Poincaré recurrence theorem in MV-algebras

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Abstract— *The classical Poincaré weak recurrence theorem states that for any probability space $(\Omega, \mathcal{S}, P)$, any P-measure preserving transformation T, and any $A \in \mathcal{S}$, almost all points of A return to A. In the present paper the Poincaré theorem is proved when the σ-algebra $\mathcal{S}$ is replaced by any σ-complete MV-algebra.*

Keywords— Measure preserving transformation, Poincaré recurrence theorem, σ-complete MV-algebra.

1 Introduction

Let $(\Omega, \mathcal{S}, P)$ be a probability space, i.e. $\Omega \neq \emptyset$, $\mathcal{S}$ is a σ-algebra of subsets of Ω (i.e. $\Omega \in \mathcal{S}; A \in \mathcal{S} \Longrightarrow \Omega \setminus A \in \mathcal{S}$, and $A_n \in \mathcal{S}$ $(n = 1, 2, ...) \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$), and $P : \mathcal{S} \to [0, 1]$ is such that $P(\Omega) = 1$, and $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$, whenever $A_n \in \mathcal{S}$ $(n = 1, 2, ...)$ and $A_n \cap A_m = \emptyset$ $(n \neq m)$. Let $T : \Omega \to \Omega$ be such that $A \in \mathcal{S}$ implies $T^{-1}(A) \in \mathcal{S}$, and

$$P(T^{-1}(A)) = P(A), \tag{1}$$

whenever $A \in \mathcal{S}$; such transformations T are called measure-preserving. The Poincaré recurrence theorem states that almost every point $x \in A$ will return to A, i.e.

$$P(A \setminus \bigcup_{n=1}^{\infty} T^{-n}(A)) = 0, \tag{2}$$

whenever $A \in \mathcal{S}$. There is also the stronger variant of the Poincaré theorem: almost all points of A will return to A infinitely many times. It means that for any $x \in A \setminus B$ (where $P(B) = 0$) and any $k \in N$ there exists $n \geq k$ such that $T^n(x) \in A$:

$$P(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} T^{-1}(A)) = 0. \tag{3}$$

The strong recurrence theorem has been proved in various connections (see [7] for Boolean algebras or [3] for topological spaces). In [8] the theorem has been proved for a special class of MV-algebras (σ-complete weakly σ-distributive MV algebras with product). In the present paper we prove the weak version for arbitrary MV-algebras.

2 MV-algebras

The notion of an MV-algebra has been introduced by Chang [1] as an algebraic system

$$(M, \oplus, \odot, \neg, u, 0)$$

where $\oplus, \odot$ are binary operations, $\neg$ is a unary oparation and $0, u$ are fixed elements. By the Mundici theorem ([4]) every MV-algebra corresponds to a unique unital l-group G (lattice ordered group with a distinguished order unit u),

$$\begin{aligned} M = [0, u] &= \{x \in G; 0 \leq x \leq u\}, \\ a \oplus b &= (a + b) \wedge u, \\ a \odot b &= (a + b - u) \vee 0, \\ \neg a &= u - a. \end{aligned} \tag{4}$$

Analogously as in quantum structures (see [2]), instead of a probability masure, a state $m : M \to [0, 1]$ can be defined.

2.1. Definition. A *state* on an MV-algebra $(M, \oplus, \odot, \neg, u, 0)$ is a mapping $m : M \to [0, 1]$ satisfying the following conditions:

(i) $m(u) = 1$;

(ii) m is additive,
i.e. $a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b)$;

(iii) m is continuous, i.e. $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a)$.

2.2. Proposition. Any state $m : M \to [0, 1]$ is strongly additive, i.e. it satisfies the following implication $a_1 + a_2 + ... + a_n \leq u \Longrightarrow m(a_1 + a_2 + ... + a_n) = \sum_{i=1}^{n} m(a_i)$.

Proof. It can be proved by induction.

The following proposition is evident.

2.3. Proposition. A mapping $m : M \to [0, 1]$ is a state if and only if the following conditions are satisfied:

(i) $m(u) = 1$;

(ii) if (a_n) is a sequence of elements of M such that $a_1 + a_2 + ... + a_n \leq u$ for any $n \in N$, then

$$m(\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} a_i) = \sum_{i=1}^{\infty} m(a_i). \tag{5}$$

3 Main result

3.1. Definition. Let M be a σ-complete MV-algebra, and $m : M \to [0, 1]$ be a state. By an *m-preserving transformation* of M we understand a mapping $\tau : M \to M$ satisfying the following conditions:

(i) $\tau(u) = u, \tau(0) = 0$;

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(ii) $\tau(a \odot b) = \tau(a) \odot \tau(b)$;

(iii) $a \leq b \Longrightarrow \tau(b - a) = \tau(b) - \tau(a)$;

(iv) $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$;

(v) $\tau(\bigvee_{n=1}^{\infty} a_n) = \bigvee_{n=1}^{\infty} \tau(a_n)$;

(vi) $m(\tau(a)) = m(a)$.

3.2. Definition. $a \setminus b = a \odot (\neg b)$.

3.3. Theorem. Let M be a σ-complete MV-algebra, $m : M \to [0,1]$ be a state. Then for any $a \in M$

$$m(a \setminus \bigvee_{i=1}^{\infty} \tau^i(a)) = 0. \tag{6}$$

Proof. Put $b_j = \bigvee_{i=j}^{\infty} \tau^i(a), b = a \setminus \bigvee_{i=1}^{\infty} \tau^i(a) = (a - \bigvee_{i=1}^{\infty} \tau^i(a)) \vee 0 = (a - b_1) \vee 0$.

By induction it can be proved that

$$b + \tau(b) + ... + \tau^n(b) \leq \bigvee_{j=0}^{n+1} \bigvee_{i=j+1}^{n+1} (\tau^j(a) - b_i) \vee 0.$$

Since

$$(\tau^i(a) - b_i) \vee 0 \leq \tau^i(a) \leq u$$

we obtain

$$b + \tau(b) + ... + \tau^n(b) \leq u.$$

By the strong additivity property

$$m(\bigvee_{i=0}^{\infty} \tau^i(b)) = \sum_{i=0}^{\infty} m(\tau^i(b)) = \sum_{i=0}^{\infty} m(b),$$

whence

$$m(b) = 0.$$

3.4. Example.

Consider a probability space $(\Omega, \mathcal{S}, \mu)$ and the MV-algebra $\mathcal{M} = \{f : \Omega \to [0,1]; f \text{ is } \mathcal{S} \text{ - measurable }\}$,

where

$$f \oplus g = \min(f + g, 1),$$
$$f \odot g = \max(f + g - 1, 0),$$
$$\neg f = 1 - f,$$

$u = 1_\Omega, 0 = 0_\Omega$. Moreover, define $m : \mathcal{M} \to [0,1]$ by the formula

$$m(f) = \int_\Omega f d\mu,$$

and

$$\tau(f) = f \circ T$$

where T is measure preserving map. By Theorem 3.3 we obtain that

$$\int_\Omega (f - \bigvee_{i=1}^{\infty} f \circ T^i) \vee 0) d\mu = 0,$$

hence μ-almost everywhere

$$(f - \bigvee_{i=1}^{\infty} f \circ T^i) \vee 0 = 0,$$

or equivalently

$$f \leq \bigvee_{i=1}^{\infty} f \circ T^i$$

μ-almost everywhere.

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