*

FM3. $M(x,z,t+s)\ge M(x,y,t)*M(y,z,s)$ *for all* $s,t\ge 0$*;*

FM4. $M(x,y,\cdot)\colon[0,+\infty)\to[0,1]$ *is left-continuous.*

If we additionally assume that M satisfies the symmetry condition:

$$M(x,y,t)=M(y,x,t) \text{ for all } t>0,$$

then $(X,M,*)$ *is called fuzzy metric space.*

Lemma 1.3 $M(x,y,\cdot)$ *is a nondecreasing function for all* $x,y\in X$.

This paper introduces the concept of a fuzzy approach space. Our definition is inspired in the concept of approach spaces introduced by Lowen [6].

Definition 1.4 *A pair* (X,δ) *where* $\delta: X\times\mathcal{P}(X)\to[0,+\infty]$ *is a function on a set X such that:*

B1. $\delta(x,\{x\})=0$ *for every* $x\in X$.

B2. $\delta(x,\emptyset)=\infty$ *for every* $x\in X$.

B3. $\delta(x,A\cup B)=\min(\delta(x,A),\delta(x,B))$ *for every* $x\in X$ *and* $A,B\in\mathcal{P}(X)$.

B4. $\delta(x,A)\le\delta(x,A^{(r)})+r$ *for every* $p\in X$, $A\in\mathcal{P}(X)$ *and* $r\in[0,\infty]$, *with* $A^{(r)}=\left\{y\in X \;/\; \delta(y,A)\le r\right\}$.

*is called an approach space (**A-space**).*

Lowen ([6], 1997) proved that at least seven different structures can be used to define an approach space, equivalently. Assuming condition B3, B4 can be rewritten as follows (see [7]):

B4′. $(A^{(r)})^{(s)}\subseteq A^{(r+s)}$ for every $A\subseteq X$ and $r,s\in[0,\infty]$.

2 KM-Fuzzy approach spaces

Following, the notions of approach space are generalized to the fuzzy setting and the concept of KM-fuzzy approach spaces is introduced.

Definition 2.1 *A* KM-fuzzy approach space *(briefly FA-space) is a pair* (X,F) *where* X *is an arbitrary set and* $F\colon X\times\mathcal{P}(X)\times[0,+\infty]\to[0,1]$ *is a (fuzzy) mapping satisfying the following conditions:*
For all $x\in X$, $A,B\subset X$ *and* $s,t\in[0,+\infty]$,

FA1. $F(x,A,0)=0$*;*

FA2. $F(x,\emptyset,t)=0$ *for all* $t<+\infty$*;*

FA3. $F(x,\{x\},t)=1$ *for all* $t>0$*;*

FA4. $F(x,A\cup B,t)\ge\max(F(x,A,t),F(x,B,t))$*;*

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FA5. $F(x,A,t+s) \geq F(x,A^{(r)},t)$, *if* $s > r \geq 0$,
$F(x,A,+\infty) \geq F(x,A^{(+\infty)},t)$,
where $A^{(r)} = \{y \in X : F(y,A,t) = 1, \forall t > r\}$,
and $A^{(+\infty)} = \{y \in X : F(y,A,+\infty) = 1\}$;

FA6. $F(x,A,\cdot)\colon [0,+\infty) \to [0,1]$ *is left-continuous for all* $x \in X$ *and* $A \subset X$.

The map F *is symmetric (or the FA-space* (X,F) *is symmetric) if*

$$F(x,\{y\},t) = F(y,\{x\},t) \text{ for all } x,y \in X,\ t > 0.$$

Firstly, we will prove that there are two families of spaces satisfying the above six properties.

Lemma 2.2 (A-space $\hookrightarrow$ FA-space) *Let* (X,δ) *be an A-space. If we define a fuzzy set* F_δ *on* $X \times \mathcal{P}(X) \times [0,+\infty]$ *as follows:*

$$F_\delta(x,A,t) = 0 \qquad \text{if } t \leq \delta(x,A),$$
$$F_\delta(x,A,t) = 1 \qquad \text{if } t > \delta(x,A),$$

then (X,F_δ) *is a FA-space.*

Proof. FA1, FA2 and FA3 are immediate. In order to prove FA4, if $A,B \subseteq X$ we can suppose $\min(\delta(x,A),\delta(x,B)) = \delta(x,A) \leq \delta(x,B)$ and, then $\max(F(x,A,t),F(x,B,t)) = F(x,A,t)$. Thus $F_\delta(x,A\cup B,t) = 0$ for $t \leq \delta(x,A)$; and $F_\delta(x,A\cup B,t) = 1$ for $t > \delta(x,A)$. This completes the proof of FA4.

For every $r \in [0,+\infty)$, the set $A^{(r)}$ is the same in **A-space** and in **FA-space**, since

$$A^{(r)}_{A-space} = \left\{ x \in X \;/\; \delta(x,A) \leq r \right\}$$
$$= \left\{ x \in X \;/\; F_\delta(x,A,t) = 1, \forall t > r \right\} = A^{(r)}_{FA-space}.$$

Let us suppose that $s > r$. Since $\delta(x,A) \leq \delta(x,A^{(r)}) + r$, thus for $t \leq \delta(x,A) - s$, we have

$$t + r < t + s \leq \delta(x,A) \leq \delta(x,A^{(r)}) + r;$$

i.e., $t < \delta(x,A^{(r)})$ and $F_\delta(x,A,t+s) = F_\delta(x,A^{(r)},t) = 0$ for $t + s \leq \delta(x,A)$.

Otherwise, if $t + s > \delta(x,A)$, then $F_\delta(x,A,t+s) = 1 \geq F_\delta(x,A^{(r)},t)$. This completes the proof.

Theorem 2.3 (FM-space $\hookrightarrow$ FA-space) *Let* $(X,M,*)$ *be a FM-space. If we define a fuzzy set* F *on* $X \times \mathcal{P}(X) \times [0,+\infty]$ *for every* $x \in X$, $t \in [0,\infty]$ *and* $A \subseteq X$ *as follows:*

$$F_M(x,A,t) = \sup_{a\in A} M(x,a,t),$$

then (X,F_M) *is a FA-space.*

Proof. By FM1 and FM2, conditions FA1 and FA3 hold.

The proof of the second condition FA2 follows from the definition of $supA$ for empty sets, $sup\emptyset = 0$. If $x \in X$ and $A,B \subseteq X$ are subsets of X, then

$$\mathbf{Max}\left(\sup_{a\in A} M(x,a,t),\ \sup_{b\in B} M(x,b,t)\right) = \sup_{a\in A\cup B} M(x,a,t).$$

Consequently FA4 is proved.

We will prove now FA5. Let $x \in X$, $A \in \mathcal{P}(X)$ and $r \in [0,\infty]$. If $A = \emptyset$, the condition FA5 is trivial. It is enough to prove that the property FA5 holds true for every $A \neq \emptyset$. Since, for every $x,y,z \in X$, $t,s > 0$

$$M(x,z,t+s) \geq M(x,y,t) * M(y,z,s);$$

in particular,

$$M(x,y,t+s) \geq M(x,a,t) * M(a,y,s)$$

for every $x \in X$, $a \in A^{(r)}$, $y \in A$, $t > 0, s > r \geq 0$.

Taking supremum over $y \in A$, then

$$F_M(x,A,t+s) = \sup_{y\in A} M(x,y,t+s)$$
$$\geq M(x,a,t) * \sup_{y\in A} M(a,y,s)$$
$$= M(x,a,t) * F_M(a,A,s).$$

Since, $a \in A^{(r)}$, then $F_M(a,A,s) = 1$ for $s > r$. Thus

$$F_M(x,A,t+s) \geq M(x,a,t).$$

Taking supremum over $a \in A^{(r)}$, we have

$$F_M(x,A,t+s) \geq F_M(x,A^{(r)},t).$$

Note that additionally the following condition is satisfied:

FA7 Let $x,y \in X$ be such that $F_M(x,\{y\},t) = 1$ for every $t > 0$, then $x = y$.

3 Characterization

Now we ask about the maps $F : X \times \mathcal{P}(X) \times [0,+\infty] \to [0,1]$ that could provide to X of a FA-space structure. In this cases, it could be interesting to weaken axioms FA4 and FA5.

Proposition 3.1 *Let* X *be a non-empty set,* $F : X \times \mathcal{P}(X) \times [0,+\infty] \to [0,1]$ *be a map satisfying axiom FA2 and such that* $F(x,A,t) = 1$ *for every* $t > 0$ *and* $x \in A \subseteq X$. *Then:*

The axiom FA4 is equivalent to

FA4′. $F(x,A\cup B,t) \geq \mathbf{Max}(F(x,A,t),F(x,B,t))$ *for* $t > 0$, *for every non-empty subsets* $A,B \subseteq X$ *such that* $A \neq B$ *and* $A\cup B \subset X$, *and for each* $x \in X\setminus(A\cup B)$.

The axiom FA5 is equivalent to

FA5′. $F(x,A,t+s) \geq F(x,A^{(r)},t)$ *for* $t > 0$, *for every* $s > r \geq 0$, $\emptyset \neq A \subset X$ *and for each* $x \in X\setminus A$.

Proof. We prove that there exists some cases in which FA4 and FA5 are always true.

Property FA4′. If A is empty, then $F(x,A\cup B,t) = F(x,B,t)$ and, by the axiom FA2

$$\mathbf{Max}(F(x,A,t),F(x,B,t)) = \mathbf{Max}(F(x,\emptyset,t),F(x,B,t))$$
$$= \mathbf{Max}(0,F(x,B,t)) = F(x,B,t).$$

A similar conclusion is valid for $B = \emptyset$. We will suppose that A and B are non-empty sets. If $x \in A\cup B$, then, by hypothesis,

$$F(x,A\cup B,t) = 1 \geq \mathbf{Max}(F(x,A,t),F(x,B,t)).$$

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We can suppose that $x \notin A \cup B$, and thus $A \cup B = X$ is impossible. So, $A \cup B \subset X$. If $A = B$, the inequality is trivial.

Property FA5$'$. If $r = +\infty$, the inequality $F(x,A,+\infty) \geq F(x,A^{(+\infty)},t)$ is obvious. Therefore we can suppose that $r < +\infty$. If A is empty, then for each $r > 0$,

$$\emptyset^{(r)} = \left\{ x \in X \ / \ F(x,\emptyset,t) = 1, \forall t > r \right\} = \emptyset,$$

and then in FA5 we have the equality. We can suppose that A is non-empty. If $x \in A$, by hypothesis, $F(x,A,t) = 1$ for all $t > 0$. Then

$$F(x,A,t+s) = 1 \geq F(x,A^{(r)},t).$$

Consequently, we can suppose that $x \notin A$.

4 Properties

The *closure* of a subset A of a $FA-space$ (X,F) can be expressed by

$$\overline{A} = A^{(0)} = \{x \in X : F(X,A,t) = 1, \forall t > 0\}.$$

Although the set X has not been endowed with a topology, the following definition can be considered.

Definition 4.1 *A is called **closed** if $\overline{A} \subseteq A$.*

Lemma 4.2 *Let (x,F) be a **FA-space**, $x \in X$ and $A \subseteq X$.*

1. *If $a \in A$, then $F(a,A,t) = 1$ for each $t > 0$.*
2. *If $a \in A$, then $F(x,a,t) \leq F(x,A,t)$ for each $t > 0$.*
3. *If $A \subseteq B \subseteq X$, then $F(x,A,t) \leq F(x,B,t)$ for each $t > 0$.*
4. $\sup_{a \in A} F(x,a,t) \leq F(x,A,t)$ *for each $t > 0$.*
5. *If $A \subseteq B \subseteq X$, then $A^{(r)} \subseteq B^{(r)}$ for each $r \geq 0$.*
6. *Let $0 \leq r \leq s$, then*
$$A \subseteq \overline{A} = A^{(0)} \subseteq A^{(r)} \subseteq A^{(s)} \subseteq \cdots \subseteq X.$$
In particular, $F(x,A,t) \leq F\left(x,\overline{A},t\right)$.
7. *A is closed if and only if $\overline{A} = A$.*
8. *If $r \geq 0$, then $\emptyset^{(r)} = \emptyset$.*
9. *$x \in A^{(r)}$ if and only if $x \in A^{(r+s)}$ for every $s > 0$.*
10. *The infimum* $\inf_{[0,+\infty)}(\{r \geq 0 \ / \ x \in A^{(r)}\})$ *is really a minimum, i.e.,*
$$A^{(r)} = \bigcap_{r<s} A^{(s)}.$$
11. *$F(x,A,\cdot)$ is a nondecreasing function.*

Proof.

1. By the axioms FA3 and FA4, if $a \in A$, we have
$$F(a,A,t) = F(a,A \cup \{a\},t)$$
$$\geq \mathbf{Max}(F(a,A,t), F(a,\{a\},t)) = \mathbf{Max}(F(a,A,t),1) = 1.$$
2. By the axiom FA4, if $a \in A$, we have
$$F(x,A,t) = F(x,B \cup \{a\},t)$$
$$\geq \mathbf{Max}(F(x,B,t), F(x,\{a\},t)) \geq F(x,\{a\},t),$$
where $B = A \backslash \{a\}$.
3. If $A \subseteq B$, then $A \cup B = B$, and thus
$$F(x,B,t) = F(x,A \cup B,t) \geq \mathbf{Max}(F(x,A,t), F(x,B,t))$$
$$\geq F(x,A,t).$$
4. By 2, if $a \in A$, then $F(x,a,t) \leq F(x,A,t)$. Taking supremum
$$\sup_{a \in A} F(x,a,t) \leq F(x,A,t).$$
5. If $A \subseteq B$, by 3
$$F(x,A,t) \leq F(x,B,t).$$
If $x \in A^{(r)}$, $F(x,A,t) = 1$ for each $t > r$ and then $F(x,B,t) = 1$ for each $t > r$. That is, $x \in B^{(r)}$.
6. If $a \in A$, by 1, $F(a,A,t) = 1$ for each $t > 0$. Thus $a \in \overline{A}$.
Let $r \leq s$. If $x \in A^{(r)}$, $F(x,A,t) = 1$, for each $t > r$, then, in particular, $F(x,A,t) = 1$ for each $t > s$. Thus $x \in A^{(s)}$.
7. Trivial.
8. By FA2, it is immediate.
9. $x \in A^{(r)}$ if and only if $F(x,A,t) = 1$ for all $t > r$. This condition is equivalent to the following for each $s > r$,
$$F(x,A,t) = 1 \text{ for all } t > s.$$
Then,
$$x \in A^{(r)} \Leftrightarrow x \in \bigcap_{r<s} A^{(s)} \Leftrightarrow x \in \bigcap_{s>0} A^{(r+s)}.$$
That is, $x \in A^{(r)}$ if and only if $x \in A^{(r+s)}$ for all $s > 0$.
10. By 3, 6 and FA5
$$F(x,A,t+s) \geq F(x,A^{(r)},t) \geq F(x,A,t),$$
if $s > 0$.

Next, the previous properties are refined.

Lemma 4.3 *Let (X,F) be a FA-space, $x \in X$, $t \geq 0$ and $A \subseteq X$.*

1. $F(x,A,t) = F\left(x,\overline{A},t\right)$.
2. *The closure $\overline{A}$ is a closed set, that is, $\overline{\overline{A}} = \overline{A}$.*
3. *A is closed if and only if for each $x \in X$ satisfying $F(x,A,t) = 1$, with $t > 0$, then $x \in A$.*
4. *$A^{(r)}$ is closed for every $r \geq 0$.*
5. *(X,F) satisfies the axiom FA7 if and only if the points are closed subsets, that is, $\overline{\{x\}} = \{x\}$.*
6. *$(A^{(r)})^{(s)} \subseteq A^{(r+s)}$ for every $r,s \geq 0$.*

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Proof.

1. Since $A \subseteq \overline{A}$, by 3 of lemma 4.2, $F(x,A,t) \leq F\left(x,\overline{A},t\right)$. Using axiom FA5 with $r=0$, it holds
$$F(x,A,t+s) \geq F(x,A^{(0)},t) = F(x,\overline{A},t)$$
for $s>0$.
Taking infimum on s, we have $F(x,A,t) \geq F(x,\overline{A},t)$.

2. By 6 of lemma 4.2, $\overline{A} \subseteq \overline{\overline{A}}$. Let $x \in \overline{\overline{A}} = A^{(0)^{(0)}}$. This is equivalent to
$$F\left(x,A^{(0)},t\right) = 1$$
for all $t>0$. Using axiom FA5 with $r=0$,
$$F(x,A,t+s) \geq F(x,A^{(0)},t) = 1$$
for $t,s>0$. Then $x \in A^{(0)} = \overline{A}$.

3. Trivial.

4. Let $x \in X$ be such that $F\left(x,A^{(r)},t\right) = 1$ for all $t>0$. By FA5,
$$F(x,A,t+s) \geq F\left(x,A^{(r)},t\right) = 1$$
for all $t>0$, $s>r\geq 0$. In particular $x \in A^{(r)}$. The previous property implies that $A^{(r)}$ is closed.

5. Trivial.

6. Let $r,s \geq 0$ and $x \in (A^{(r)})^{(s)}$, that is,
$$F\left(x,A^{(r)},t\right) = 1$$
for all $t>s$. By FA5,
$$F(x,A,t+s) \geq F\left(x,A^{(r)},t\right) = 1$$
for $t>s>r\geq 0$. Thus $x \in A^{(r+s)}$.

The following proposition is an interesting property related to closures of subsets of FA-spaces.

Proposition 4.4 *Let (X,F) be a FA-space, $r \geq 0$ and $A,B \subseteq X$ subsets of X.*

1. $A^{(r)} \cup B^{(r)} \subseteq (A\cup B)^{(r)}$.

2. *If (X,F) verifies the equality in FA4, then* $(A\cup B)^{(r)} = A^{(r)} \cup B^{(r)}$.

Proof. Since $A \subseteq A\cup B$ and $B \subseteq A \cup B$, by 5 in lemma 4.2,
$$A^{(s)} \subseteq (A\cup B)^{(s)}, \qquad B^{(s)} \subseteq (A\cup B)^{(s)}.$$
Then $A^{(s)} \cup B^{(s)} \subseteq (A\cup B)^{(s)}$.

Suppose that the equality holds in FA4. If $x \in (A\cup B)^{(r)}$, then
$$\mathbf{Max}(F(x,A,t), F(x,B,t)) = F(x,A\cup B,t) = 1,$$
for all $t>r$.

Then, for each $t>r$,
$$F(x,A,t) = 1 \text{ or } F(x,B,t) = 1.$$
Suppose that $t>r$ is such that $F(x,A,t)=1$. If $s>0$, then
$$F(x,A,t+s) = F(x,A^{(r)},t) \geq F(x,A,t) = 1.$$
Then $F(x,A,t)=1$ for all $t>r$; i.e., $x \in A^{(r)}$.

Consequently, $x \in A^{(r)}$ or $x \in B^{(r)}$and then $x \in A^{(r)} \cup B^{(r)}$.

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