Decision Making in Competitive Location using Fuzzy Sets

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Abstract*—This paper deals with the competitive location problems using fuzzy sets. The basic notions on fuzzy optimization and linear programming using fuzzy sets are briefly reviewed. The standard leader-follower location problem and its linear mathematical programming formulation are described. The works appeared in the literature concerning the use of fuzzy sets are analyzed.*

Keywords— Fuzzy Optimization, Competitive Location, Leader-Follower problem.

1 Introduction

Competitive location models represent the competition between two or more firms which provide goods or services to customers. The natural objective of each firm is to maximize its profit, which is often replaced by the maximization of the market share. That is, each firm tries to capture as many customers (or demand) as possible. The problem of locating facilities in a competitive environment has been addressed in several papers in the field of Operations Research, Management Science, Regional Economy and Economic Geography, see e.g. Craig et al. (1984) [9], Drezner (1995) [13], Eiselt and Laporte (1989,1996) [14][15], Friesz et al. (1988) [16], Hakimi (1990) [17], Plastria (2001) [22], Santos-Peñate et al. (2007) [23], Spaulding and Cromley (2007) [25], Till (1992, 2000) [26][27], and the references therein.

In many cases, the real competitive contexts cannot be described with precision. The complex, subjective, and dynamic behavior of real customers produces a high degree of uncertainty related to their decision making process. Consequently the most common scenario corresponds to problems where the information of the parameters and variables is vague or imprecise. Fuzzy sets theory is a rigorous and effective instrument to treat problems where the necessary information is imprecise. In fuzzy optimization two families of problems exist. On the one hand we have problems where the objective function and the constraints are fuzzy and, on the other hand, those where the coefficients are fuzzy. Fuzzy functions are characterized by their membership function. Fuzzy decisions are the intersection of fuzzy sets corresponding to the objective function and to the constraints. Linguistic variables can also be considered (See Zadeh [34]).

In this work we consider the application of fuzzy optimization methodologies to competitive location problems. The paper is organized as follows. In section 2, we describe the fuzzy optimization models. The basic concepts of competitive location are presented in Section 3. Section 4 is devoted to the solution approaches to competitive location problems using fuzzy optimization and fuzzy sets. Finally, Section 5 contains the conclusions.

2 Fuzzy Optimization Problems

Optimization in their most general form involves finding optimal solutions according to stated criteria. This task is usually formulated as optimization problems using objective functions and constraints. The procedures use mathematical properties of the objective and constraints functions. In practice, however, many situations lack the exact information that is needed in the problem, including its objective and constraints, or in other cases, where it is unreasonable to access such specific constraints or clearly defined objective functions. In these situations it is advantageous to model and solve the problem using soft computing and fuzzy techniques.

An *optimization problem* consists of finding the value of the *decision variables* so that an *objective function* is minimized or maximized when the possible values of the variables are subject to a set of *constraints*. The objective function, denoted by f, is defined on a set of solutions denoted by X and the constraints are given by a vector function g in such a way that the problem is formulated as follows:

ISBN: 978-989-95079-6-8

Minimize

$$f(\vec{x})$$

subject to: (1)

$$g(\vec{x}) \leq \vec{0}$$

$$\vec{x} \in X$$

where $\vec{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the vector of decision variables, f and g are defined on $\mathbb{R}^n$ and X is a subset of $\mathbb{R}^n$. The mathematical formulations are presented in this paper using "minimize" operator because the other case, the maximization case, is analogous.

Among the optimization problems that are included in Mathematical Programming, the models that have received the most attention and have offered the most useful applications in different areas are Linear Programming (LP) models, which is the single objective linear case.

The classic problem of LP is to find the maximum or minimum values of a linear function subject to constraints that are represented by linear equations or inequalities. The most general formulation of this problem is:

Minimize

$$\vec{c}^t \vec{x}$$

subject to: (2)

$$A\vec{x} \leq \vec{b}$$

$$\vec{x} \geq \vec{0}$$

where $\vec{c} = (c_1, c_2, ..., c_n) \in \mathbb{R}^n$ is the objective vector, $\vec{b} = (b_1, b_2, ..., b_m) \in \mathbb{R}^m$ is the right hand side vector, and $A = [a_{ij}]$ is an *(n,m)*-matrix where a_{ij} is the coefficient of variable x_j in constraint *i*. In an economic context where the aim is the minimization of the total cost with a limitation of the resources, $\vec{c}$ is the cost vector, $\vec{b}$ is the vector of resources and A is known as the technological matrix. An equivalent formulation of the linear problem is the following:

Minimize

$$\sum_{j=1}^{n} c_i x_j$$

subject to: (3)

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, 2, ...m$$

$$x_j \geq 0, \quad j = 1, 2, ..., n$$

In many real situations not all the constraints and objective functions can be valued in a precise way and in these situations we are dealing with the general problem form of Fuzzy Linear Programming (FLP). FLP is characterized as follows: a_{ij}, b_i and c_j can be expressed as *fuzzy numbers*, x_i as variables whose states are fuzzy numbers, addition and multiplication operates with fuzzy numbers, and the inequalities are among fuzzy numbers. Different FLP models can be considered according to the elements that contain imprecise information, this is the criterion used in the classification proposed in Verdegay (1995) [33] and Cadenas and Verdegay (1999) [1].

If we take as a basis the classification proposed in Verdegay [33], we have models with the fuzzy feasible sets, models with fuzzy goals, models with fuzzy coefficients of objective function, models with coefficients of the technological matrix and fuzzy right hand sides, and totally fuzzy models where all the elements of the problem are fuzzy.

1. Models with fuzzy objective function

These models are those whose objective function is not fully known. In LP problems where the costs are known with imprecision they are represented by an *n*-dimensional fuzzy vector $\utilde{c} = (\utilde{c}_1, \utilde{c}_2, ..., \utilde{c}_n)$, leading to the following model:

Minimize

$$z = \utilde{\vec{c}}^t x$$

subject to (4)

$$A\vec{x} \leq \vec{b}$$

$$\vec{x} \geq \vec{0}$$

Evidently, z is also a fuzzy number, but $\vec{x}$ can be a vector of fuzzy or non-fuzzy numbers, and each fuzzy cost is described by its corresponding membership function $\mu_j(c)$. Each coefficient $\utilde{c}_{kj}$ of the objective function is a plane fuzzy number of the L-R type with modal interval $\left[\underline{c}_j, \overline{c}_j\right]$ and membership functions g_j and h_j (which can be linear, parabolic, etc.). Delgado et al. (1989) [11] prove that the solution can be obtained with the multi-objective auxiliary model:

Minimize

$$z = \left[\vec{c}_1^t \vec{x}, \vec{c}_2^t \vec{x}, ..., \vec{c}_{2^n}^t \vec{x}\right]$$

subject to

$$A\vec{x} \leq \vec{b} \qquad (5)$$

$$\vec{x} \geq \vec{0},$$

$$\alpha \in [0,1], c_{kj} \in \left\{g_j^{-1}(1-\alpha), h_j^{-1}(1-\alpha)\right\}$$

$$k = 1, ..., 2^n, j = 1, ..., n.$$

2. Models with feasible fuzzy set (fuzzy constraints)

This is the case where constraints can be satisfied, and consequently the feasible region, can be defined as a fuzzy set; it should be defined by means of a membership function $\mu : \mathbb{R}^n \to [0,1]$. In such a situation, for each constraint *i*, a desirable quantity b_i is considered, but the possibility that it is greater is accepted until a maximum $b_i + t_i$ (t_i is referred

ISBN: 978-989-95079-6-8

to as a violation *tolerance level*). This model is represented by

Minimize

$z = \vec{c}^{\,t}\vec{x}$

subject to: (6)

$A\vec{x} \lesssim \vec{b}$

$\vec{x} \geq \vec{0}$

where the symbol $\lesssim$ indicates the imprecision of the constraints and where each fuzzy constraint $\vec{a}_i^t x \lesssim \vec{b}_i$ is specified by a membership function in the form:

$$\mu_i\left(\vec{a}_i^t\vec{x}\right) = \begin{cases} 1 & \text{if } \vec{a}_i^t\vec{x} < b_i \\ f_i\left(\vec{a}_i^t\vec{x}\right) & \text{if } b_i \leq \vec{a}_i^t\vec{x} \leq b_i + t_i \\ 0 & \text{if } b_i + t_i < \vec{a}_i^t\vec{x} \end{cases} \quad (7)$$

where $\vec{a}_i = \left(a_{i1}, a_{i2}, \ldots, a_{in}\right) \in \mathbb{R}^n$ is the vector of coefficients for constraint *i*. Expression (7) means that, for each constraint *i*, given the level of tolerance t_i, each point (*n*-dimensional vector) $\vec{x}$ *is* associated with a number $\mu_i(\vec{x}) \in [0,1]$ which is known as the degree of fulfillment (or verification) of the constraint *i*. The functions f_i are assumed continuous and monotonous non-decreasing functions. In particular, Verdegay (1982) [32], using the representation theorem for fuzzy sets, proves that the solutions for the case of linear functions f_i can be obtained from the auxiliary model:

Minimize

$z = \vec{c}^{\,t}\vec{x}$

subject to (8)

$A\vec{x} \leq \vec{b} + \left(1-\alpha\right)\vec{t}^{\,t}$

$\vec{x} \geq \vec{0}, \alpha \in \left[0,1\right]$

where $\vec{t} = \left(t_1, t_2, \ldots, t_m\right)$.

3. Models with fuzzy goals

A fuzzy optimization problem with fuzzy goals is one whose goal set is fuzzy, that is, it allows the objective function value to be slightly below the minimum goal for a maximization problem, and similarly for a minimization problem. The corresponding linear model is expressed in the following way:

$\widetilde{\text{Minimize}}$

$z = \vec{c}^{\,t}\vec{x}$

subject to (9)

$A\vec{x} \leq \vec{b}$

$\vec{x} \geq 0$

If t_0 is the maximum quantity that the function objective should be inferior to the minimum goal c_0, then each vector $\vec{x}$ is associated with a number $\mu_0\left(\vec{x}\right)$, which represents the degree that the decision maker considers to be an achieved goal. It is defined according to the following function:

$$\mu_0\left(\vec{x}\right) = \begin{cases} 1 & \text{if } \vec{c}^{\,t}\vec{x} > c_0 \\ f_0\left(\vec{c}^{\,t}\vec{x}\right) & \text{if } c_0 - t_0 \leq \vec{c}^{\,t}\vec{x} \leq c_0 \\ 0 & \text{if } \vec{c}^{\,t}\vec{x} < c_0 - t_0 \end{cases} \quad (10)$$

where f_0 is a continuous monotonous non-decreasing function. The corresponding satisfactory solutions can be obtained by solving the equivalent problem when a level for the α-cuts is chosen.

4. Models with fuzzy coefficients in the technological matrix

Consider a problem of this type:

Minimize

$z = \vec{c}^{\,t}\vec{x}$

subject to (11)

$\underset{\sim}{A}\vec{x} \lesssim \underset{\sim}{\vec{b}}$

$\vec{x} \geq \vec{0}$

where the values of the technological matrix and the right hand sides are fuzzy numbers. Fuzzy constraints can also be included. Delgado et al. (1987) [10] also include imprecision in the constraints. They consider the fuzzy relations of the constraints with the application of a ranking function *g* to compare the fuzzy terms. This new formulation is expressed by the auxiliary problem:

Minimize

$z = \vec{c}^{\,t}\vec{x}$

subject to (12)

$\underset{\sim}{a}_i\vec{x} \leq_g \underset{\sim}{b}_i + \underset{\sim}{t}_i(1-\alpha), \; i = 1, \ldots m$

$\vec{x} \geq \vec{0}, \alpha \in \left[0,1\right]$

3 Competitive Location

In competitive location problems two or more firms compete by mean of their locations for providing services or products to customers. Several scenarios may be considered:

- No firm operates in the market and firm F_1 wants to enter the market with p_1 facilities.
- There are *s* firms F_i operating in the market, each firm F_i has p_i facilities and a new firm F_{s+1} wants to enter the market with p_{s+1} facilities.
- There are *s* firms F_i operating in the market, each firm F_i has p_i facilities and a firm F_k wants to open $\overline{p}_k$ facilities or to close $\underline{p}_k$ facilities.

In particular, the Stackelberg location model is a standard two-stage problem where a firm, the leader, chooses its location points and then a competitor, the follower, enters

ISBN: 978-989-95079-6-8

the market and decides its locations taking into account the leader position. The problem of the follower is to find the optimal locations given the position of the leader, the problem of the leader is to determine the best locations taking into account the reaction of the follower to any possible strategy of a competitor.

Different objectives can be considered:

- The maximization of the market share.
- The minimization of the competitor market share.
- The maximization of the difference between its market share and the competitor market share.

For modelling the customer's behaviour in the problem we can use different rules:

- *Binary choice rule*. A choice rule where the customers patronize their closest facility.
- *Partially binary choice rule*. A choice rule where each customer patronizes the closest facility of each firm.
- *Proportional choice rule.* With this choice rule, customers patronize all the facilities according to a non-increasing function of the travel distance.
- *Threshold choice rule.* A customer choice rule where a threshold-sensitive behaviour is assumed.

Moreover, whether goods are essential or not, demand are said to be inelastic (constant demand) or elastic (demand varies with distance). The customer behaviour is modelled by their optimization problems that result from the application of the choice rule for them.

For binary inelastic demand, the leader, follower and customer decision problems can be formulated as linear optimization models (see Campos-Rodríguez et al. [6]). The problems are stated as follows.

Let $m = |L|$ be the number of possible facility locations and $n = |C|$ be the number of customer locations. Let $d_{ki} = d(c_k,f_i)$ be the distance between the k-th customer location c_k and the i-th facility point f_i. In addition, let h_k be the total demand of the customers located at c_k. Finally, let $K = \{1, 2, ..., n\} = [1..n]$ denote the index set for the customer locations and $I = \{1, 2, ..., m\} = [1..m]$ denote the index set for the facility locations.

A set Z of location points is identified by a binary vector z with size m. This vector is $z = (z_1, z_2,..., z_m) = (z_i, i \in [1..m])$ where $z_i = 1$ if $f_i \in Z$ and $z_i = 0$ otherwise. Thus the corresponding set is given by $Z = \{ f_i \in L: z_i = 1 \}$.

The decision variables in the leader and follower location problems are the m-vectors x and y of 0-1 or binary decision variables corresponding to the sets X and Y.

For the customer problem, let z_{ki} be the 0-1 decision variables indicating whether the customers located at the k customer location c_k prefer the location f_i for the facility. However, in each customer problem, the sets X and Y are data that are represented by corresponding m-vectors of 0-1 values $\overline{x}$ and $\overline{y}$ for the variables x and y.

For the follower location problem we have an m-vector of values $\overline{x}$ and an m-vector of variables y. The binary choice rule oriented to the leader implies that the scalar product $\overline{x} \cdot \overline{y}$ is zero.

Using the coefficients b_{ij}^k and c_{ij}^k given, for any $i,j \in [1..m]$, $k \in [1..n]$, respectively, by:

$$b_{ij}^k = \begin{cases} 1 & d_{ki} \le d_{kj} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

and

$$c_{ij}^k = \begin{cases} 1 & d_{ki} < d_{kj} \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

The customer decision problem of the customers at location c_k, for every $k \in [1..n]$, is the linear feasible problem in the variables z_{ki}, for any $i \in [1..m]$, given by:

$$\begin{aligned} &\sum_{i=1}^{m} z_{ki} = 1 \\ &z_{ki} \le \overline{x}_i + \overline{y}_i && i \in [1..m] \\ &z_{ki} \le 1 - c_{ji}^k \overline{y}_j - b_{ji}^k \overline{x}_j - b_{ij}^k \overline{x}_i && i, j \in [1..m] \\ &z_{ki} \in \{0,1\} && i \in [1..m] \end{aligned} \tag{15}$$

For the follower problem consider the coefficients c_{ki} given, for any $k \in [1..n]$ and $i \in [1..m]$, by

$$c_{ki} = \begin{cases} 1 & d_{ki} < \min\{d_{kj} : \overline{x}_j = 1\} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

Then, the follower problem is the linear optimization problem in the variables z_{ki}, $k \in [1..n]$, $i \in [1..m]$, given by:

$$\begin{aligned} &\text{maximize} \\ &\quad \sum_{i=1}^{m} \sum_{k=1}^{n} h_k z_{ki} \\ &\text{such that:} \\ &\quad \sum_{i=1}^{m} y_i = r \\ &\quad \sum_{i=1}^{m} z_{ki} \le 1 && k \in [1..n] \\ &\quad z_{ki} - c_{ki} y_i \le 0 && k \in [1..n]; i \in [1..m] \\ &\quad z_{ki}, y_i \in \{0,1\} && k \in [1..n]; i \in [1..m] \end{aligned} \tag{17}$$

Finally, consider the set J of all the possible selection for the follower. The leader problem is the linear optimization problem in the variables w and for any $k \in [1..n]$, $i \in [1..m]$ and $j \in J$, by

ISBN: 978-989-95079-6-8

$$z_{ki}^{j} = \begin{cases} 1 & d_{ki} \le \min\{d_{kj} : j \in J\} \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

given by:

minimize

w

such that:

$$\sum_{i=1}^{m} x_i = p$$
$$\sum_{k=1}^{n} h_k - \sum_{i=1}^{m}\sum_{k=1}^{n} h_k z_{ki}^{j} \le w \quad j \in J \quad (19)$$
$$\sum_{i=1}^{m} z_{ki}^{j} \le 1; j \in J \quad k \in [1..n]$$
$$z_{ki}^{j} - c_{ki}^{j} x_i \le 0 \quad k \in [1..n]; i \in [1..m]; j \in J$$
$$z_{ki}^{j}, x_i \in \{0,1\} \quad k \in [1..n]; i \in [1..m]; j \in J$$
$$w \ge 0$$

See Campos-Rodríguez et al. (2009) [6] for details.

4 Solution Approaches

Several kind of uncertainty have been considered in Location problems (Benati (2000) [2], Benati and Hansen (2000) [1], Devletoglou (1965) [12], and Sikdera et al. (2007) [24]). Fuzzy sets have been used in several location problems (see Canós et al. (1998) [7]), In general, the user preferences have uncertainty and the corresponding utilities are uncertain. Therefore their choice must be described using a membership function and the competitive models become fuzzy. On the other hand, travel times to reach the facilities are also uncertain and they must be expressed as triangular fuzzy number. The distances d_{ki} should be replaced in the corresponding model by a triangular fuzzy distance $\tilde{d}_{ki} = (d_{ki}^1, d_{ki}^2, d_{ki}^3)$. The using of the α-cuts will deal to a crisp model and the solution will be a function of this parameter α. There are several works in Literature using fuzzy methods for competitive location problems.

Liang *et al.* [18] analyze the optimum output quantity decision analysis of a duopoly market under a fuzzy decision environment. To efficiently handle the fuzziness of the decision variables, the linguistic values, subjectively represented by the trapezoidal fuzzy numbers, are used to act as the evaluation tool of decision variables such as fixed cost and unit variable cost.

Osumi *et al.* [21] investigate a competitive facility location problem where there are two facilities on a linear market. Customers at a demand point utilize the facility which seems to be the nearest one from them. They do not distinguish the small difference between two distances. This preference is formulated by introducing relative fuzzy difference based on the actual distance between two points. This paper considers the problems to find the optimal location for the follower and for the leader.

Uno and Katagiri [31] study a new optimal location problem, called defensive location problem (DLP). In the DLPs, a decision maker locates defensive facilities in order to prevent her/his enemies from reaching an important site, called a core; the DLPs are formulated as bilevel 0-1 programming problems to find Stackelberg solutions. an interactive fuzzy satisfying method is proposed,

Uno *et al.* [28] y [29] analyse competitive facility location problems and consider the case where the set of customers is divided into several subsets or types by investigating their preferences and criteria. Since the preferences and criteria often include the vagueness of human's judgement they express such parameter by fuzzy numbers.

Uno and Masatoshi [31] propose a multi-objective approach for competitive facility location models with fuzzy numbers. In cases where the objective of each firm that locates its own facilities is only to maximize its reward, the location of all facilities is usually crowded on some regions which have many populations or are a hub for all regions. Such a location forces the firms to compete extremely and the market about facilities is insecure. Therefore, they formulate a multi-objective problem in which the decision maker is an arbitrator and whose objectives are rewards for all firms. By using a solution to the problem, they find a good restriction such that more firms can survive.

Considering the imprecision or vagueness in the customer choice rule, the appropriate model for dealing problem (19) is that with fuzzy coefficients in the technological matrix since the coefficients c_{ki}^{j}. Several relaxed choice rules have been proposed to allow a soft behavior of the customers. These rules will result in corresponding membership functions for the coefficients c_{ki}^{j} that provide a particular linear programming problem with fuzzy technological coefficients. The problem is then solved by the corresponding auxiliary problem.

5 Conclusions and further research

Fuzzy sets constitute an appropriate approach to manage the uncertainty that appears in real competitive location problems. Fuzzy competitive location problems need to be clearly formalized and classified. There are several research works in some competitive location problems with fuzzy elements. However, practical solution procedures have to be tested for real problems Chance-constrained or Possibility theory and the classical use of α-cuts.

Acknowledgements

The authors have supported by projects TIN2008-06873-C04-01, TIN2008-06873-C04-02, ECO2008-05589 of Spanish govern and FEDER, and PI042005/044 of Canary govern.

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