Optimistic Fuzzy Weighted Average

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Abstract *– The fuzzy weighted average (FWA) is used in many engineering problems where aggregation of fuzzy information is dealt with. In this framework, many algorithms have been proposed to compute efficiently the FWA according to Zadeh's extension principle. However, due to fuzzy interval calculus, the exact solution presents a characteristic that may be viewed as an important drawback. Indeed, replacing each individual by the fuzzy weighted average in the assessment of the population score leads to a result different from the original one. The presented work is an attempt to propose an optimistic counterpart for the FWA that eliminates the mentioned characteristic, possibly undesirable. Actually, the optimistic FWA is computed by using a modified division operator. When existing, the latter is the inverse operator of the fuzzy multiplication. Contrary to the conventional FWA, the optimistic one can be computed by a sequence of elementary arithmetic operations.*

Keywords *– Fuzzy arithmetic, Fuzzy weighted average, Gradual real numbers*

1 Introduction

The average is probably the easiest and the most widespread solution to aggregate information. The average is defined as the single value which all the individuals of a population should have so that their total is unchanged. When the sample contains several times the same individuals, it is possible to reformulate the average as a weighted average by introducing coefficients related to each individual. So a weighted average is an average in which a weight w_i is assigned to each quantity x_i to be averaged. The weights determine the relative importance of each quantity on the average. Considering N numbers x_i with associated weights w_i, the weighted average y is expressed as follows:

$$y = f(x_1, x_2, ..., x_N, w_1, w_2, ..., w_N) = \frac{x_1 w_1 + x_2 w_2 + ... + x_N w_N}{w_1 + w_2 + ... + w_N} \quad (1)$$

In an imprecise environment where information are poorly defined, it may be appropriate to represent scores and weighting coefficients by fuzzy numbers $\tilde{x}_i$ and $\tilde{w}_i$. In this case, the weighted average becomes a fuzzy number $\tilde{y}$ as well, i.e. the fuzzy weighted average (FWA).

If we follow the traditional fuzzy set theory, the fuzzy average $\tilde{y} = f(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_N, \tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_N)$ is obtained according to the extension principle that is:

$$\mu_{\tilde{y}}(y) = \sup_{\substack{x_1, .., x_N, w_1, .., w_N \ / \\ y = f(x_1, .., x_N, w_1, .., w_N)}} \min(\mu_{\tilde{x}_1}(x_1), .., \mu_{\tilde{x}_N}(x_N), \mu_{\tilde{w}_1}(w_1), .., \mu_{\tilde{w}_N}(w_N)) \quad (2)$$

Many authors proposed computational algorithms providing a discrete but exact solution of (2). All methods are based on the α-cut representation of fuzzy sets and interval analysis as initially suggested by Dong and Wong [5]. In interval computation, efficient algorithms exist for specific classes of functions, especially for the class of fractionally-linear functions [12]. The weighted average is an example of such functions and the problem of computing exactly the FWA is thus practically solvable. Liou and Wang [14] were the first to observe that since the x_i appear only in the numerator of (1), only the smallest values of the x_i are used to find the smallest value of (1), and only the largest values of the x_i are used to find the largest value of (1). This decomposition is used to reduce the algorithmic complexity in recent proposed approaches ([4], [8], [9], [12], [13], [15]).

Even if the above mentioned iterative procedures determine the exact solution for the FWA and so cope with multiple appearance of variables in the expression of the function f in (1), there is no known closed-form formula for computing $\tilde{y}$. Moreover, there is no available algorithm that could be directly implemented using elementary arithmetic operations between fuzzy operands. This aspect is studied in [16] where the authors are interested in designing a symbolic engine that could transform any given function into a sequence of elementary operations for which fuzzy interval computation would achieve exact result (without overestimation). Unfortunately, it was proven by Nguyen *et al.* [16] that operations with one or two fuzzy operands are not sufficient to describe generic functions on fuzzy sets.

In addition to the difficulty of obtaining the exact solution, the fuzzy weighted average has an important drawback directly related to fuzzy interval calculus. Indeed, replacing each individual by the fuzzy weighted average in the assessment of the population score leads to a result different from the original one. Actually, when the fuzzy average is viewed as expressing a requirement of maximal tolerance on a variable y which is itself the result of a computation involving quantities $x_1, x_2, ..., x_N$ whose values are implicitly constrained by this calculation, it is desirable to solve the fuzzy

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equation:

$$B \otimes \tilde{y} = A \tag{3}$$

with $B = \tilde{w}_1 \oplus \tilde{w}_2 \ ..\oplus..\ \tilde{w}_N$ (4)

and $A = (\tilde{w}_1 \otimes \tilde{x}_1) \oplus (\tilde{w}_2 \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{w}_N \otimes \tilde{x}_N)$

for determining $\tilde{y}$. For example, such a procedure would be probably well suited to the case where the FWA is used for aggregating rule contributions in a Sugeno-like fuzzy system that would deal with fuzzy inputs.

In this context of «optimistic fuzzy interval calculus» [6], standard fuzzy operations can not be used directly for computing $\tilde{y}$. It means that a modified division denoted $\ominus_{\approx}$ must be searched for so that computing $\tilde{y}$ as $A \ominus_{\approx} B$ guarantees the satisfaction of the equality constraint (3). In the chosen context, both fuzzy quantities A and B are computed with the same weights $\tilde{w}_i$ according to (4) but the computation of $\tilde{y}$ does not handle explicitly weight interaction. The proposed approach is thus different from constrained fuzzy arithmetics as dealt with by Klir in [10] whose objective is an appropriate handling of interactive variables.

The motivation of this paper is twofold. Indeed, the presented work is an attempt:

- to propose an optimistic counterpart for the fuzzy weighted average,
- to compute the latter by a sequence of elementary operations.

This paper is organized as follows. Section 2 introduces fuzzy intervals with emphasis on their profile representation and their combination by arithmetic operators. In section 3, a new modified division operator is proposed for directly solving fuzzy equations. Then, section 4 is devoted to the implementation of the optimistic fuzzy weighted average with the presentation of two examples taken from the literature.

2 Fuzzy arithmetic operators

2.1 Fuzzy Intervals and Profiles

Let us consider a unimodal fuzzy interval A with kernel value K_A and support $S_A = [S_A^-, S_A^+]$ where S_A^- and S_A^+ are respectively the lower and upper bounds, of the interval S_A.

In order to specify the fuzzy interval shape, two additional functions are used to link the support with the kernel value. These functions, called left and right profiles, respectively denoted A^- and A^+, are defined from the membership function μ_A by:

$$\begin{aligned} A^-(\lambda) &= \text{Inf } \{x \mid \mu_A(x) \geq \lambda; x \geq S_A^-\} \\ A^+(\lambda) &= \text{Sup } \{x \mid \mu_A(x) \geq \lambda; x \leq S_A^+\} \end{aligned} \tag{5}$$

where $\lambda \in [0, 1]$. The profile A^- (resp. A^+) is an increasing (resp. decreasing) mapping that corresponds to the left (resp. right) part of the fuzzy interval A. It can be easily stated that:

$$K_A = A^-(1) = A^+(1), \tag{6}$$

$$S_A = [A^-(0), A^+(0)]. \tag{7}$$

Finally, the fuzzy interval A is univoquely defined by its left and right profiles. Thus, in the same way that the conventional interval S_A is denoted $[S_A^-, S_A^+]$, the fuzzy interval A will be denoted $[A^-, A^+]$. Equivalently, A can be viewed as the family of nested intervals $A(\lambda) = [A^-(\lambda), A^+(\lambda)]$, $\lambda \in [0, 1]$, when an explicit formulation with respect to λ is preferred.

For simplicity, the following additional notations are used in the remaining part of the paper. Given the interval $S = [S^-, S^+]$, its midpoint $M(S)$, its radius $R(S)$ and its relative extent $Rex(S)$ are respectively defined by:

$$M(S) = (S^+ + S^-) / 2, \tag{8}$$

$$R(S) = (S^+ - S^-) / 2, \tag{9}$$

$$Rex(S) = R(S) / M(S). \tag{10}$$

2.2 Arithmetic operators

Let $A = [A^-, A^+]$ and $B = [B^-, B^+]$ be two fuzzy intervals, the classical four arithmetic operations are expressed by:

$$A \oplus B = [A^- + B^-, A^+ + B^+] \tag{11}$$

$$A \ominus B = [A^- - B^+, A^+ - B^-] \tag{12}$$

$$A \otimes B = [min\ Z, max\ Z] \quad \text{where } Z = \{A^-B^-, A^-B^+, A^+B^-, A^+B^+\} \tag{13}$$

$$A \oslash B = [A^-, A^+] \otimes [1/B^+, 1/B^-] \quad \text{for } B \text{ such that } 0 \notin S_B \tag{14}$$

In order to cope with the twofold objective of solving equation (3) for determining the optimistic fuzzy average $\tilde{y}$ and of computing the fuzzy solution by a sequence of elementary operations, it is quite natural to search for a representation of $\tilde{y}$ in the form:

$$\tilde{y} = A \oslash B \tag{15}$$

where A and B are defined according to equation (4).

However, using the usual division and multiplication operators given by (14) and (13), it can be easily stated that substituting $\tilde{y}$ computed according to (15) into (3) gives a result more imprecise than the original A. Figure 1 illustrates this characteristic using two triangular fuzzy numbers A and B, with $K_A = 2.4$, $S_A = [0.8, 5.8]$ and $K_B = 1$, $S_B = [0.4, 1.9]$.

At best $A \subseteq B \otimes (A \oslash B)$ which means that the desired equality is generally not achieved. This problem is related to the lack of inverses in the calculus of fuzzy quantities.

Thus, a way around overestimation problems must be searched for outside standard arithmetic operations. One may think of using fuzzy arithmetic with requisite equality constraints as proposed in [10]. Klir's idea consists in doing fuzzy arithmetic with constraints dictated by the context of the problem. In practice, constraints are achieved by a-priori knowing that the α-cuts from two variables are the same. The approach is thus efficient for avoiding overestimation due to the occurrence of interactive variables. However, even when using constrained arithmetic, the calculus of fuzzy quantities is still pessimistic about the precision.

In the context of optimistic fuzzy calculus, we propose to use the modified division operator presented in next section.

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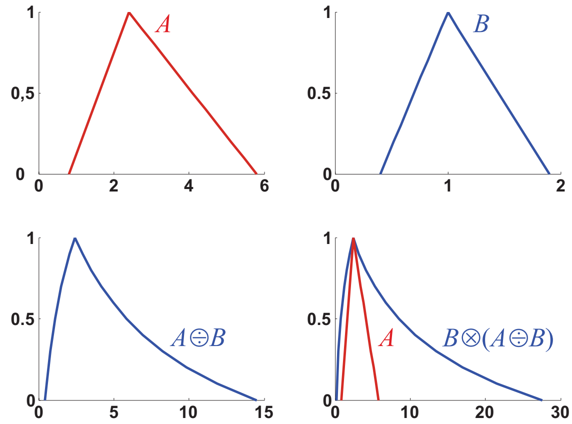

Figure 1: Fuzzy division and multiplication

3 The modified $\ominus_{\approx}$ Operator

3.1 Definition

Let $A = [A^-, A^+]$ and $B = [B^-, B^+]$ be two unimodal fuzzy intervals with $0 \notin S_B$. The result of the modified division $A \ominus_{\approx} B$ is defined from two gradual real numbers [7], denoted Φ^- and Φ^+, defined by their assignment functions from [0, 1] to the reals, as :

$$\Phi^-(\lambda) = \frac{Num^-(\lambda)}{Den^-(\lambda)} = \frac{M(A(\lambda)) - R(A(\lambda)) \cdot sign(M(S_B))}{M(B(\lambda)) - R(B(\lambda)) \cdot sign(Num^-(\lambda))} \tag{16}$$

and

$$\Phi^+(\lambda) = \frac{Num^+(\lambda)}{Den^+(\lambda)} = \frac{M(A(\lambda)) + R(A(\lambda)) \cdot sign(M(S_B))}{M(B(\lambda)) + R(B(\lambda)) \cdot sign(Num^+(\lambda))} \tag{17}$$

It is clear that, given A and B, both gradual numbers Φ^- and Φ^+ can always be computed. It is thus always possible to define $A \ominus_{\approx} B$ as the ordered pair (Φ^-, Φ^+) considering Φ^- as a left profile and Φ^+ as a right profile. Such entities are studied in [11] where it is proposed to add an extra feature, called "orientation", to pairs of profiles so as to build "ordered fuzzy numbers".

It is shown in [2] that the modified division $\ominus_{\approx}$ is the inverse operator of $\otimes$, that is $B \otimes (A \ominus_{\approx} B) = (A \ominus_{\approx} B) \otimes B = A$. Further details on the way equations (16) and (17) were established can be found in [3].

There is no guarantee that the ordered pair (Φ^-, Φ^+) is a gradual interval [7]. In other words, computations of Φ^- and Φ^+ according to (16) and (17) do not guarantee that $\forall \lambda \in [0, 1]$, $\Phi^-(\lambda) \leq \Phi^+(\lambda)$. It is however possible to determine a necessary and sufficient condition on operands A and B, under which $A \ominus_{\approx} B = (\Phi^-, \Phi^+)$ is the gradual interval $[\Phi^-, \Phi^+]$.

For unimodal operands, requiring that $\Phi^- \leq \Phi^+$ at the kernel and support levels, that is :

$$\Phi^-(1) = \Phi^+(1) = K_\Phi$$
$$\Phi^-(0) \leq K_\Phi \leq \Phi^+(0)$$

leads to the following condition on A and B:

$$\frac{1 - |Rex(S_A)|}{1 - S \cdot |Rex(S_B)|} \leq \frac{\delta K}{\delta M} \leq \frac{1 + |Rex(S_A)|}{1 + |Rex(S_B)|} \tag{18}$$

where $\delta K = K_A / K_B$, $\delta M = M(S_A) / M(S_B)$ and $S = sign(1 - |Rex(S_A)|)$. Extending (18) to all λ-cuts leads to a necessary and sufficient condition under which $[\Phi^-, \Phi^+]$ is a gradual interval. Without any additional monotonicity assumption, the obtained interval is not always a fuzzy interval in the sense that λ-cuts are not necessarily nested.

Next subsection illustrates different behaviors of the pair (Φ^-, Φ^+).

3.2 Modified division of fuzzy triangular sets

Using the modified division operator for computing the division $A \ominus_{\approx} B$ where A and B are the triangular fuzzy subsets dealt with previously in figure 1, the fuzzy subset plotted in figure 2 is obtained. It can be stated that the latter is less imprecise than the standard division $A \ominus B$. As desired, the computed result is the exact solution of the fuzzy equation $B \otimes X = A$. It induces that X is implicitly less imprecise than A which is clearly verified in figure 2.

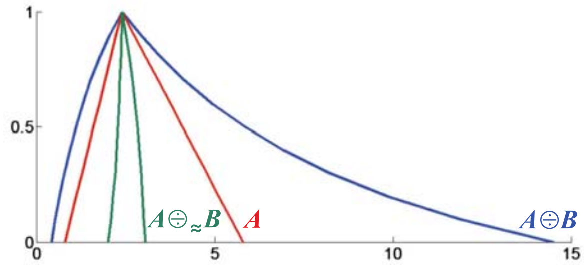

Figure 2: Modified fuzzy division

One may try to compute $1 \ominus_{\approx} A$ in order to determine the inverse of A according to $\ominus_{\approx}$ where 1 is the degenerated fuzzy subset associated with the crisp value 1. Considering the fuzzy subset A defined previously in figure 1, the result of the computation using equations (16) and (17) is given in figure 3. It appears clearly that (Φ^-, Φ^+) is not a gradual interval since the left and right profiles are exchanged, that is: $\Phi^+(\lambda) < \Phi^-(\lambda)$ for $\lambda \neq 1$.

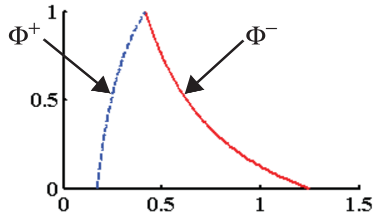

Figure 3: Computation of Φ^- and Φ^+ for $1 \ominus_{\approx} A$

It would have been possible to detect the «non-inversibility» of A prior to calculus according to the violation of condition (18).

The «non-inversibility» statement can be generalized to any

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non degenerated fuzzy subset. Indeed, let us suppose a fuzzy unimodal interval A, $0 \notin S_A$, such that $1 \ominus_{\approx} A$ exists. In this case, condition (18) has to be verified. As $K_1 = M(S_1) = 1$ and $R(S_1) = Rex(S_1) = 0$, condition (18) becomes:

$$\frac{1}{1-|Rex(S_A)|} \leq \frac{M(S_A)}{K_A} \leq \frac{1}{1+|Rex(S_A)|} \tag{19}$$

As $0 \notin S_A$, $M(S_A)$ and K_A have the same sign. Let us first consider that $M(S_A)$ and K_A are both positive. Then, inequality (19) is rewritten as follows:

$$\frac{M(S_A)}{M(S_A)-R(S_A)} \leq \frac{M(S_A)}{K_A} \leq \frac{M(S_A)}{M(S_A)+R(S_A)} \tag{20}$$

otherwise expressed as:

$$S_A^+ = M(S_A)+R(S_A) \leq K_A \leq M(S_A)-R(S_A) = S_A^- \tag{21}$$

The single case where condition (21) holds, i.e. A is invertible, is for A being a crisp positive number. The case when $M(S_A)$ and K_A are both negative leads to a similar conclusion.

3.3 Modified division for computing FWA

In the context of computing weighted average, weights are usually supposed to be positive numbers. Extending to fuzzy weighted average, fuzzy positive weights $\tilde{w}_i$ are also assumed. Thus, computing the FWA $\tilde{y}$ as $A \ominus_{\approx} B$ with B being the sum of the weights, i.e. $B = \tilde{w}_1 \oplus \tilde{w}_2 \ ..\oplus.. \ \tilde{w}_N$, one can restrict the closed form of the modified division to the case of B positive. Equations (16) and (17) are then rewritten as $A \ominus_{\approx} B = [\Phi^-, \Phi^+]$, with

$$\Phi^-(\lambda) = \frac{A^-(\lambda)}{M(B(\lambda)) - R(B(\lambda)) \cdot sign(A^-(\lambda))} \tag{22}$$

and

$$\Phi^+(\lambda) = \frac{A^+(\lambda)}{M(B(\lambda)) + R(B(\lambda)) \cdot sign(A^+(\lambda))}. \tag{23}$$

4 Examples

4.1 Two-term average

To illustrate the proposed method for computing the fuzzy weighted average, a two-term example discussed in [9] is considered. Hence, we focus on the computation of:

$$\tilde{y} = (\tilde{x}_1 \otimes \tilde{w}_1 \oplus \tilde{x}_2 \otimes \tilde{w}_2) \ominus_{\approx} (\tilde{w}_1 \oplus \tilde{w}_2) \tag{24}$$

where fuzzy triangular scores $\tilde{x}_1$, $\tilde{x}_2$ and fuzzy triangular weights $\tilde{w}_1$, $\tilde{w}_2$ are illustrated in figure 4. Using the profile representation, it follows:

$\tilde{x}_1 = [\lambda, 2-\lambda]$,
$\tilde{x}_2 = [\lambda+2, 4-\lambda]$,
$\tilde{w}_1 = [0.3\lambda, 0.9-0.6\lambda]$,
$\tilde{w}_2 = [0.3\lambda+0.4, 1-0.3\lambda]$.

The computation of $\tilde{y}$ according to (24) leads to the fuzzy result plotted in figure 5. As both fuzzy variables $\tilde{x}_1$ and $\tilde{x}_2$ are positive, according to (22) and (23) the developed forms of the left and right profiles of $\tilde{y}$ are given by :

$$\tilde{y}^- = \frac{\tilde{x}_1^- \cdot \tilde{w}_1^- + \tilde{x}_2^- \cdot \tilde{w}_2^-}{\tilde{w}_1^- + \tilde{w}_2^-} \tag{25}$$

$$\tilde{y}^+ = \frac{\tilde{x}_1^+ \cdot \tilde{w}_1^+ + \tilde{x}_2^+ \cdot \tilde{w}_2^+}{\tilde{w}_1^+ + \tilde{w}_2^+} \tag{26}$$

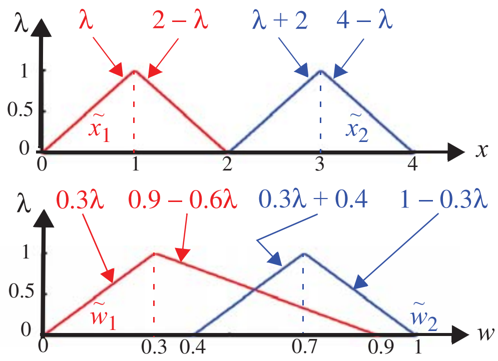

Figure 4: Fuzzy quantities and weights

Althrough the λ parametrization has been omitted in equations (25) and (26), all involved profiles are fonctions from the unit interval [0, 1] to the real line. It means that the addition, multiplication and division operations are function operations.

It can be observed that the computed optimistic FWA is a gradual interval. However, that one is not a fuzzy interval since its left profile is not an increasing function with respect to λ.

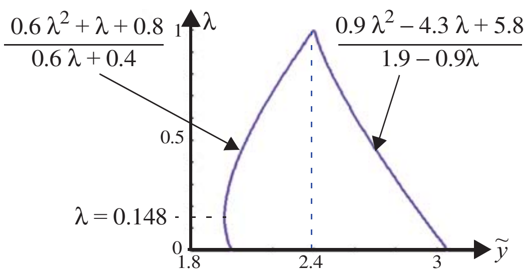

Figure 5: Optimistic fuzzy weighted average

For comparison purpose, figure 6 regroups the exact FWA according to Zadeh's extension principle with its lower and upper approximations, respectively obtained with the modified division operator and the conventional one. At the kernel level (λ = 1), all approaches give the same result, i.e. the weighted average for precise weights and numbers. At the other levels, the following inclusion is obtained:

$$\text{FWA}_{\text{optimistic}} \subseteq \text{FWA}_{\text{exact}} \subseteq \text{FWA}_{\text{arithmetic}}. \tag{27}$$

In the very simple example under consideration, it can be observed that the $\tilde{x}_i$ variables are sorted. Then, according to [13], the analytical closed form of the exact FWA is expressed by:

$$\tilde{y}^- = \frac{\tilde{x}_1^- \cdot \tilde{w}_1^+ + \tilde{x}_2^- \cdot \tilde{w}_2^-}{\tilde{w}_1^+ + \tilde{w}_2^-} \tag{28}$$

$$\tilde{y}^+ = \frac{\tilde{x}_1^+ \cdot \tilde{w}_1^- + \tilde{x}_2^+ \cdot \tilde{w}_2^+}{\tilde{w}_1^- + \tilde{w}_2^+} \tag{29}$$

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However, in the general case it may not be possible to order the $\tilde{x}_i$ fuzzy variables and numerical methods have to be applied for discretized λ values. In this context, mathematical programming techniques may be preferred ([9], [8]). Methodologies based on the handling of gradual numbers are also sensitive to the ordering problem. It is thus suggested in [7] to divide the λ-range in several parts for which specific orders are defined.

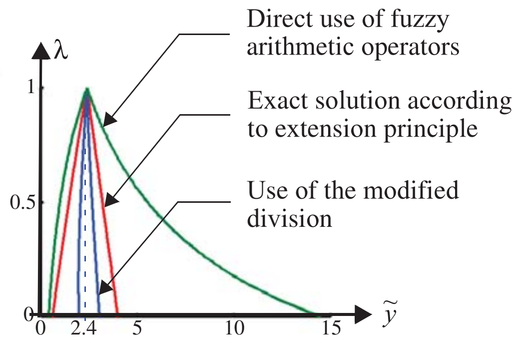

Figure 6: Fuzzy weighted average

4.2 Five-term example

This example taken from [13] consists in computing a five-term FWA, with the fuzzy triangular numbers $\tilde{x}_i$ and $\tilde{w}_i$, i = 1, ..., 5, given in table 1 and illustrated in figure 7.

Table 1: Fuzzy numbers and fuzzy weights

$[\, S_{\tilde{x}}^{-} \;\; K_{\tilde{x}} \;\; S_{\tilde{x}}^{+} \,]$	$[\, S_{\tilde{w}}^{-} \;\; K_{\tilde{w}} \;\; S_{\tilde{w}}^{+} \,]$
[1 2 3]	[1 2 5]
[2 5 7]	[2 2.5 3]
[6 8 9]	[4 7 9]
[7 9 10]	[3 4 7]
[10 11 12]	[2 3 4]

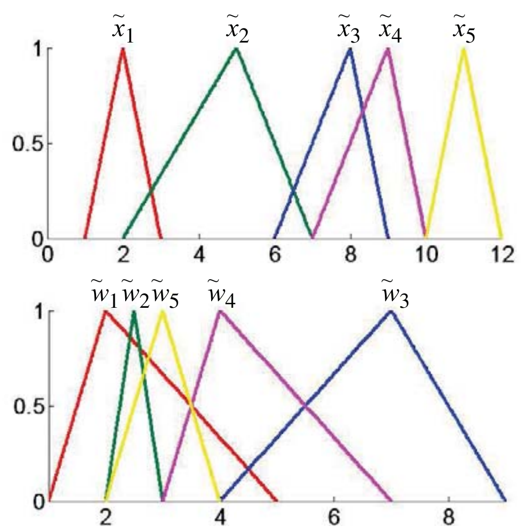

Figure 7: Fuzzy numbers and fuzzy weights

Using the modified division operator to compute the FWA, the result plotted in figure 8 is obtained. It can be observed that the optimistic FWA is here a fuzzy number, i.e. both left and right profiles are monotonic. As expected, the inclusion property (27) is valid.

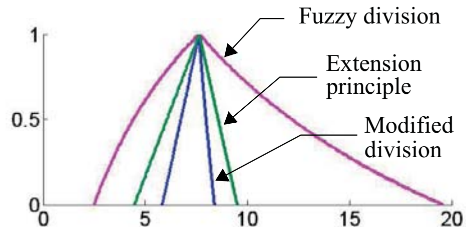

Figure 8: Five-term FWA

5 Conclusion

Based on the use of a modified fuzzy division operator, an optimistic counterpart of the usual fuzzy weighted average was proposed. The modified division is the inverse operation of the fuzzy multiplication. Consequently, the computed optimistic fuzzy average can replaced all individuals without modifying the weighted sum of the population. Moreover, the computation can be achieved by a sequence of elementary arithmetic operations. Unfortunately, the optimistic average may not be a fuzzy number but only an interval of gradual numbers. Actually, the modified division definition only guarantees that the result can be expressed in the form of an ordered pair of gradual numbers without further requirement on the profile monotonicity.

Two simple examples from the fuzzy literature have been used for illustration. More complicated and realistic cases must be further tested. For example, the proposed optimistic average may be used in the context of determining cluster centers for linguistic fuzzy C-means ([1]). Another possible use may be found in the aggregation of Sugeno-like rule consequents. More conceptual works have also to be developed in order to correctly position the optimistic FWA with respect to the exact one.

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