A construction of an L-fuzzy valued measure of L-fuzzy sets

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Abstract— *We develope a construction of an L-fuzzy valued measure by extending a measure defined on a σ-algebra of crisp sets to an L-fuzzy valued T_M-countably additive measure defined on a T_M-clan of L-fuzzy sets in the case when operations for L-fuzzy sets and L-fuzzy real numbers are based on the minimum t-norm T_M.*

Keywords— L-fuzzy set, L-fuzzy real number, L-fuzzy valued measure.

1 Introduction

There are many works devoted to measures of fuzzy sets. The most important concepts and results concerning this topic are considered by D. Butnariu and E.P. Klement [2]. Our interest is in developing a theory of measure and integral where not only sets are fuzzy, but also measure and integral take fuzzy real values. To do this we need a concept of a fuzzy real number. There are several (quite different) ways to define a fuzzy real number. For our purposes we use fuzzy real numbers as they were first defined by B. Hutton [3], and then studied thoroughly in a series of papers (see e.g.[4],[5]). The preference of using this approach for defining fuzzy real numbers is motivated by our intention to develop the investigations from [6],[7].

In our previous work [1] we suggested the construction of a T-measure of L-fuzzy sets by extension of a measure defined on a σ-algebra of crisp sets to a T-measure on a T-tribe in the case of the minimum t-norm T_M and $L = [0,1]$. The main purpose of the present paper is to show that the construction still can be generalized in the case when L is complete and completely distributive lattice $L(\wedge, \vee, 0_L, 1_L)$. It was achieved in the case of the minimum t-norm T_M: $T_M(x,y) = x \wedge y$.

2 Preliminaries

Let X be a nonempty set. The class of all L-fuzzy subsets of X (in the sequel L-fuzzy sets) is denoted L^X. The operations for L-fuzzy sets $A, B \in L^X$ such as intersection, union and difference are defined by using a minimum triangular norm T_M, its corresponding triangular conorm S_M and an involution N:
$(A \cap B)(x) = A(x) \wedge B(x)$, $(A \cup B)(x) = A(x) \vee B(x)$,
$(A \setminus B)(x) = A(x) \wedge N(B(x))$.

2.1 Classes of L-fuzzy sets

In order to define an L-fuzzy valued function of L-fuzzy sets we consider such classes of L-fuzzy sets as T_M-semirings (defined by analogy with classical case, see e.g. [8]) and T_M-clans (see e.g. [2]).

Definition 2.1. *A subclass $\wp \subset L^X$ is called a T_M-semiring on X if the following properties are satisfied:*

(i) $\emptyset \in \wp$;

(ii) for all $A, B \in \wp$ *we have* $A \wedge B \in \wp$;

(iii) for all $A, B \in \wp$ *there exist such T_M-disjoint L-fuzzy sets* $A_1, A_2, ..., A_n \in \wp$ *that* $A \setminus B = \bigvee_{i=1}^{n} A_i$.

A finite family of L-fuzzy sets $A_1, A_2, \ldots, A_n$ is said to be T_M-*disjoint* (see e.g.[2]) iff for each $k \in \{1, \ldots, n\}$

$$\Big(\bigvee_{j=1, j\neq k}^{n} A_j\Big) \wedge A_k = \emptyset.$$

A countable family of L-fuzzy sets is said to be T_M-disjoint iff every finite subfamily of this family is T_M-disjoint.

Definition 2.2. *A subclass $\mathcal{A} \subset L^X$ is called a T_M-clan on X iff the following properties are satisfied:*

(i) $\emptyset \in \mathcal{A}$;

(ii) for all $A \in \mathcal{A}$ *we have* $N(A) \in \mathcal{A}$;

(iii) for all $A, B \in \mathcal{A}$ *we have* $A \wedge B \in \mathcal{A}$.

2.2 L-fuzzy real numbers

For our purposes we use the L-fuzzy real numbers as they were first defined by B. Hutton [3].

Definition 2.3. *An L-fuzzy real number is a function $z : \mathbb{R} \to L$ such that*

(i) z is non-increasing;

(ii) $\bigwedge_x z(x) = 0_L$, $\bigvee_x z(x) = 1_L$;

(iii) z is left semi-continuous, i.e. $\bigwedge_{t<x} z(t) = z(x)$.

The set of all L-fuzzy real numbers is called *the L-fuzzy real line* and it is denoted by $\mathbb{R}(L)$. An L-fuzzy number z is called *non-negative* if $z(0) = 1_L$. The set of all non-negative L-fuzzy real numbers we denote by $\mathbb{R}_+(L)$.

The ordinary real line $\mathbb{R}$ can be identified with the subspace

$$\{z_a \mid a \in \mathbb{R}\} \subset \mathbb{R}(L)$$

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by assigning to a real number $a \in \mathbb{R}$ the fuzzy real number z_a defined by

$$z_a(x) = \begin{cases} 1_L, \text{ if } x \le a, \\ 0_L, \text{ if } x > a. \end{cases}$$

The operation of L-fuzzy addition $\oplus$ defined by

$$(z_1 \oplus z_2)(x) = \bigvee_t \left(\{z_1(t) \wedge z_2(x-t)\}\right)$$

whenever $z_1, z_2 \in \mathbb{R}(L)$, is a jointly continuous extension of addition $+$ from the real line $\mathbb{R}$ to the L-fuzzy real line $\mathbb{R}(L)$.

Given a sequence of non-negative L-fuzzy real numbers $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+(L)$ we consider the countable sum

$$\bigoplus_{n\in\mathbb{N}} z_n = \bigvee_{n\in\mathbb{N}} (z_1 \oplus z_2 \oplus \ldots \oplus z_n).$$

2.3 L-fuzzy valued functions

Let $\aleph$ be a class of L-fuzzy sets. Within this section some basic properties of an L-fuzzy valued function $\eta : \aleph \to \mathbb{R}_+(L)$ are considered.

Definition 2.4. *An L-fuzzy valued function η is called T_M-additive iff*

for all $A, B \in \aleph$ such that $A \wedge B = \emptyset, A \vee B \in \aleph$ it holds $\eta(A \vee B) = \eta(A) \oplus \eta(B)$.

Definition 2.5. *An L-fuzzy valued function η is called T_M-countably additive iff*

for all $(A_n)_{n\in\mathbb{N}} \subset \aleph$ such that $(\forall i, j \in \mathbb{N} : i \neq j \Rightarrow A_i \wedge A_j = \emptyset)$ and $\bigvee_{n=1}^{\infty} A_n \in \aleph$ it holds $\eta(\bigvee_{n=1}^{\infty} A_n) = \bigoplus_{n\in\mathbb{N}} \eta(A_n)$.

Definition 2.6. *Let $\wp$ be a T_M-semiring. A function $m : \wp \to \mathbb{R}_+(L)$ is called an L-fuzzy valued elementary T_M-measure if it satisfies the following conditions:*

(i) $m(\emptyset) = z_0$;

(ii) m is T_M-additive.

Definition 2.7. *Let $\mathcal{A}$ be a T_M-clan. A function $\mu : \mathcal{A} \to \mathbb{R}_+(L)$ is called an L-fuzzy valued T_M-countably additive measure if it satisfies the following conditions:*

(i) $\mu(\emptyset) = z_0$;

(ii) μ is T_M-countably additive.

3 A construction of an L-fuzzy valued elementary T_M-measure

Let Φ be a σ-algebra of "crisp" subsets of X and ν is a finite measure $\nu : \Phi \to [0, +\infty[$. Our aim is to construct an L-fuzzy valued T_M-countably additive measure $\tilde{\mu}$ on a T_M-clan by extension a crisp measure ν. To achieve this we generalize the well known construction of "classic" measure theory (see e.g. [8]) to the L-fuzzy case.

3.1 T_M-semiring $\tilde{\wp}$ of L-fuzzy sets

To realise the construction we use the special type of L-fuzzy sets with respect to σ-algebra Φ of crisp sets on X. For $M \in \Phi$, $\alpha \in L$ we define L-fuzzy set $A(M, \alpha)$:

$$(A(M,\alpha))(x) = \begin{cases} \alpha, x \in M, \\ 0_L, x \notin M. \end{cases}$$

For convenience we denote this special type of L-fuzzy sets by $A(M, \alpha)$ (the choice of a letter in the notation is not crucial: $B(N, \beta)$ is still the same type of an L-fuzzy set).

Proposition 3.1. *The class of L-fuzzy sets*

$$\tilde{\wp} = \{A(M,\alpha) | M \in \Phi \text{ and } \alpha \in L\}$$

is a T_M-semiring.

Proof. For all $A(M, \alpha), B(K, \beta) \in \tilde{\wp}$ we have
$A(M, \alpha) \wedge B(K, \beta) = C(M \cap K, \alpha \wedge \beta) \in \tilde{\wp}$ and
$A(M, \alpha) \setminus B(K, \beta) = A_1(M \setminus K, \alpha) \vee A_2(M \cap K, \alpha \wedge N(\beta))$,
where $A_1(M \setminus K, \alpha), A_2(M \cap K, \alpha \wedge N(\beta)) \in \tilde{\wp}$.
Taking into account that also $\emptyset = A(\emptyset, 0_L) \in \tilde{\wp}$, it is proved that $\tilde{\wp}$ is a T_M-semiring. □

3.2 L-fuzzy valued elementary T_M-measure $\tilde{m}$

We define an L-fuzzy valued function $\tilde{m}$ on the T-semiring $\tilde{\wp}$ by the formula $\tilde{m}(A(M, \alpha)) = z_{\nu(M),\alpha}$, where

$$z_{\nu(M),\alpha}(t) = \begin{cases} 1_L, t \le 0, \\ \alpha, 0 < t \le \nu(M), \\ 0_L, t > \nu(M), \end{cases}$$

is an L-fuzzy real number.

Proposition 3.2. *$\tilde{m}$ is an L-fuzzy valued elementary T_M-measure.*

Proof. It is easy to see that

$$\tilde{m}^*(\emptyset) = z_{\nu(\emptyset),0_L} = z_0$$

and the equality

$$\tilde{m}(A(M,\alpha)) \oplus \tilde{m}(B(K,\beta)) = \tilde{m}(A(M,\alpha) \vee B(K,\beta))$$

is true if $A(M, \alpha) = \emptyset$ or $B(K, \beta) = \emptyset$.

Let us consider $A(M, \alpha), B(K, \beta) \in \tilde{\wp}$ such that

$$A(M,\alpha) \wedge B(K,\beta) = \emptyset,\ A(M,\alpha) \cup B(K,\beta) \in \tilde{\wp} \text{ and } \alpha \neq 0_L, \beta \neq 0_L.$$

It follows that $M \cap K = \emptyset$ or $\alpha \wedge \beta = 0_L$.

If $M \cap K = \emptyset$ then $\alpha = \beta$ and in this case

$$\tilde{m}(A(M,\alpha)) \oplus \tilde{m}(B(K,\alpha)) = z_{\nu(M),\alpha} \oplus z_{\nu(K),\alpha} =$$
$$= z_{\nu(M)+\nu(K),\alpha} = z_{\nu(M\cup K),\alpha} = \tilde{m}(A(M,\alpha) \vee B(K,\alpha)).$$

If $M \cap K \neq \emptyset$ then it is sufficient to consider the case when $\alpha \wedge \beta = 0_L$. In this case α and β are incomparable (due to the assumptions that $\alpha \neq 0_L, \beta \neq 0_L$). Because of

$$A(M,\alpha) \vee B(K,\beta) =$$
$$= C_1(M \cap K, \alpha \vee \beta) \vee C_2(M \setminus K, \alpha) \vee C_3(K \setminus M, \beta) \in \tilde{\wp}$$

we obtain that $K \setminus M = \emptyset$, $M \setminus K = \emptyset$ and hence $M = K$. Then

$$\tilde{m}(A(M,\alpha)) \oplus \tilde{m}(B(M,\beta)) = z_{\nu(M),\alpha} \oplus z_{\nu(M),\beta} =$$
$$= z_{\nu(M),\alpha\vee\beta} = C(M, \alpha \vee \beta) = \tilde{m}(A(M,\alpha) \vee B(M,\beta)).$$

By this we prove that $\tilde{m}$ is T_M-additive. □

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4 Measurable L-fuzzy sets

4.1 An extension of an L-fuzzy valued elementary T_M-measure

We define an L-fuzzy valued function $\tilde{m}^* : L^X \to \mathbb{R}_+(L)$ by the following formula

$\tilde{m}^*(E) = \bigwedge\{\bigoplus_{n=1}^{\infty} \tilde{m}(E_n) \mid (E_n)_{n\in\mathbb{N}} \subset \tilde{\wp}:$

$E \le \bigvee_{n=1}^{\infty} E_n$ and $(E_n)_{n\in\mathbb{N}}$ is T_M-disjoint$\}, E \in L^X$.

Remarks.

(i) For every $E \in L^X$ there always exists such a sequence $(E_n)_{n\in\mathbb{N}} \subset \tilde{\wp}$ that $E \le \bigvee_{n=1}^{\infty} E_n$. It is enough to take $E_1(X, 1_L)$. Thus $\tilde{m}^*$ is bounded from above in the following sense: $\tilde{m}^*(E) \le z_{\nu(X),1_L}$ for all $E \in L^X$.

(ii) For all $E \in \tilde{\wp}$ we have $\tilde{m}^*(E) = \tilde{m}(E)$.

Proposition 4.1. *For all $A, B \in L^X$ we have*

$$\tilde{m}^*(A) \oplus \tilde{m}^*(B) \ge \tilde{m}^*(A \wedge B) \oplus \tilde{m}^*(A \vee B).$$

Proof. For a given L-fuzzy sets A and B we consider two T_M-disjoint sequences:

$$(C_i(M_i, \alpha_i))_{i\in\mathbb{N}} \subset \tilde{\wp} : A \le \bigvee_{i=1}^{\infty} C_i,$$

$$(D_j(K_j, \beta_j))_{j\in\mathbb{N}} \subset \tilde{\wp} : B \le \bigvee_{j=1}^{\infty} D_j$$

and define a new T_M-disjoint sequence

$(H_{ij}(L_{ij}, \gamma_{ij}))_{i,j\in\mathbb{N}}$, where $L_{ij} = M_i \cap K_j$ and $\gamma_{ij} = \alpha_i \wedge \beta_i$.

Obviously, $A \wedge B \le \bigvee_{i\in\mathbb{N}} \bigvee_{i\in\mathbb{N}} H_{ij}$.

As to L-fuzzy set $A \vee B$ we can cover it with elements of three sequences:

$$(\tilde{H}_{ij}(L_{ij}, \lambda_{ij}))_{i,j\in\mathbb{N}}, \quad \text{with} \quad L_{ij} = M_i \cap K_j, \lambda_{ij} = \alpha_i \vee \beta_i,$$

$$(\tilde{C}_i(\tilde{M}_i, \alpha_i))_{i\in\mathbb{N}}, \quad \text{with} \quad \tilde{M}_i = M_i \setminus \bigcup_{j\in\mathbb{N}} K_j,$$

$$(\tilde{D}_j(\tilde{K}_j, \beta_j))_{j\in\mathbb{N}}, \quad \text{with} \quad \tilde{K}_j = K_j \setminus \bigcup_{i\in\mathbb{N}} M_i.$$

Now let us transform the sum $\bigoplus_{i=1}^{\infty} \tilde{m}^*(C_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(D_j)$. We use the following notation:

$$F_{ij}^{\alpha} = (L_{ij}, \alpha_i), F_{ij}^{\beta} = (L_{ij}, \beta_j), ij \in \mathbb{N}.$$

Because of σ-additivity of measure ν we get

$$\nu(M_i) = \nu(\tilde{M}_i) + \sum_{j=1}^{\infty} \nu(L_{ij}), i \in \mathbb{N},$$

$$\nu(K_j) = \nu(\tilde{K}_j) + \sum_{i=1}^{\infty} \nu(L_{ij}), j \in \mathbb{N}.$$

It follows

$$\tilde{m}^*(C_i) = z_{\nu(M_i),\alpha_i} = z_{\nu(\tilde{M}_i)+\sum_{j=1}^{\infty}\nu(L_{ij}),\alpha_i} =$$

$$= z_{\nu(\tilde{M}_i),\alpha_i} \oplus \bigoplus_{j=1}^{\infty} z_{\nu(L_{ij}),\alpha_i} = \tilde{m}^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(F_{ij}^{\alpha}),$$

$$\tilde{m}^*(D_j) = z_{\nu(K_j),\beta_j} = z_{\nu(\tilde{K}_j)+\sum_{i=1}^{\infty}\nu(L_{ij}),\beta_j} =$$

$$= z_{\nu(\tilde{K}_j),\beta_j} \oplus \bigoplus_{i=1}^{\infty} z_{\nu(L_{ij}),\beta_j} = \tilde{m}^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} \tilde{m}^*(F_{ij}^{\beta}).$$

Now we obtain

$$\bigoplus_{i=1}^{\infty} \tilde{m}^*(C_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(D_j) =$$

$$= \bigoplus_{i=1}^{\infty}(\tilde{m}^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(F_{ij}^{\alpha})) \oplus \bigoplus_{j=1}^{\infty}(\tilde{m}^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty} \tilde{m}^*(F_{ij}^{\beta})) =$$

$$= \bigoplus_{i=1}^{\infty} \tilde{m}^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty}\bigoplus_{j=1}^{\infty} \tilde{m}^*(F_{ij}^{\alpha}) \oplus \bigoplus_{j=1}^{\infty}\bigoplus_{i=1}^{\infty} \tilde{m}^*(F_{ij}^{\beta}) =$$

$$= \bigoplus_{i=1}^{\infty} \tilde{m}^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty}\bigoplus_{j=1}^{\infty} (\tilde{m}^*(F_{ij}^{\alpha}) \oplus \tilde{m}^*(F_{ij}^{\beta})).$$

Since

$$\tilde{m}^*(F_{ij}^{\alpha}) \oplus \tilde{m}^*(F_{ij}^{\beta}) = \tilde{m}^*(H_{ij}) \oplus \tilde{m}^*(\tilde{H}_{ij}),$$

we continue

$$\bigoplus_{i=1}^{\infty} \tilde{m}^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty}\bigoplus_{j=1}^{\infty} (\tilde{m}^*(F_{ij}^{\alpha}) \oplus \tilde{m}^*(F_{ij}^{\beta})) =$$

$$= \bigoplus_{i=1}^{\infty} \tilde{m}^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty}\bigoplus_{j=1}^{\infty} \tilde{m}^*(\tilde{H}_{ij}) \oplus \bigoplus_{i=1}^{\infty}\bigoplus_{j=1}^{\infty} \tilde{m}^*(H_{ij}).$$

Now taking into account the fact that

$$A \wedge B \le \bigvee_{i=1}^{\infty}\bigvee_{j=1}^{\infty} H_{ij} \text{ and } A \vee B \le (\bigvee_{i=1}^{\infty} C_i) \vee (\bigvee_{j=1}^{\infty} D_j) \vee (\bigvee_{i=1}^{\infty}\bigvee_{j=1}^{\infty} \tilde{H}_{ij}),$$

we get

$$\bigoplus_{i=1}^{\infty} \tilde{m}^*(\tilde{C}_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(\tilde{D}_j) \oplus \bigoplus_{i=1}^{\infty}\bigoplus_{j=1}^{\infty} \tilde{m}^*(\tilde{H}_{ij}) \oplus \bigoplus_{i=1}^{\infty}\bigoplus_{j=1}^{\infty} \tilde{m}^*(H_{ij}) \ge$$

$$\ge \tilde{m}^*(A \wedge B) \oplus \tilde{m}^*(A \vee B).$$

So independent of the choice of sequences $(C_i)_{i\in\mathbb{N}}$, $(D_j)_{j\in\mathbb{N}}$ it holds

$$\bigoplus_{i=1}^{\infty} \tilde{m}^*(C_i) \oplus \bigoplus_{j=1}^{\infty} \tilde{m}^*(D_j) \ge \tilde{m}^*(A \wedge B) \oplus \tilde{m}^*(A \vee B).$$

Finally, by taking infimums of sums we get

$$\tilde{m}^*(A) \oplus \tilde{m}^*(B) \ge \tilde{m}^*(A \wedge B) \oplus \tilde{m}^*(A \vee B).$$

□

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4.2 $\tilde{m}^*$-measurable L-fuzzy sets

We generalize the concept of measurability in the sense of Caratheodory (see e.g. [8]).

Definition 4.2. *A set $A \in L^X$ is called an $\tilde{m}^*$-measurable L-fuzzy set, if it satisfies the following conditions for all sets $E \in L^X$:*

(i) $\tilde{m}^*(A) \oplus \tilde{m}^*(E) = \tilde{m}^*(A \wedge E) \oplus \tilde{m}^*(A \vee E)$,

(ii) $\tilde{m}^*(N(A)) \oplus \tilde{m}^*(E) = \tilde{m}^*(N(A) \wedge E) \oplus \tilde{m}^*(N(A) \vee E)$.

We denote $\mathcal{A}_{\tilde{m}^*}$ the class of all $\tilde{m}^*$-measurable L-fuzzy sets. Some obvious properties of $\mathcal{A}_{\tilde{m}^*}$:

$$\emptyset, X \in \mathcal{A}_{\tilde{m}^*},$$

$$A \in \mathcal{A}_{\tilde{m}^*} \Longrightarrow N(A) \in \mathcal{A}_{\tilde{m}^*}.$$

Theorem 4.3. *Class $\mathcal{A}_{\tilde{m}^*}$ is a T_M-clan.*

Proof. For given L-fuzzy sets $A_1, A_2 \in \mathcal{A}_{\tilde{m}^*}$ and $E \in L^X$ we will prove the equality

$$\begin{aligned} \tilde{m}^*(E) \oplus \tilde{m}^*(A_1 \wedge A_2) = \\ \tilde{m}^*(E \wedge (A_1 \wedge A_2)) \oplus \tilde{m}^*(E \vee (A_1 \wedge A_2)) \end{aligned} \qquad (1)$$

Since A_1, A_2 are $\tilde{m}^*$-measurable we have

$$\tilde{m}^*(A_1) \oplus \tilde{m}^*(A_2) = \tilde{m}^*(A_1 \wedge A_2) \oplus \tilde{m}^*(A_1 \vee A_2),$$

$$\tilde{m}^*(A_1 \wedge E) \oplus \tilde{m}^*(A_1 \vee E) = \tilde{m}^*(A_1) \oplus \tilde{m}^*(E).$$

Now by summing up these two equalities and adding one more additional summand $\tilde{m}^*(A_1 \wedge (E \vee A_2))$ to both sides of the equality we obtain

$$\begin{aligned} \tilde{m}^*(A_1) \oplus \tilde{m}^*(A_2) \oplus \tilde{m}^*(A_1 \wedge E) \oplus \\ \oplus \tilde{m}^*(A_1 \vee E) \oplus \tilde{m}^*(A_1 \wedge (E \vee A_2)) = \\ = \tilde{m}^*(A_1 \wedge A_2) \oplus \tilde{m}^*(A_1 \vee A_2) \oplus \tilde{m}^*(A_1) \oplus \\ \oplus \tilde{m}^*(E) \oplus \tilde{m}^*(A_1 \wedge (E \vee A_2)). \end{aligned} \qquad (2)$$

Let us transform now the left part of (2). To do this we use (3) and (4):

$$\begin{aligned} \tilde{m}^*(A_1) \oplus [\tilde{m}^*(A_2) \oplus \tilde{m}^*(E \wedge A_1)] = \\ = \tilde{m}^*(A_1) \oplus [\tilde{m}^*(E \wedge A_1 \wedge A_2) \oplus \tilde{m}^*((E \wedge A_1) \vee A_2)] = \\ = \tilde{m}^*(E \wedge A_1 \wedge A_2) \oplus \tilde{m}^*((E \wedge A_1) \vee A_2 \vee A_1) \oplus \\ \oplus \tilde{m}^*(((E \wedge A_1) \vee A_2) \wedge A_1) = \\ = \tilde{m}^*(E \wedge A_1 \wedge A_2) \oplus \tilde{m}^*(A_1 \vee A_2) \oplus \tilde{m}^*((E \vee A_2) \wedge A_1) \end{aligned} \qquad (3)$$

and

$$\begin{aligned} \tilde{m}^*(E \vee A_1) \oplus \tilde{m}(A_1 \wedge (E \vee A_2)) = \\ = \tilde{m}^*(E \vee (A_1 \wedge A_2) \vee A_1) \oplus \tilde{m}^*(E \vee A_1 \wedge A_2 \wedge A_1) = \\ = \tilde{m}^*(E \vee (A_1 \wedge A_2)) \oplus \tilde{m}^*(A_1). \end{aligned} \qquad (4)$$

Next we substitute (3) and (4) in (2).

$$\tilde{m}^*(E \wedge A_1 \wedge A_2) \oplus \tilde{m}^*(A_1 \vee A_2) \oplus \tilde{m}^*(A_1 \wedge (E \vee A_2)) \oplus$$

$$\oplus \tilde{m}^*(E \vee (A_1 \wedge A_2)) \oplus \tilde{m}^*(A_1) =$$

$$= \tilde{m}^*(A_1 \wedge A_2) \oplus \tilde{m}^*(A_1 \vee A_2) \oplus \tilde{m}^*(A_1) \oplus \tilde{m}^*(E) \oplus$$

$$\oplus \tilde{m}^*(A_1 \wedge (E \vee A_2)).$$

Finally, canceling like terms we obtain (1).
By analogy it can be proved that

$$\tilde{m}^*(E) \oplus \tilde{m}^*(A_1 \vee A_2) = \tilde{m}^*(E \wedge (A_1 \vee A_2)) \oplus \tilde{m}^*(E \vee (A_1 \vee A_2)).$$

Taking into account that $N(A_1 \vee A_2) = N(A_1) \wedge N(A_2)$ and $N(A_1 \wedge A_2) = N(A_1) \vee N(A_2)$, we get that $A_1 \vee A_2$ and $A_1 \wedge A_2$ are $\tilde{m}^*$-measurable. □

Proposition 4.4. $\tilde{\wp} \subset \mathcal{A}_{\tilde{m}^*}$

Proof. First we will show that for a given $A(H, \alpha) \in \tilde{\wp}$ and all $E \in L^X$ it holds

$$\tilde{m}^*(A) \oplus \tilde{m}^*(E) = \tilde{m}^*(A \vee E) \oplus \tilde{m}^*(A \wedge E).$$

Because of Proposition 4.1 it is sufficient to prove the inequality

$$\tilde{m}^*(A) \oplus \tilde{m}^*(E) \leq \tilde{m}^*(A \vee E) \oplus \tilde{m}^*(A \wedge E).$$

We consider two T_M-disjoint sequences $(C_k(M_k, \beta_k))_{k \in \mathbb{N}} \subset \tilde{\wp}$ and $(D_k(K_k, \gamma_k))_{k \in \mathbb{N}} \subset \tilde{\wp}$ such that

$$A \vee E \leq \bigvee_{k=1}^{\infty} C_k \text{ and } A \wedge E \leq \bigvee_{k=1}^{\infty} D_k$$

and define two the new sequences

$$(F_k(M_k \setminus H, \beta_k))_{k \in \mathbb{N}} \text{ and } (G_k(H \cap M_k, \beta_k))_{k \in \mathbb{N}}.$$

By T_M-additivity of $\tilde{m}$ it holds

$$\tilde{m}(G_k) \oplus \tilde{m}(F_k) = \tilde{m}(C_k),\ k \in \mathbb{N}.$$

Taking into account

$$A \leq \bigvee_{k=1}^{\infty} G_k \text{ and } E \leq (\bigvee_{k=1}^{\infty} F_k) \vee (\bigvee_{k=1}^{\infty} G_k),$$

we obtain

$$\tilde{m}^*(A) = \tilde{m}(A) \leq \bigoplus_{k=1}^{\infty} \tilde{m}(G_k),$$

$$\tilde{m}^*(E) \leq \bigoplus_{k=1}^{\infty} \tilde{m}(F_k) \oplus \bigoplus_{k=1}^{\infty} \tilde{m}(D_k).$$

Summing up

$$\tilde{m}^*(A) \oplus \tilde{m}^*(E) \leq \bigoplus_{k=1}^{\infty} \tilde{m}(G_k) \oplus \bigoplus_{k=1}^{\infty} \tilde{m}(F_k) \oplus \bigoplus_{k=1}^{\infty} \tilde{m}(D_k) =$$

$$= \bigoplus_{k=1}^{\infty} (\tilde{m}(G_k) \oplus \tilde{m}(F_k)) \oplus \bigoplus_{k=1}^{\infty} \tilde{m}(D_k) = \bigoplus_{k=1}^{\infty} \tilde{m}(C_k) \oplus \bigoplus_{k=1}^{\infty} \tilde{m}(D_k).$$

So independent of a choice of sequences $(C_k)_{k \in \mathbb{N}}$ and $(D_k)_{k \in \mathbb{N}}$ it holds

$$\bigoplus_{k=1}^{\infty} \tilde{m}(C_k) \oplus \bigoplus_{k=1}^{\infty} \tilde{m}(D_k) \geq \tilde{m}^*(A) \oplus \tilde{m}^*(E).$$

Finally, by taking infimums of sums we get

$$\tilde{m}^*(A) \oplus \tilde{m}^*(E) \leq \tilde{m}^*(A \vee E) \oplus \tilde{m}^*(A \wedge E).$$

Since

$$N(A) = B_1(H, N_\alpha) \vee B_2(X \setminus H, 1_L),$$

where $B_1(H, N_\alpha), B_2(X \setminus H, 1_L) \in \tilde{\wp}$, the equality

$$\tilde{m}^*(N(A)) \oplus \tilde{m}^*(E) = \tilde{m}^*(N(A) \wedge E) \oplus \tilde{m}^*(N(A) \vee E)$$

can be proved by analogy with the proof of Theorem 4.3. □

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5 A construction of an L-fuzzy valued T_M-countably additive measure

Let us denote by $\tilde{\mu}$ the restriction of $\tilde{m}^*$ to $\mathcal{A}_{\tilde{m}^*}$:

$$\tilde{\mu} : \mathcal{A}_{\tilde{m}^*} \to \mathbb{R}_+(L).$$

Theorem 5.1. *$\tilde{\mu}$ is T_M-countably additive L-fuzzy valued function.*

Proof. First we will show that for a T_M - disjoint sequence $A_1, A_2, ..., A_n \in \mathcal{A}_{\tilde{m}^*}$ it holds $\tilde{\mu}(\bigvee_{i=1}^{n} A_i) = \bigoplus_{i=1}^{n} \tilde{\mu}(A_i)$.

For $n = 2$ by usinq $\tilde{m}^*$-measurability and T_M - disjointness of A_1, A_2 we obtain

$$\tilde{\mu}(A_1) \oplus \tilde{\mu}(A_2) = \tilde{\mu}(A_1 \vee A_2) \oplus \tilde{\mu}(A_1 \wedge A_2) =$$
$$= \tilde{\mu}(A_1 \vee A_2) \oplus \tilde{\mu}(\emptyset) = \tilde{\mu}(A_1 \vee A_2).$$

Now we assume that the equality holds for a given $n \in \mathbb{N}$ and we prove it for $n+1$ T_M-disjoint sets $A_1, A_2, ..., A_{n+1} \in \mathcal{A}_{\tilde{m}^*}$

$$\tilde{\mu}(\bigvee_{k=1}^{n+1} A_k) = \tilde{\mu}((\bigvee_{k=1}^{n} A_k) \vee A_{n+1}) = \tilde{\mu}(\bigvee_{k=1}^{n} A_k) \oplus \tilde{\mu}(A_{n+1}) =$$
$$= \bigoplus_{k=1}^{n} \tilde{\mu}(A_k) + \tilde{\mu}(A_{n+1}) = \bigoplus_{k=1}^{n+1} \tilde{\mu}(A_k).$$

Now let us consider a T_M-disjoint sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_{\tilde{m}^*}$ such that $\bigvee_{n=1}^{\infty} A_n \in \mathcal{A}_{\tilde{m}^*}$. To prove the inequality

$$\tilde{\mu}(\bigvee_{n=1}^{\infty} A_n) \le \bigoplus_{n=1}^{\infty} \tilde{\mu}(A_n)$$

we take T_M-disjoint sequences $(B_k^n)_{k\in\mathbb{N}} \in \tilde{\wp}$ such that $A_n \le \bigvee_{k=1}^{\infty} B_k^n, n \in \mathbb{N}$. Then $\bigvee_{n=1}^{\infty} A_n \le \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{\infty} B_k^n$ and hence

$$\tilde{m}^*(\bigvee_{n=1}^{\infty} A_n)) \le \bigoplus_{n=1}^{\infty} \bigoplus_{k=1}^{\infty} \tilde{m}(B_k^n).$$

Taking into account that this inequality holds independent of the choice of sequences $(B_k^n)_{k\in\mathbb{N}}$ we obtain

$$\tilde{m}^*(\bigvee_{n=1}^{\infty} A_n) \le \bigoplus_{n=1}^{\infty} \tilde{m}^*(A_n).$$

Now we will prove inverse inequality. For a given $n \in \mathbb{N}$ we have

$$\tilde{\mu}(\bigvee_{k=1}^{n} A_k) \le \tilde{\mu}(\bigvee_{k=1}^{\infty} Ak).$$

and

$$\tilde{\mu}(\bigvee_{k=1}^{n} A_k)) = \bigoplus_{k=1}^{n} \tilde{\mu}(A_k),$$

hence

$$\bigoplus_{k=1}^{n} \tilde{\mu}(A_k) \le \tilde{\mu}(\bigvee_{k=1}^{\infty} A_k).$$

Finally by taking supremum we obtain

$$\bigoplus_{k=1}^{\infty} \tilde{\mu}(A_k) \le \tilde{\mu}(\bigvee_{k=1}^{\infty} A_k).$$

□

As the final result, by extension of a crisp measure ν we obtain L-fuzzy valued T_M-countably additive measure $\tilde{\mu} : \mathcal{A}_{\tilde{m}^*} \to \mathbb{R}_+(L)$ such that

(i) $\tilde{\mu}/\tilde{\wp} = \tilde{m}$;

(ii) $\tilde{\mu}/\Phi = \nu$.

The last equality means that for every $M \in \Phi$ it holds $\tilde{\mu}(A(M, 1_L)) = z_{\nu(M)}$.

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