Multiple Negations in Fuzzy Interval Logic

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Abstract— *This paper continues our study in fuzzy interval logic based on the Checklist Paradigm(CP) semantics of Bandler and Kohout. We investigate the fuzzy interval system of negation which was defined by the Sheffer(*NAND*), the Nicod(*NOR*) and the implication connectives of m_1 interval system in depth. The top-bottom(*TOP-BOT*) pair of fuzzy negation interval shows non-involutive property; however, it shows various interesting properties as the negation is iterated to each* TOP *and* BOT *fuzzy negation connective. As the iteration reaches to infinity, the* TOP-BOT *pair of iterated negation shows nearly involutive property.*

Keywords— Checklist Paradigm, Fuzzy Interval, Fuzzy Negation, Non-Involution, Gap-Theorem.

1 Introduction

The convention that fuzzy logics default into 2-valued logic for the values 0 and 1 is generally accepted. The question how connectives of the same type differ outside the boundary points, within the open interval $(0,1)$ has been well researched. For example the implications such as Lukasiewicz($\overset{L}{\to}$), Kleene-Dienes($\overset{KD}{\to}$), Reichenbach($\overset{KDL}{\to}$), Goguen($\overset{G43}{\to}$), Gödel($\overset{S*}{\to}$), Early-Zadeh($\overset{EZ}{\to}$) and Wilmott($\overset{W}{\to}$) are interrelated by the following inequalities [1]:

$$a \overset{S*}{\to} b \ \leq a \overset{G43}{\to} b \ \leq a \overset{L}{\to} b, \text{ and}$$

$$a \overset{W}{\to} b \ \leq a \overset{EZ}{\to} b \ \leq a \overset{KD}{\to} b \ \leq a \overset{KDL}{\to} b \ \leq a \overset{L}{\to} b, \ \text{ where}$$

$a \overset{L}{\to} b = min(1, 1-a+b)$; $a \overset{KDL}{\to} b = min(1, 1-a+ab)$; $a \overset{KD}{\to} b = max(1-a, b)$, etc[1]. However, much less is known about inter-relations of interval systems of connectives.

In 1979 Bandler and Kohout derived five interesting systems of fuzzy logic, $m_1, m_2, \ldots, m_5$, based on the Checklist paradigm [2]. Since then, the systems of connectives that can be generated from implicational intervals by group transformations have been investigated systematically; in particular, the m_1 interval logic system of 16 connectives has been investigated in depth. Among them, ten 2-argument connectives such as $\wedge, \vee, \to, \ldots \equiv$, *oplus*(XOR) yield the interval pairs of connectives ($conbot \leq m_1 \leq contop$) where its implication($\to$) yields Lukasiewicz and Kleene-Dienes implication for its TOP-BOT pair of interval, in particular. However, a unary connective, a negation, did not yield interval but just singleton: i.e. $\neg a = 1-a$. From this question, we proposed the alternative negations in [3] by the Sheffer(NAND), the Nicod(NOR) and a pair of <implication, 0>, which fuzzifies $\neg a$, generating the intervals like the fuzzified 2-ary connectives. The TOP-BOT pair of generated negation interval lacks of involutive property on the surface, but a question still remains if they show a nearly involutive property converging to a certain value as an iteration of negation proceeds.
Thus, we further investigate this non-involutive property of TOP-BOT pair of interval negations as well as the relationship between three definitions of negation which are equivalent in two-valued logic but bifurcate into different negations in interval many-valued logic in this paper.

[1]For the rest of definitions, refer to [1],[2]

2 Interval logic system generated by the checklist paradigm

The structure of five fuzzy interval systems m_1 - m_5, based on the Checklist paradigm by Bandler and Kohout in [2] is generated by a distinct measure that performs the *summarization* of the information contained in certain well-defined binary structures called *fine structures*. The interval produced by a measure m_i pair of connectives of one type can be generically characterized by the following inequality:

$$CON_{Bot} \leq m_i \leq CON_{Top}, \ \ i \in \{1,2,\ldots,5\}$$

For the details of the checklist paradigm and its various uses, see the papers [2],[4],[5],[6],[7],[8].

2.1 Five Implication Operator Based Interval Systems of Bandler and Kohout

The following five inequalities linking the interval bounds for implication operators $[\to_{bot}, \to_{top}]$ with corresponding measures, m_i, $i = \{1,2,3,4,5\}$ have been listed in [2].

1. The Kleene-Dienes implication(KD) and Łukasiewicz implication (Ł) respectively, are attainable lower and upper bounds of m_1:
   $max(1-a,b) \leq m_1(\to) \leq min(1, 1-a+b)$.

2. A certain new function of (a,b) and the Goguen-Gaines (G43) implication (the left-hand side) are respectively attainable lower and upper bounds of m_2:
   $max(0, (a+b-1)/a) \leq m_2(\to) \leq min(1, b/a)$.

3. Another function of (a,b) and the Early Zadeh implication (EZ) are respectively attainable lower and upper bounds of m_3:
   $max(a+b-1, 1-a) \leq m_3(\to) \leq max[min(a,b), 1-a]$.

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4. Still another function of (a,b) and the Wilmott implication (W) respectively, are attainable lower and upper bounds of m_4:
$min[max(a+b-1,1-a),max(b,1-a-b)] \le m_4(\rightarrow)$
$\le min[max(1-a,b),(max(a,(1-b),min(b,1-a)))]$.
5. Yet another function of (a,b) and one of G43 respectively, are attainable lower and upper bounds of m_5:
$max[(a+b-1)/a,1-a] \le m_5(\rightarrow) \le max[min(1,b/a),1-a]$.

In m_1 system, a Kleene-Dinese logic system forms a a lower bound of the interval$(m1(\rightarrow_{bot}))$ while a Łukasiewicz logic system forms a upper bound of the interval $(m1(\rightarrow_{top}))$.

From this implicational interval pair $[m1(\rightarrow_{bot}),m1(\rightarrow_{top})]$, one can generate all 16 pairs of connectives of corresponding interval systems by Kohout-Bandler group of logic transformations. 10 of these pairs of connectives are genuine interval pairs and remaining 6 connectives collapse into a single point, not an interval. It has been investigated systematically over the years in the series of papers [9], [10]. These 16 connectives of measure $m_1,\ldots,\ m_5$ can be generated by group transformation. 8 of these interval pair of connectives of m_1 system are shown in section 3.

Table 1: Group transformation of m_1 system: $<\rightarrow_{top},\rightarrow_{bot}>$.

Transformation of Connective	Type of Interval Bound
$g_{1t} = I(\rightarrow_{Top}) = min(1,1-a+b)$	$\rightarrow_{Top}$
$g_{2t} = C(\rightarrow_{Top}) = min(1,1+a-b)$	$\leftarrow_{Top}$
$g_{3t} = D(\rightarrow_{Top}) = max(0,b-a)$	$\not\leftarrow_{Bot}$
$g_{4t} = N(\rightarrow_{Top}) = max(0,a-b)$	$\not\rightarrow_{Bot}$
$g_{5t} = LC(\rightarrow_{Top}) = min(1,a+b)$	$\vee_{Top}$
$g_{6t} = LD(\rightarrow_{Top}) = max(0,1-a-b)$	$\downarrow_{Bot}$
$g_{7t} = RC(\rightarrow_{Top}) = min(1,2-a-b)$	$\vert_{Top}$
$g_{8t} = RD(\rightarrow_{Top}) = max(0,a+b-1)$	$\wedge_{Bot}$
$g_{1b} = I(\rightarrow_{Bot}) = max(1-a,b)$	$\rightarrow_{Bot}$
$g_{2b} = C(\rightarrow_{Bot}) = max(a,1-b)$	$\leftarrow_{Bot}$
$g_{3b} = D(\rightarrow_{Bot}) = min(1-a,b)$	$\not\leftarrow_{Top}$
$g_{4b} = N(\rightarrow_{Bot}) = min(a,1-b)$	$\not\rightarrow_{Top}$
$g_{5b} = LC(\rightarrow_{Bot}) = max(a,b)$	$\vee_{Bot}$
$g_{6b} = LD(\rightarrow_{Bot}) = min(1-a,1-b)$	$\downarrow_{Top}$
$g_{7b} = RC(\rightarrow_{Bot}) = max(1-a,1-b)$	$\vert_{Bot}$
$g_{8b} = RD(\rightarrow_{Bot}) = min(a,b)$.	$\wedge_{Top}$

3 Characterization of logics by group transformation

Definition 1 *Group Transformations in Logic:*
Let f be any one of the 10 two-argument propositional connectives of a logic system and $\neg$ be an involutive negation. Then, we define the following transformations over f:

1. $I(f) = f(x,y)$: *Identity transformation*
2. $D(f) = \neg f(\neg x,\neg y)$: *Dual transformation*
3. $C(f) = f(\neg x,\neg y)$: *Contradual transformation*
4. $N(f) = \neg f(x,y)$: *Negation transformation*

It is well known that for the crisp (2-valued) logic these $\mathcal{T} = \{I,D,C,N\}$ transformations determine the Piaget group which is a realization of the abstract Klein 4-element group.

The new 4 non-symmetrical transformations below which are added to the above 4 basic transformations by Bandler and Kohout in 1979 [11],[12] enriches the algebraic structure of logical transformations.

Definition 2 *8-element Group Transformations: Bandler-Kohout*

5. $LC(f) = f(\neg x,y)$: *Left Contradual*
6. $RC(f) = f(x,\neg y)$: *Right Contradual*
7. $LD(f) = \neg f(\neg x,y)$: *Left Dual*
8. $RD(f) = \neg f(x,\neg y)$: *Right Dual*

This enlarged set of transformations $\mathcal{T} = \{I,\ D,\ C,\ N,\ LC,\ RC,\ LD,\ RD\}$ forms 8-element group $\{T,\circ\}$ called $\mathcal{S}_{2\times2\times2}$ group. The equation, $N^2 = C^2 = D^2 = LC^2 = LD^2 = RC^2 = RD^2 = I$, provides sufficient information to identify this group. Its group operations are shown in [13] in detail.

When $\mathcal{T} = \{I,\ D,\ C,\ N,\ LC,\ RC,\ LD,\ RD\}$ are applied to Łukasiewicz implication$(\rightarrow_{top})$ and to Kleene-Dienese implication$(\rightarrow_{bot})$ of m_1 system of logic, respectively, they yield the closed set of connectives as below [13],[10].

The definitions of group transformation use an involutive negation $\neg a = 1-a$ which satisfies all of four axioms below. An involutive negation, $\neg$, can be generated by a pair of Łukasiewicz implication$(\rightarrow_{Top})$ and a constant 0 or by that of Kleene-Dienese implication$(\rightarrow_{Bot})$ and constant(0):

$$\neg a = \begin{cases} a \rightarrow_{Top} 0 = a \overset{Ł}{\rightarrow} 0 = min(1,1-a+0) = 1-a \\ a \rightarrow_{Bot} 0 = a \overset{KD}{\rightarrow} 0 = max(1-a,0) = 1-a \end{cases}$$

However, both of them fall into a single point; there is no genuine interval of negation.

Axiom 3 *Axioms of Fuzzy Negation [1]:*

Let a negation $\neg$ a be defined by a function

$$\neg : [0,1] \rightarrow [0,1]$$

A function $\neg$ should satisfy at least two of axiomatic requirements below:

A1. $\neg\, 0 = 1$ *and* $\neg\, 1 = 0$. : *boundary condition.*
A2. *For all* $a,b \in [0,1]$, *if* $a \le b$, *then* $\neg a \ge \neg b$. : *monotonicity.*
A3. $\neg$ *is a continuous function.* : *continuity.*
A4. $\neg$ *is involutive, i.e.* $\neg(\neg\, a) = a,\ \forall a \in [0,1]$. : *involution.*

In the next section, we define various negations which yield intervals and investigate their properties.

4 Interval Negations in Many-Valued Logics

In [3], we have shown that the negations can also be defined by the Sheffer or the Nicod connectives as well as by the implication:

Definition 4 *Negation [3]:*

1. $\neg_S\, a = a \mid a$: by Sheffer
2. $\neg_N\, a = a \downarrow a$: by Nicod
3. $\neg_{PLY}\, a = a \rightarrow 0$: by Implication

In classical logic, negations by above definitions are all equal: $\neg\, a = \neg_S\, a = \neg_N\, a = \neg_{PLY}\, a = 1-a$. In many_valued logic, however, they do not yield the equal results – $\neg_S\, a \ne \neg_N\, a \ne \neg_{PLY}\, a$ – but these negations in checklist paradigm m_1 system generate a genuine interval negation [3].

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4.1 Negation on m_1 Defined by the Sheffer Connective

Since the Sheffer connective, $a \mid b$, of the interval system m_1 appears a TOP-BOT pair by means of RD transformation of $<\rightarrow_{Top}, \rightarrow_{Bot}>$ as it is shown at the table in section 3,

$$a \mid b = \begin{cases} a \mid_{Top} b = min(1, 2-a-b) \\ a \mid_{Bot} b = max(1-a, 1-b), \end{cases}$$

the connective type $\neg_S\, a$ also also appears in two forms:

$$\neg_S\, a = \begin{cases} a \mid_{Top} a = min(1, 2(1-a)) \\ a \mid_{Bot} a = 1-a \end{cases}$$

Thus, $\neg_S\, a$ defined by *Sheffer* generates the interval

$$[\, \mid_{Bot},\ \mid_{Top}\,] = [\, 1-a,\ \ min(1, 2(1-a))\,].$$

4.2 Negation in m_1 Defined by the Nicod Connective

Since the Nicod connective, $a \downarrow b$, appears as a TOP-BOT pair of connectives in the interval system m_1 , similarly

$$a \downarrow b = \begin{cases} a \downarrow_{Top} b = min(1-a, 1-b) \\ a \downarrow_{Bot} b = max(0, 1-a-b), \end{cases}$$

the connective type $\neg_N\, a$ also also appears in two forms:

$$\neg_N\, a = \begin{cases} a \downarrow_{Top} a = 1-a & = D(a \mid_{Bot} a) \\ a \downarrow_{Bot} a = max(0, 1-2a) & = D(a \mid_{Top} a). \end{cases}$$

Thus, the interval of $\neg_N\, a$ by *Nicod* is:

$$[\downarrow_{Bot},\ \downarrow_{Top}] = [max(0, 1-2a),\ 1-a].$$

4.3 Negation in m_1 Defined by the Implication Connective

Since $a \overset{KD}{\rightarrow} b \leq m_1(\rightarrow) \leq a \overset{Ł}{\rightarrow} b$, in $a \rightarrow b$ of m_1 system, i.e. the BOT connective is the Kleene-Dienes implication operator while the TOP id Łukasiewicz, we have:

$$\neg_{PLY}\ a = \begin{cases} a \rightarrow_T 0 = a \overset{Ł}{\rightarrow} 0 = min(1, 1-a+0) = 1-a \\ a \rightarrow_B 0 = a \overset{KD}{\rightarrow} 0 = max(1-a, 0) = 1-a \end{cases}$$

4.4 Interval of Negation in m_1 logic system

Two intervals of negation defined by the *Sheffer* connective and by the *Nicod* connective may be combined to yield a genuine interval of negation since $1-a$ is the lower bound of *Sheffer* negation and the upper bound of *Nicod* negation, respectively. These two intervals are concatenated as below:

$$\begin{aligned} [\neg_{BOT}, \neg_{TOP}] &= [\downarrow_{Bot}, \downarrow_{Top}] \cup [\, \mid_{Bot}, \mid_{Top}\,] \\ &= [\ \max(0, 1\text{-}2a),\ \min(1, 2(1\text{-}a)\ ] \\ &= \begin{cases} [\, 0,\ 2(1-a)\,], & 1 \geq a \geq .5 \\ [\, 1-2a,\ 1\,] & .5 > a \geq 0 \end{cases} \\ \neg_{MID} &= \neg_{PLY}\, a = a \downarrow_{Top} a = a \mid_{Bot} a = \ 1-a. \end{aligned}$$

Thus, *Sheffer*($\mid$) and *Nicod*($\downarrow$) form the TOP system of negation and the BOT system of negation, respectively. The $\neg_{PLY}$ $a = 1-a$, is a median value of the interval negation, $\neg_{Mid}$. Since both $\neg_{Top}$ and $\neg_{Bot}$ satisfy A1 - A3 of *Axioms of fuzzy negation* except A4 of sec.3, they are non-involutive TOP-BOT pair of fuzzy interval negations while $\neg_{Mid}$ is an involutive fuzzy negation.

Theorem 5 *Gap Theorem 1. (Bandler and Kohout [6])*

$$\begin{aligned} a \text{ AND}_{Top}\, b - a \text{ AND}_{Bot}\, b &= a \text{ OR}_{Top}\, b - a \text{ OR}_{Bot}\, b \\ &= a \text{ PLY}_{Top}\, b - a \text{ PLY}_{Bot}\, b \\ &= min(\varphi a,\ \varphi b). \\ a \text{ IFF}_{Top}\, b - a \text{ IFF}_{Bot}\, b &= a \text{ EOR}_{Top}\, b - a \text{ EOR}_{Bot}\, b \\ &= 2min(\varphi a,\ \varphi b). \end{aligned}$$

Similarly, we can describe Gap Theorem for a TOP-BOT pair of negation.

Theorem 6 *Gap Theorem 2.*

$$\neg_{TOP}\ a - \neg_{BOT}\ a = min(2a, 2(1-a)) = 2\varphi a$$

Hence, the margins of imprecision can be directly measured by the degree of fuzziness φ where $\varphi a = min(a, 1-a)$.

To further investigate a non-involutive properties of $[\neg_{Bot}, \neg_{Top}]$ pair, we apply multiple nagation. It leads to the iterative negations, interrupting mathematical properties and mathematical limit, but yields an interesting sequence of values of interval. In the following sections, we use the symbols $\neg_T$ and $\neg_B$ for $\neg_{Top}$ and $\neg_{Bot}$, respectively, for simplicity.

5 Multiple negations

As both $\neg_{Bot}(= \neg_B)$ and $\neg_{Top}(= \neg_T)$ are non-involutive, a question arise what would happen if a negation is iterated to $\neg_B$ and $\neg_T$, respectively: namely, a multiple negation.

Definition 7 *Multiple Negations.*

1. $\neg^1\ a = \neg\ a.$
2. $\neg^n\ a = (\neg\ a)^n = \neg\ (\neg^{n-1}\ a)$
   $= \neg(\neg(\neg^{n-2}\ a))$
   $= \underbrace{\neg(\neg(\neg \ldots, (\neg}_{n}\ a))), \quad 1 \leq n$

Thus, multiple negations will be applied to $\neg_T$ and $\neg_B$, respectively in the following subsections in order to explore their properties from the computational results. The following two subsections summarize the computational results of multiple negations of $\neg_{Top} a$ and $\neg_{Bot} a$, respectively .

5.1 Multiple negation of $\neg_{TOP}$

$$\begin{aligned} \neg_T a = a \mid_T a \quad &= min(1, 2(1-a)) \\ &= \begin{cases} 2(1-a), & a \in [\frac{1}{2},\ 1] \\ 1 & a \in [0, \frac{1}{2}) \end{cases} \end{aligned} \tag{1}$$

$$\begin{aligned} \neg_T^2 a = \neg_T(\neg_T a) \quad &= min(1,\ 2(1-\neg_T a)) \\ &= \begin{cases} 1, & a \in (\frac{3}{4}, 1] \\ 2(2a-1), & a \in [\frac{1}{2},\ \frac{3}{4}] \\ 0 & a \in [0,\ \frac{1}{2}) \end{cases} \end{aligned} \tag{2}$$

$$\begin{aligned} \neg_T^3 a = \neg_T(\neg_T^2 a) \quad &= min(1,\ 2(1-\neg_T^2 a))) \\ &= \begin{cases} 0, & a \in (\frac{3}{4},\ 1] \\ 2(3-4a), & a \in [\frac{5}{8},\ \frac{3}{4}] \\ 1 & a \in [0,\ \frac{5}{8}) \end{cases} \end{aligned} \tag{3}$$

$$\begin{aligned} \neg_T^4 a = \neg_T(\neg_T^3 a) \quad &= min(1,\ 2(1-\neg_T^3 a))) \\ &= \begin{cases} 1, & a \in (\frac{11}{16},\ 1] \\ 2(8a-5), & a \in [\frac{5}{8},\ \frac{11}{16}] \\ 0 & a \in [0,\ \frac{5}{8}) \end{cases} \end{aligned} \tag{4}$$

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$$\neg_T^5 a = \neg_T(\neg_T^4 a) \quad = min(1,\ 2(1-\neg_T^4 a))) = \begin{cases} 0, & a \in (\frac{11}{16},\ 1] \\ 2(11-16a), & a \in [\frac{21}{32},\ \frac{11}{16}] \\ 1 & a \in [0,\ \frac{21}{32}] \end{cases} \tag{5}$$

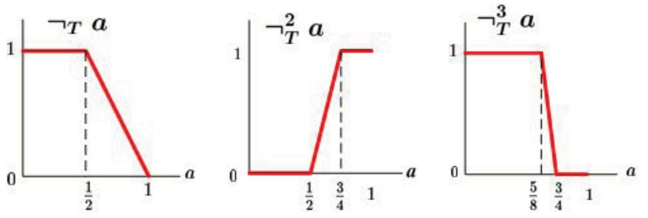

Figure 1: $\neg_T\ a,\ \neg_T^2\ a,\ \neg_T^3\ a$

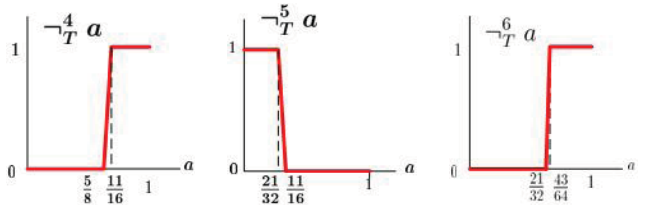

Figure 2: $\neg_T^4\ a,\ \neg_T^5\ a,\ \neg_T^6\ a$

If we further generalize it to $\neg_T^n\ a$ both for $n = 2k$ and for $n = 2k+1, \forall k \geq 1$, respectively, we get the following formulas (6)-(7).

$$\neg_T^{2k}\ a = \neg_T(\neg_T^{2k-1} a) \quad = \begin{cases} 1, & a \in (u_1,\ 1] \\ v_0, & a \in [u_2,\ u_1] \\ 0 & a \in [0,\ u_2) \end{cases} \tag{6}$$

where $v_0 = 2^{2k}a - \frac{2}{3}(2^{2k}-1)$,

$u_1 = \frac{2\cdot 4^k+1}{3\cdot 4^k}$,

$u_2 = \frac{2(4^k-1)}{3\cdot 4^k}$,

and

$$\neg_T^{2k+1}\ a = \neg_T(\neg_T^{2k} a) \quad = \begin{cases} 0, & a \in (u_3,\ 1] \\ v_1 & a \in [u_4,\ u_3] \\ 1 & a \in [0,\ u_4) \end{cases} \tag{7}$$

where $v_1 = \frac{2}{3}(2^{2k+1}+1) - 2^{2k+1}a$,

$u_3 = u_1 = \frac{2\cdot 4^k+1}{3\cdot 4^k}$,

$u_4 = \frac{4^{k+1}-1}{6\cdot 4^k}$.

Thus, any multiple negations in the odd sequence of $\neg_T^{2k+1}$, $\forall k \geq 0$, i.e. $< \neg_T a, \neg_T^3\ a, \ldots, \neg_T^{2k+1}\ a, \ldots >$, are monotonically decreasing(**Axiom A2**) while those in the even sequence of $\neg_T^{2k}$, $\forall k \geq 1$, $< \neg_T^2 a, \neg_T^4\ a, \ldots, \neg_T^{2k}\ a, \ldots >$, are monotonically increasing(**A2**). All of multiple negations satisfy the boundary condition(**A1**) and continuity(**A3**) .

5.2 *Multiple negation of* $\neg_{BOT}$

$$\neg_{bot} a = a \downarrow_B a \quad = max(0, 1-2a) = \begin{cases} 0, & a \in (\frac{1}{2},\ 1] \\ 1-2a & a \in [0,\ \frac{1}{2}] \end{cases} \tag{8}$$

$$\neg_B^2 a = \neg_B(\neg_B a) \quad = max(0, 1-2\cdot\neg_B a) = \begin{cases} 1, & a \in (\frac{1}{2},\ 1] \\ 4a-1, & \in [\frac{1}{4},\ \frac{1}{2}] \\ 0 & a \in [0,\ \frac{1}{4}) \end{cases} \tag{9}$$

$$\neg_B^3 a = \neg_B(\neg_B^2\ a) \quad = max(0, 1-2\cdot\neg_B^2\ a) = \begin{cases} 0, & a \in (\frac{3}{8},\ 1] \\ 3-8a, & a \in [\frac{1}{4},\ \frac{3}{8}] \\ 1 & a \in [0,\ \frac{1}{4}) \end{cases} \tag{10}$$

$$\neg_B^4 a = \neg_B(\neg_B^3\ a) \quad = max(0,\ 1-2\cdot\neg_B^3 a) = \begin{cases} 1, & a \in (\frac{3}{8},\ 1] \\ 16a-5, & a \in [\frac{5}{16},\ \frac{3}{8}] \\ 0 & a \in [0,\ \frac{5}{16}) \end{cases} \tag{11}$$

$$\neg_B^5 a = \neg_B(\neg_B^4\ a) \quad = max(0,\ 1-2\cdot\neg_B^4 a) = \begin{cases} 0, & a \in (\frac{11}{32}\ 1] \\ 11-32a, & a \in [\frac{5}{16},\ \frac{11}{32}] \\ 1 & a \in [0,\ \frac{5}{16}) \end{cases} \tag{12}$$

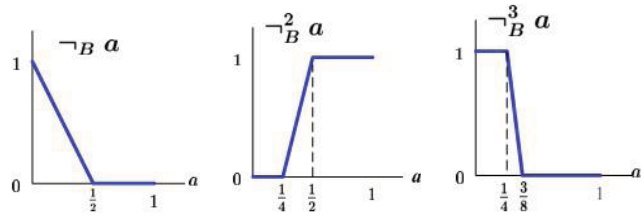

Figure 3: $\neg_B\ a,\ \neg_B^2\ a,\ \neg_B^3\ a$

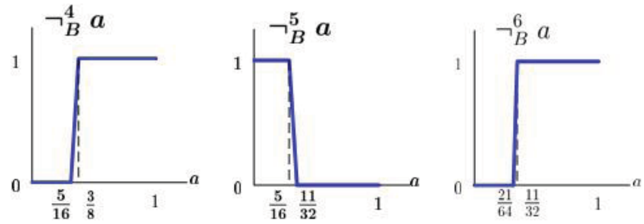

Figure 4: $\neg_B^4\ a,\ \neg_B^5\ a,\ \neg_B^6\ a$

If we further generalize it to $\neg_T^n\ a$ both for $n = 2k$ and for $n = 2k+1, \forall k \geq 1$, respectively, we get the following formulas (13)-(14).

$$\neg_B^{2k}\ a = \neg_B(\neg_B^{2k-1} a) \quad = \begin{cases} 1, & a \in (w_1,\ 1] \\ y_0, & a \in [w_2,\ w_1] \\ 0 & a \in [0,\ w_2) \end{cases} \tag{13}$$

where $y_0 = 2^{2k}a - \frac{1}{3}(2^{2k}-1)$,

$w_1 = \frac{4^k+2}{3\cdot 4^k}$,

$w_2 = \frac{4^k-1}{3\cdot 4^k}$,

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and

$$\neg_B^{2k+1} a = \neg_B(\neg_B^{2k} a) \quad = \begin{cases} 0, & a \in (w_3,\ 1] \\ y_1 & a \in [w_4,\ w_3] \\ 1 & a \in [0,\ w_4) \end{cases} \qquad (14)$$

where $y_1 = \frac{1}{3}(2^{2k+1}+1) - 2^{2k+1}a,$

$w_3 = \frac{2\cdot 4^k+1}{6\cdot 4^k},$

$w_4 = w_2 = \frac{4^k-1}{3\cdot 4^k}.$

Similar to TOP system of negation in sec. 5.1., any multiple negations in the odd sequence are monotonically decreasing while those in the even sequence are monotonically increasing. All of them satisfy the boundary condition and continuity. Both TOP-BOT systems of negation also show the following property in the interval of a.

Property 8 *The width of interval of a:*
The width of interval of a in eq.(6)-(7) and eq.(13)-(14) is:

1. *For* $n = 2k$: $\neg_T^{2k}\ a$ *and* $\neg_B^{2k}\ a$,
   $|\ u_1 - u_2\ | = |\ w_1 - w_2\ | = \frac{1}{2^{2k}} = \frac{1}{2^n},$
2. *For* $n = 2k+1$: $\neg_T^{2k+1}\ a$ *and* $\neg_B^{2k+1}\ a$,
   $|\ u_3 - u_4\ | = |\ w_3 - w_4\ | = \frac{1}{2^{2k+1}} = \frac{1}{2^n}.$

Theorem 9 *A relationship between* $\neg^n$ *and* $\neg^{n+1}$:

(1) $1 - \neg_T^n\ a = \frac{1}{2}\neg_T^{n+1}\ a,\ \ \forall n \in N,\ a \in [max(u_2,u_4),\ u_1]$ *in eq.(6)-(7).*

(2) $1 - \neg_B^{n+1}\ a = 2\cdot\neg_B^n\ a,\ \ \forall n \in N,\ a \in [w_2,\ min(w_1,w_3)]$ *in eq.(13)-(14).*

Proof:

(1) (*a*) Let $n = 2k,\ \forall k \geq 1.$

$$\begin{aligned} 1 - \neg_T^{2k} a &= 1 - (2^{2k}a - \tfrac{2}{3}(2^{2k}-1)) \\ &= \tfrac{1}{3}(2^{2k+1}+1) - 2^{2k}a \\ &= \tfrac{1}{2}(\tfrac{2}{3}(2^{2k+1}+1) - 2^{2k+1}a) \\ &= \tfrac{1}{2}\cdot\neg_T^{2k+1}\ a \end{aligned}$$

(*b*) Let $n = 2k+1, \forall k \geq 0.$

$$\begin{aligned} 1 - \neg_T^{2k+1} a &= 1 - (\tfrac{2}{3}(2^{2k+1}+1) - 2^{2k+1}a) \\ &= 2^{2k+1}a - \tfrac{1}{3}(2^{2k+2}-1) \\ &= \tfrac{1}{2}(2^{2k+2}a - \tfrac{2}{3}(2^{2k+2}-1)) \\ &= \tfrac{1}{2}\neg_T^{2k+2}\ a \end{aligned}$$

Therefore, $1 - \neg_T^n\ a = \frac{1}{2}\neg_T^{n+1}\ a,\ ,\ \forall n \in N.$ □

(2) (*a*) $n = 2k-1, \forall k \geq 1.$

$$\begin{aligned} 1 - \neg_B^{2k} a &= 1 - (2^{2k}a - \tfrac{1}{3}(2^{2k}-1)) \\ &= \tfrac{2}{3}(2^{2k-1}+1) - 2^{2k}a \\ &= 2(\tfrac{1}{3}(2^{2k-1}+1) - 2^{2k-1}a) \\ &= 2\cdot\neg_B^{2k-1}\ a \end{aligned}$$

(*b*) $n = 2k, \forall k \geq 1.$

$$\begin{aligned} 1 - \neg_B^{2k+1} a &= 1 - (\tfrac{1}{3}(2^{2k+1}+1) - 2^{2k+1}a) \\ &= 2^{2k+1}a - \tfrac{2}{3}(2^{2k}-1) \\ &= 2(2^{2k}a - \tfrac{1}{3}(2^{2k}-1)) \\ &= 2\cdot\neg_B^{2k}\ a \end{aligned}$$

Therefore, $1 - \neg_B^{n+1}\ a = 2\cdot\neg_B^n\ a,\ \forall n \in N.$ □

5.3 *Convergence of multiple negations*

In order to investigate a convergence of multiple negations, let us apply lim to $\neg_T^{2k}\ a$, $\neg_T^{2k+1}\ a$, $\neg_B^{2k}\ a$ and to $\neg_B^{2k+1}\ a$ as $k \to \infty$, respectively. It derives the following formula of convergence in each case:

$$\lim_{k\to\infty} \neg_T^{2k}\ a = \begin{cases} 1, & a \in (\frac{2}{3}^+,\ 1] \\ z_0, & a \in [\frac{2}{3}^-,\ \frac{2}{3}^+] \\ 0 & a \in [0,\ \frac{2}{3}^-) \end{cases} \qquad (15)$$

where $z_0 = \lim_{k\to\infty} v_0 = \lim_{k\to\infty} 2^{2k}a - \dfrac{2}{3}(2^{2k}-1)$, in eq.(6)
$z_0 = \frac{2}{3}$ if $a = \frac{2}{3}$, in particular,
and $\frac{2}{3}^+ = \lim_{k\to\infty} u_1 = \frac{2}{3} + \varepsilon_1,$
$\frac{2}{3}^- = \lim_{k\to\infty} u_2 = \frac{2}{3} - \varepsilon_1$, for a very small $\varepsilon_1 > 0.$

$$\lim_{k\to\infty} \neg_T^{2k+1}\ a = \begin{cases} 0, & a \in (\frac{2}{3}^+,\ 1] \\ z_1 & a \in [\frac{2}{3}^-,\ \frac{2}{3}^+] \\ 1 & a \in [0,\ \frac{2}{3}^-) \end{cases} \qquad (16)$$

where $z_1 = \lim_{k\to\infty} v_1 = \lim_{k\to\infty} \dfrac{2}{3}(2^{2k+1}+1) - 2^{2k+1}a$, in eq.(7)
$z_1 = \frac{2}{3}$ if $a = \frac{2}{3}$, in particular,
and $\frac{2}{3}^+ = \lim_{k\to\infty} u_3 = \frac{2}{3} + \varepsilon_2,$
$\frac{2}{3}^- = \lim_{k\to\infty} u_4 = \frac{2}{3} - \varepsilon_2$, for a very small $\varepsilon_2 > 0$
where $\varepsilon_2 < \varepsilon_1$.

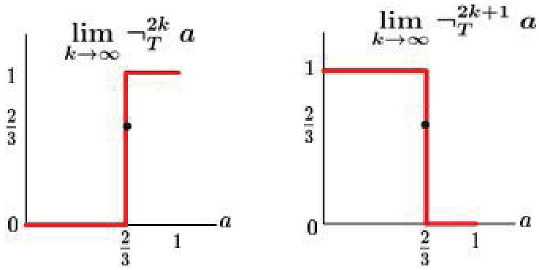

Figure 5: $\lim_{k\to\infty} \neg_T^{2k}\ a$ and $\lim_{k\to\infty} \neg_T^{2k+1}\ a$

As $k \to \infty$. both lim $\neg_T^{2k}\ a$ and lim $\neg_T^{2k+1}\ a$ converge either to 0 or to 1, depending on the values of a in the most of intervals, except $a \in [\frac{2}{3} - \varepsilon_i,\ \frac{2}{3} + \varepsilon_i]$ for $i = 1,2$ where ε_i is in eq.(15)-(16). For an a in the interval $[\frac{2}{3} - \varepsilon_i,\ \frac{2}{3} + \varepsilon_i]$, the values of $\lim_{k\to\infty} \neg_T^{2k}\ a$ or those of $\lim_{k\to\infty} \neg_T^{2k+1}\ a$ increases or decreases drastically in monotonicity, respectively.

We can investigate the limit negations of $\neg_B^{2k}\ a$ and $\neg_B^{2k+1}\ a$ of BOT system, similarly.

$$\lim_{k\to\infty} \neg_B^{2k}\ a = \begin{cases} 1, & a \in (\frac{1}{3}^+,\ 1] \\ z_2, & a \in [\frac{1}{3}^-,\ \frac{1}{3}^+] \\ 0 & a \in [0,\ \frac{1}{3}^-) \end{cases} \qquad (17)$$

where $z_2 = \lim_{k\to\infty} y_0 = \lim_{k\to\infty} 2^{2k}a - \dfrac{1}{3}(2^{2k}-1)$, in eq.(13)
$z_2 = \frac{1}{3}$ if $a = \frac{1}{3}$, in particular,
and $\frac{1}{3}^+ = \lim_{k\to\infty} w_1 = \frac{1}{3} + \delta_1,$
$\frac{1}{3}^- = \lim_{k\to\infty} w_2 = \frac{1}{3} - \delta_1$, for a very small $\delta_1 > 0.$

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$$\lim_{k\to\infty} \neg_B^{2k+1} a = \begin{cases} 0, & a \in (\frac{1}{3}^+,\ 1] \\ z_3, & a \in [\frac{1}{3}^-,\ \frac{1}{3}^+] \\ 1 & a \in [0,\ \frac{1}{3}^-) \end{cases} \quad (18)$$

where $z_3 = \lim_{k\to\infty} y_1 = \lim_{k\to\infty} \frac{1}{3}(2^{2k+1}+1) - 2^{2k+1}a$, in eq.(14)
$z_3 = \frac{1}{3}$ if $a = \frac{1}{3}$, in particular,
and $\frac{1}{3}^+ = \lim_{k\to\infty} w_3 = \frac{1}{3} + \delta_2$,
$\frac{1}{3}^- = \lim_{k\to\infty} w_4 = \frac{1}{3} - \delta_2$, for a very small $\delta_2 > 0$ where $\delta_2 < \delta_1$.

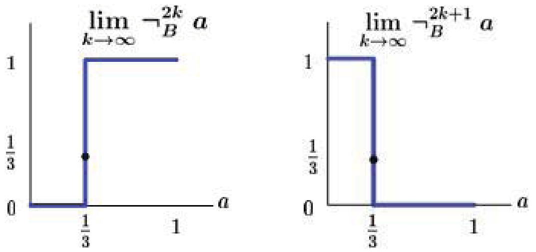

Figure 6: $\lim_{k\to\infty} \neg_B^{2k}\ a$ and $\lim_{k\to\infty} \neg_B^{2k+1}\ a$

As $k \to \infty$. both $\lim \neg_B^{2k}\ a$ and $\lim \neg_B^{2k+1}\ a$ converge either to 0 or to 1 for the most of $a \in [0,\ 1]$, except $a \in [\frac{1}{3} - \delta_i,\ \frac{1}{3} + \delta_i]$, for $i = 1, 2$ where δ_i is in eq.(17)-(18). For an a in $[\frac{1}{3} - \delta_i,\ \frac{1}{3} + \delta_i]$, the values of $\lim_{k\to\infty} \neg_B^{2k}\ a$ or those of $\lim_{k\to\infty} \neg_B^{2k+1}\ a$ are in the drastic monotonic increase or decrease, respectively. Thus, we can further describe the following properties.

Property 10 *For $n \in N$,*

P1. $\lim_{n\to\infty} \neg_T^{n+1}\ a \approx 1 - \lim_{n\to\infty} \neg_T^{n}\ a,\quad a \in [0,\ 1].$

P2. $\lim_{n\to\infty} \neg_B^{n+1}\ a \approx 1 - \lim_{n\to\infty} \neg_B^{n}\ a,\quad a \in [0,\ 1].$

P3. $\lim_{k\to\infty} \neg_T^{n}\ a \approx \lim_{n\to\infty} \neg_B^{n}\ (a + \frac{1}{3}),\quad a \in [0,\ 1].$

P4. $\lim_{n\to\infty} \neg_B^{n}\ a \approx \lim_{k\to\infty} \neg_T^{n}\ (a - \frac{1}{3}),\quad a \in [0,\ 1].$

From the above **P1** and **P2**, we can notice that both $\neg_T$ and $\neg_B$ are *nearly limit involutive* because $\lim_{n\to\infty} \neg_T^{n+1}\ a \approx 1 - \lim_{n\to\infty} \neg_T^{n}\ a \approx 1 - (1 - \lim_{n\to\infty} \neg_T^{n-1}\ a) \approx \lim_{n\to\infty} \neg_T^{n-1}\ a,\quad a \in [0,\ 1]$, i.e. $\neg_T(\neg_T(\lim_{n\to\infty} \neg_T^{n-1}) \approx \lim_{n\to\infty} \neg_T^{n-1}$. It holds for $\neg_B$, similarly.

6 Conclusions

The involutive fuzzy negation, $\neg a = 1 - a$, has been collapsed into a single point while ten of binary connectives yield TOP-BOT interval pairs in m_1 logic system of 16 connectives by group of logic transformations since Bandler-Kohout proposed five checklist paradigm based logic systems $m_1 - m_5$. A BOT-TOP pair of connectives of fuzzy negation interval $[\neg_{BOT},\ \neg_{TOP}]$, however, could be successfully generated from the interval of *Nicod* system $[\downarrow_{top},\ \downarrow_{bot}]$ and that of *Sheffer* system $[|_{bot},\ |_{top}]$. These pairs of negation connective are non-involutive negation pairs by themselves. When a negation is applied iteratively, those multiple negations, $\neg_T^n\ a, \neg_B^n\ a$ change their values from 0 to 1 or 1 to 0 drastically within a very small interval whose width is $\frac{1}{2^n}$ and reveals a property of nearly limit involution.

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