L-Prime Spectrum of Modules

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Abstract: *We investigate the Zariski Topology on the L-prime spectrum of modules consisting of the collection of all prime L-submodules and prove some useful results.*

Keywords: L- ideal, L-submodule, L-prime submodule, Special L-submodule, L-Irreducible submodule, L-Prime Spectrum, Zariski topology, Generic point.

1 Introduction

Throughout this paper *R* is a commutative ring with identity and *M* is a unitary *R*-module. The prime spectrum *Spec(R)* and the topological space obtained by introducing Zariski topology on the set of prime ideals of a commutative ring with identity play an important role in the fields of commutative algebra, algebraic geometry and lattice theory. Also, recently the notion of prime submodules and Zariski topology on *Spec(M)*, the set of all prime submodules of a module *M* over a commutative ring with identity *R*, are studied by many authors (for example see [1,2], [3]) that is in [3] a module with Zariski topology is called *top module* and it is shown that every multiplication module is a topmodule. As it is well known [4], introduced the notion of a fuzzy subset μ of a nonempty set *X* as a function from *X* to unit real interval $I=[0,1]$. [5] replaced I by a complete lattice *L* in the definition of fuzzy sets and introduced the notion of *L*-fuzzy sets. the notion of fuzzy groups was introduced by Rosenfeld [6]); and fuzzy sub-modules of *M* over *R* were first introduced by Negoita and Ralescu [7]. İn [8] Pan studied fuzzy finitely generated modules and fuzzy quotient modules (also see [9]). In the recent five years a remarkable amount of work has been done on fuzzy ideals in general and prime fuzzy ideals in particular, and some interesting topological properties of the spectrum of fuzzy prime ideals of a ring are obtained (see [10], [11], [12], [13], [14], [15] and [16,17].

Finally R.Ameri and R. Mahjoob 2007 studied Zariski topology on $L-Spec(M)$ defining all prime submodules of M. They investigated some basic properties of prime L-submodules and characterize the prime L-submodules of M. They established the relationship between primeless and L-primeless for a given module via the role of the lattice L. Finally they investigated the Zariski topology on $L-Spec(M)$. They showed that for L-top modules Zariski topology on $L-Spec(M)$ exists. In this paper we contınue the paper [18] R.Ameri and R. Mahjoob 2007. $L(M)$ defines all $L-$submodules of M. We define L-special submodule of M to construct a topology on L-spec(M). Then we examine the relationship between with the L-special submodule and L-irreducible submodule. L_T_p module is defined to construct a topology. We introduce that If $L(M)$ is L_T_p module then L-Spec(M) is a T_0 space. We show that L-Spec(M) is a T_1 space if and only if for every $\mu \in \text{L-Spec(M)}$ is maximal. The topological space $(\text{L-Spec(M)} = \aleph, T)$ where M is a Noetherian $R-$module is compact. We show that where Y is a closed subset of $\aleph$, If $J(Y)$ is L-prime submodule of M then Y is irreducible. And if $\eta \in Y$ is a generic point of Y then Y is irreducible.

2 Preliminaries.

Throughout of this paper, by *R* we mean a commutative ring with identity, and *M* is a unital *R*-module and *L* denotes a complete lattice. By an *L*-subset μ of a non-empty set *X*, we mean a function μ from *X* to *L* and if *L* = [0, 1], then μ is called a fuzzy subset of *X*. *LX* denotes the set of all *L*-subsets of *X*.

Definition 2.1: A fuzzy ideal of R is a fuzzy subset of R such that;
i- $\mu(x-y) \geq \min(\mu(x), \mu(y)) \quad \forall x, y \in R$
ii- $\mu(xy) \geq \mu(x) \quad \forall x, y \in R$
The set of all *L*-ideals of *R* is denoted by *L I (R)*.

Definition 2.2: If μ is a fuzzy subset of M, then for any $t \in [0,1]$ the set $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is called a level subset of M with respect to μ.

Definition 2.3([19]): Let μ and σ be a fuzzy ideal of R Then $\forall x \in R$ there exists $y, z \in R$ s.t. $\mu.\sigma(x,y) = \sup_{x=yz}\{\min(\mu(y), \sigma(z))\}$.

Definition 2.4: Let R be a ring and $\mu \in LI(R)$. A non constant μ is called L-prime ideal for every $\eta, \beta \in LI(R)$, $\eta\beta \subseteq \mu$ implies that $\eta \subseteq \mu$ or $\beta \subseteq \mu$.

By L-Spec(R) we mean the set of all prime *L*-ideals of *R*.

Definition 2.5: A fuzzy submodule of M is a fuzzy subset of M such that
1- $\mu(0) = 1$

2- $\mu(rx) \geq \mu(x) \quad \forall r \in R$ and $\forall x \in M$

3- $\mu(x+y) \geq \min(\mu(x), \mu(y)) \quad \forall x, y \in M$

L-(M) denotes the set of all *L*-submodules of M .

Theorem 2.6: μ is a fuzzy submodule of M if and only if μ_t is a submodule of M for all $t \in \operatorname{Im}\mu$.

Definition 2.7 ([20])**:** For $\mu, \nu \in L^M$ and $\eta \in L^R$, define $\mu:\eta \in L^M$ as follows: $\mu:\eta = \bigcup\{\nu \in L^M \mid \eta\nu \subseteq \mu\}$.

Definition 2.8: Let M be an R module and $\mu \in L(M)$. A non constant μ is called L-prime submodule for every $\eta \in LI(R)$ and $\beta \in L(M)$, $\eta\beta \subseteq \mu$ implies that $\eta \subseteq \mu$ or $\beta \subseteq \mu:1_M$.

By L-Spec(M) we mean the set of all *L*- prime submodules of M .

3 L-prime Spectrum on Modules

Example3.1:

$$\mu(x,y)=\begin{cases}1 & \text{if } (x,y)=(0,0)\\ 1/3 & \text{if } (x,y)\in(\mathbb{Z},0)-\{(0,0)\}\\ 0 & \text{otherwise}\end{cases}$$

is L-submodule of $(\mathbb{Z},0)$,

$$\sigma(x,y)=\begin{cases}1 & \text{if } (x,y)=(0,0)\\ 1/2 & \text{if } (x,y)\in(0,\mathbb{Z})-\{(0,0)\}\\ 0 & \text{otherwise}\end{cases}$$

is L-submodule of $(0,\mathbb{Z})$ and

$$\beta(x,y)=\begin{cases}1 & \text{if } (x,y)=(0,0)\\ 0 & \text{otherwise}\end{cases}$$

is L-prime submodule of $(\mathbb{Z},\mathbb{Z})$. $\mu \not\subset \beta$ and $\nu \not\subset \beta$ but $\mu \cap \nu \subseteq \beta$. We show that $\eta \cap \nu \subseteq \mu \Rightarrow \eta \subseteq \mu \text{ or } \nu \subseteq \mu$ is not always true for L-prime submodules in general; Every prime submodule is not L-Special submodule.

Definition 3.2: A submodule μ of M is called L-Special submodule, if for any η,ν L-submodule of M such that $\eta \cap \nu \subseteq \mu \Rightarrow \eta \subseteq \mu \text{ or } \nu \subseteq \mu.$

Example 3.3: Every prime L-ideal of M is L-Special submodule of R module R.

Definition 3.4: Let μ be a L-subset of M . $V(\mu)$ is a subset of all L-prime submodules of M defined by $V(\mu)=\{\eta \mid \eta \text{ is } L-\text{prime submodule of } M,\ \mu \subseteq \eta\}$

For any L-submodule μ of M , it is easy to show that $V(1_M)=\varnothing$, $V(0_M)=L-Spec(M)$, $\bigcap_{i\in I} V(\mu_i)=V(\sum_{i\in I}\mu_i)$ and $V(\eta)\cup V(\nu)\subseteq V(\eta\cap\nu)$.

Proposition 3.5: Let M be an R module. For any L-submodule η,ν of M, $V(\eta\cap\nu)=V(\eta)\cup V(\nu)$ if and only if every L-prime ideal μ of M is L-Special submodule.

Proof: Suppose that for all $\eta,\nu \in L(M)$, $V(\eta\cap\nu)=V(\eta)\cup V(\nu).$

Let $\eta\cap\nu\subseteq\mu$ for all L-prime ideal μ of M.

$$\eta\cap\nu\subseteq\mu \Rightarrow \mu\in V(\eta\cap\nu)=V(\eta)\cup V(\nu)$$
$$\Rightarrow \mu\in V(\eta) \text{ or } \mu\in V(\nu)$$
$$\Rightarrow \eta\subseteq\mu \text{ or } \nu\subseteq\mu.$$

Conversely, let any every L-prime ideal μ of M be L-Special submodule. Then

$$\mu\in V(\eta\cap\nu) \Leftrightarrow \eta\cap\nu\subseteq\mu$$
$$\Leftrightarrow \eta\subseteq\mu \text{ or } \nu\subseteq\mu$$
$$\Leftrightarrow \mu\in V(\eta)\cup V(\nu).$$

Definition 3.6: A L-submodule μ of M is called L-irreducible submodule, if $\eta\cap\nu=\mu \Rightarrow \eta=\mu$ or $\nu=\mu$ for any η,ν L-submodule of M .

Proposition 3.7: Every L-special submodule μ of M is L-irreducible module.

Proof: Let $\eta\cap\nu=\mu.$ Then $\eta\subseteq\mu \text{ or } \nu\subseteq\mu$. And also $\mu\subseteq\eta \text{ or } \mu\subseteq\nu$ then $\eta=\mu \text{ or } \nu=\mu.$

Proposition 3.8: Let μ be L-submodule of M , ν be prime L-submodule of M . If $[\eta:\mu]\subseteq[\nu:\mu]$ implies $\eta\subseteq\nu$ for each η L-submodule of M , then ν is L-special submodule of M.

Proof: Let $\eta\cap\mu\subseteq\nu.$ Since $[\nu:1_M]$ is L-prime ideal by R. Ameri ([1]) theorem 3.6

$$[\eta\cap\mu:1_M]\subseteq[\nu:1_M]\Rightarrow[\eta:1_M]\cap[\mu:1_M]\subseteq[\nu:1_M]$$
$$\Rightarrow[\eta:1_M]\subseteq[\nu:1_M] \text{ or } [\mu:1_M]\subseteq[\nu:1_M]$$
$$\Rightarrow \eta\subseteq\nu \text{ or } \mu\subseteq\nu.$$

Definition 3.9: An L-submodule of μ of M is called T' L-submodule, if

i) $(\eta\cap\nu)\mu=\eta\mu\cap\nu\mu \qquad \forall\eta,\nu\in$ LI(R)

ii) $[\eta:\mu]\subseteq[\nu:\mu]\Rightarrow\eta\subseteq\nu \quad \forall\eta,\nu\in L(M)$.

Theorem 3.10: Let μ be T' L-submodule of M. Then η is L-special submodule if and only if for ($\eta\subseteq\mu$) $[\eta:\mu]$ is a special L-ideal.

Proof: Let $\nu,\beta\in LI(R)$ and $\mu,\eta\in L(M)$ such that $\nu\cap\beta\subseteq[\eta:\mu]$. Then since

$[\eta:\mu]\mu\subseteq\eta$,

$(\nu\cap\beta)\mu\subseteq[\eta:\mu]\mu\Rightarrow\nu\mu\cap\beta\mu\subseteq\eta$. Since η is L-special submodule, $\nu\mu\subseteq\eta$ *or* $\beta\mu\subseteq\eta$. Then $\nu\subseteq[\eta:\mu]$ or $\beta\subseteq[\eta:\mu]$.

Conversely, let $\eta\cap\nu\subseteq\mu$ for all $\nu,\eta\in LI(R)$ and $\mu,\beta\in L(M)$ Then $[\eta\cap\nu:\beta]\subseteq[\mu:\beta]\Rightarrow[\eta:\beta]\cap[\nu:\beta]\subseteq[\mu:\beta]$. Therefore for T' L-submodule then $[\eta:\beta]\subseteq[\mu:\beta]$ *or* $[\nu:\beta]\subseteq[\mu:\beta]$. Then $\eta\subseteq\mu$ or $\nu\subseteq\mu$.

For any L-submodule μ of M, $V(\mu)$ denotes the set of all prime submodules of M containing μ, that is; $V(\mu)=\{\eta\in\aleph \mid \mu\subseteq\eta\}$. It is clear that; $V(1_M)=\{\ \}$ and $V(0_M)=\aleph$. Also we can show that for any family of submodules of $\{\mu_i\}_{i\in I}$ of M and $\mu,\nu\in L(M)$, $\bigcap_{i\in I}V(\mu_i)=V(\sum_{i\in I}\mu_i)$, $V(\mu)\bigcup V(\nu)\subseteq V(\mu\bigcap\nu)$. Therefore if $\zeta(M)$ denotes the collection of all subsets $V(\mu)$ of $L-Spec(M)$ then $\zeta(M)$ contains empty set and itself. Also $\zeta(M)$ closed under arbitrary intersection. But $\zeta(M)$ is not closed under finite union. Therefore on the set of all prime submodules Zariski topology does not exist since third rule of Zariski topology is not satisfied, i.e; $V(\mu\cap\nu)\neq V(\mu)\cup V(\nu)$. If this inequality replaces with equality these modules can be called as L_T_p modules.

Definition 3.11: Let μ,ν be L-submodule of M. $L(M)$ is called L_T_p module if and only if $V(\mu\cap\nu)=V(\mu)\cup V(\nu)$.

Lemma 3.12: $L(M)$ is L_T_p module if and only if every L-prime submodule μ is L-special submodule of M.

Proof: From proposition 2.5.

Lemma 3.13: Let μ,β be an L-submodule of M. If $(\alpha\subseteq\mu)$ such that $\langle\alpha\rangle=\beta$, then $V(\alpha)=V(\beta)$.

Proof:Let μ,β be an L-submodule of M and $\langle\alpha\rangle=\beta$. Since $\alpha\subseteq\beta$, $V(\beta)\subseteq V(\alpha)$.

Conversely Let $\mu\in V(\alpha)$. Then $\alpha\subseteq\mu$. This implies that $\langle\alpha\rangle\subseteq\langle\mu\rangle$ and so $\beta\subseteq\langle\mu\rangle$. Since μ is L-prime submodule of M, then $\beta\subseteq\mu$. Therefore $\mu\in V(\beta)$. This completes the proof.

Definition 3.14: $L(M)$ is distrubutive if $\eta,\nu,\mu\in L(M)$

i) $(\eta+\nu)\cap\mu=(\eta\cap\mu)+(\nu\cap\mu)$

ii) $(\eta\cap\nu)+\mu=(\eta+\mu)\cap(\nu+\mu)$.

Definition 3.15: Let Y be a subset of $L(M)$. $J(Y)$ is the intersection of all prime submodules which belongs to Y.

Definition 3.16: Let μ be a L-subset of M.

$p.rad(\mu)=\cap\{\eta \mid \mu\subseteq\eta, \eta \text{ is prime L-submodule}\}$

is called a radical of μ.

Definition 3.17: Let M be an R module.

$\aleph$=L-Spec(M)=$\{\mu \mid \mu \text{ is prime L-submodule of } M\}$

is called L-prime spectrum of M.

Definition 3.18: $V(x)=\{\eta\in\aleph \mid \eta(x)=1\}$ for all $x\in M$.

If we could not find any L-prime submodule ν which contains L-prime submodule μ, then $p.rad(\mu)=1_M$.

Theorem 3.19: Let $L(M)$ be L_T_p module with $p.rad(\mu)=\mu$. Then $L(M)$ is distrubutive.

Proof: $\eta,\nu,\mu\in L(M)$.

$$
\begin{aligned}
(\eta+v)\cap\mu &= p.rad((\eta+v)\cap\mu)\\
&= J(V((\eta+v)\cap\mu))\\
&= J(V(\eta+v)\cup V(\mu))\\
&= J(V(\eta\cup v)\cup V(\mu))\\
&= J(V(\eta\cup v)\cup V(\mu))\\
&= J((V(\eta)\cap V(v))\cup V(\mu))\\
&= J((V(\eta)\cup V(\mu))\cap(V(v)\cup V(\mu)))\\
&= J((V(\eta\cap\mu)\cap(V(v\cap\mu)))\\
&= J((V(\eta\cap\mu)\cup(v\cap\mu)))\\
&= J((V(\eta\cap\mu)+(v\cap\mu)))\\
&= p.rad((\eta\cap\mu)+(v\cap\mu))\\
&= ((\eta\cap\mu)+(v\cap\mu))
\end{aligned}
$$

Therefore $L(M)$ is distrubutive.

Theorem 3.20: Let $L(M)$ be L_T_p module, μ be L-submodule of M, and Y be a subset of L-Spec(M). Then

i) $V(\mu)$ is closed in L-Spec(M) and $J(Y)$ is an L-submodule of M equal to $p.rad(J(Y))$.

ii) $V(J(Y))$ is the closure of Y in L-Spec(M).

Proof: i) It is clear that $V(\mu)$ is closed in L-Spec(M) and $J(Y)$ is an L-submodule of M. Finally

$$
\begin{aligned}
p.rad(J(Y)) &= p.rad(\cap\{\eta|\eta \text{ is prime L-submodule in } Y\}\\
&= \cap\{\eta|\eta\in Y\}\\
&= J(Y).
\end{aligned}
$$

ii) Let $V(\mu)$ be closed set containing Y. That is $Y\subseteq V(\mu)$. Consequently, $V(J(Y))\subseteq V(\mu)$. Since $Y\subseteq V(J(Y))$, then $V(J(Y))$ is the smallest closed subset of L-Spec(M). Thus $\overline{Y}=V(J(Y))$.

Proposition 3.21: Let Y be a subset of $L-Spec(M)$. Then

i) $J(V(J(Y)))=p.rad(J(Y))=J(Y)$

ii) $V(J(V(Y)))=V(p.rad(Y))=V(Y)$.

Corollary 3.22: For every family $\{Y_i\}_{i\in\Lambda}$ be a closed subsets of L-Spec(M), $J(\cap_{i\in\Lambda}Y_i)=p.rad(\sum J(Y_i))$.

Proof: Since $\{Y_i\}_{i\in\Lambda}$ be a closed subsets of L-Spec(M), then every $Y_i=V(Y_i)$ for each $i\in\Lambda$. So

$$
\begin{aligned}
J(\cap_{i\in\Lambda}Y_i) &= J(\cap_{i\in\Lambda}V(J(Y_i)))\\
&= J(V(\cup_{i\in\Lambda}J(Y_i)))\\
&= J(V(\textstyle\sum J(Y_i)))\\
&= p.rad(\textstyle\sum J(Y_i)) \text{ by proposition 2.20.}
\end{aligned}
$$

Corollary 3.23: Let $\mu\in Y\subseteq$ L-Spec(M). Then $\overline{\{\mu\}}$ the closure of μ is the set $V(\mu)$. We say that $\{\mu\}$ is the closed point of L-Spec(M) if and only if μ is maximal submodule in L-Spec(M).

Proof: Let $Y=\{\mu\}$ then $V(J(Y))=\overline{\{\mu\}}$ by theorem 2.19 and $V(J(Y))=V(J(\mu))=V(\mu)$.

$\{\mu\}$ is the closed, that is $\{\mu\}=\overline{\{\mu\}}$. This implies $V(J(\mu))=\mu$. So μ is maximal submodule in L-Spec(M).

Conversely now suppose that μ is maximal in L-Spec(M), then $V(\mu)=\{\mu\}=Y$ and so $Y=\overline{Y}$. Hence Y is closed.

Proposition 3.24: If $L(M)$ is L_T_p module and $\mu,v\in$ L-Spec(M). Then $\mu\in\overline{\{v\}}$ if and only if $\mu\subseteq v$.

Proposition 3.25: Let $L(M)$ be L_T_p module and L-Spec(M) is a T_0 space.

Proof: Let $\mu,v\in$ L-Spec(M) be two distinct points. We have two cases;

i) $\mu\in\overline{\{v\}}\Rightarrow v\subseteq\mu.$ Since $\mu\neq v$, $v\notin\overline{\{\mu\}}$, then $v\in\overline{\{\mu\}}^c$. Therefore $\overline{\{\mu\}}^c$ is an open set which contains v but not μ.

ii) $v\in\overline{\{\mu\}}\Rightarrow\mu\subseteq v.$ Since $\mu\neq v$, $\mu\notin\overline{\{v\}}$, then $\mu\in\overline{\{v\}}^c$. Therefore $\overline{\{v\}}^c$ is an open set which contains μ but not v.

Proposition 3.26: L-Spec(M) is a T_1 space if and only if for every $\mu\in$ L-Spec(M) is maximal.

Proof: Let $\forall\mu \in$ L-Spec(M) be maximal $\Rightarrow$ $\overline{\{\mu\}} = V(J(\mu)) = V(\mu) \Rightarrow$ Since μ is maximal, $\mu = \overline{\{\mu\}}$. This means that $\{\mu\}$ is the closed. Thus, L-Spec(M) is a T_1 space.

Conversely vice versa. (L-Spec(M) is a T_1 space. $\mu = \overline{\{\mu\}}$.)

Definition 3.27: $\aleph(\mu) = \aleph - V(\mu)$ is called the complement of $V(\mu)$ in L-Spec(M).

Proposition 3.28: Let Y be a subset of $\aleph$ and $\overline{Y}$ denote the closure of Y. Then $\overline{Y} \subseteq V(1_N)$ where $N = \bigcap_{\beta\in Y} \beta_*$.

Proof: Clearly $1_N(x) = 1 \Leftrightarrow \beta(x) = 1 \ \ \forall\beta \in Y$. Therefore if $\eta \in Y$, then $1_N \subseteq \eta$ and consequently $\eta \in V(1_N)$. Therefore the closed set $V(1_N)$ containing Y, contains $\overline{Y}$.

Definition 3.29: Let x be element of M, $V(x) = \{\mu \in \text{L-Spec(M)} \mid \mu(x)=1\}$.

Theorem 3.30: Let $f : M \to M'$ module epimorphism and $\overline{f} : \text{L-Spec(M}') \to \text{L-Spec(M)}$ be a function defined by $\overline{f}(\eta) = f^{-1}(\eta)$. Then $\overline{f}$ is continuous, injective and L-Spec(M$'$) is homeomorphic to the closed subset $V(1_{\ker f})$. If f is an isomorphism, then $\overline{f}$ is homeomorphism.

Proof: Let μ and ν be an L-prime submodules of M'. Since f is surjective, $\overline{f}(\mu) = \overline{f}(\nu) \Rightarrow f^{-1}(\mu)(m) = f^{-1}(\nu)(m) \ \ \forall m \in M.$ Note that; R. Ameri([1]) theorem 3.14 $f^{-1}(\nu)$ is prime submodule. Then by definition, $\mu(\overline{f}(m)) = \nu(\overline{f}(m)) \ \forall m \in M.$ This implies that, $\mu = \nu$. So $\overline{f}$ is injective.

If $V(m)$ is a basic closed set in L-Spec(M), then $\overline{f}^{-1}(V(m))$ is a basic closed set in L-Spec(N). Because

$$\begin{aligned}\overline{f}^{-1}(V(m)) &= \left\{\mu \in \text{L-Spec(M}') \,\middle|\, \overline{f}(\mu)(m) = 1\right\} \\ &= \left\{\mu \in \text{L-Spec(M}') \,\middle|\, f^{-1}(\mu)(m) = 1\right\} \\ &= \left\{\mu \in \text{L-Spec(M}') \,\middle|\, \mu(\overline{f}(m)) = 1\right\} \\ &= V'(f(m)_1).\end{aligned}$$

Hence $\overline{f}$ is continuous. Let μ and ν be an L-prime submodules of M'. $f^{-1}(\mu)$ is constant on $Kerf$. Indeed $f^{-1}(\mu)(m) = \mu(f(m)) = \mu(0) = 1$ for all $m \in Kerf$. Then $f^{-1}(\mu) \in V(1_{Kerf})$. If $\nu \in V(1_{Kerf})$ then $\nu \in$ L-Spec(M) constant on $Kerf$. Since f is an isomorphism it follows from theorem 3.5.11 that $f(\nu) \in$ L-Spec(M$'$). This defines a function $\overline{g} : V(1_{Kerf}) \to \text{L-Spec(M}')$ where $\overline{g}(\nu) = f(\nu)$.

Clearly $\overline{g} = \overline{f}^{-1}$. To prove the continuity of $\overline{g}$, $V'(f(m))$ be a closed set in L-Spec(M$'$). Then

$$\begin{aligned}\overline{g}^{-1}(V'(f(m))) &= \overline{f}(V'(f(m)_1) \\ &= \left\{\overline{f}(\nu) \,\middle|\, \nu(f(m)) = 1\right\} \\ &= \left\{f^{-1}(\nu) \,\middle|\, f^{-1}(\nu)(m) = 1\right\} \\ &= V(m) \cap V(1_{Kerf}).\end{aligned}$$

which is closed subset of L-Spec(M).

Finally, suppose that f is an isomorphism. Then $Kerf = \{0\}$ and

$V(1_{Kerf}) = V(0_M) = \text{L-Spec(M)}.$

Corollary 3.31: If N is submodule M such that $N \subseteq \bigcap_{\eta\in\aleph} \eta_*$, then L-Spec(M) and L-Spec(M/N) are homeomorphic.

Proof: Let f be a natural homomorphism of M onto M/N. Then each $\eta \in \aleph$ is constant on $Kerf$.

Theorem 3.32: Let M be a Noetherian $R-$module. The topological space $(\text{L-Spec(M)} = \aleph, T)$ is compact.

Proof: If $L - \{1\}$ has no prime element, then $\aleph = \varnothing$ proof is complete. Suppose that $L - \{1\}$ have prime elements and a be a prime element of $L-\{1\}$. Let $\{\aleph((m_i)_t) \mid i\in\Lambda, t\in K\subseteq L-\{0\}\}$ be a cover of $\aleph$ by its basic open sets. Let $\vee\{t \mid t\in K\}=b.$ Then $\{\aleph((m_i)_b) \mid i\in\Lambda\}$ also cover $\aleph$. Therefore $\aleph=\cup\{\aleph((m_i)_b) \mid i\in\Lambda\}=\aleph(\bigcup_{i\in\Lambda}(m_i)_b)).$ Hence $V(\bigcup_{i\in\Lambda}(m_i)_b))=\varnothing.$ Let N be a prime submodule of $M.$ Define $\eta: M\to L$, defined by $\eta(m)=\begin{cases}1 & \text{if } m\in N\\ 0 & \text{otherwise}\end{cases}$. So η is L-prime submodule then $\eta\in\{\aleph((m_i)_b) \mid i\in\Lambda\}$ and so $\bigcup_{i\in\Lambda}(m_i)_b\not\subset\eta.$ This implies that $\exists i\in\Lambda$ s.t. $(m_i)_b\not\subset\eta.$ Hence either $b>\eta(m_i)$ or b and $\eta(m_i)$ are noncomparable. In either case $m_i\notin N$. Therefore $\{m_i \mid i\in\Lambda\}$ is not contained in any proper submodule of M. $\langle\{m_1,m_2,\ldots,m_k\}\rangle=M$ since M is Noetherian. Suppose $V(\cup_{i=1}^{k}(m_i)_b)\neq\varnothing$ let $\beta\in V(\cup_{i=1}^{k}(m_i)_b).$ Then $\cup_{i=1}^{k}(m_i)_b\subseteq\beta\Rightarrow\beta(m_i)\geq b$ for all $i=1,\ldots k$. Suppose $\exists i,$ s.t. $\beta(m_i)\neq 1.$ Now $\beta(m)=\beta(\sum_{i=1}^{k}r_i m_i)\geq\wedge_{i=1}^{k}\beta(m_i)\geq\wedge_{i=1}^{k}b=b.$ Now $\beta\in\aleph$ and so $\beta\in\{\aleph((m_i)_b) \mid i\in\Lambda\}.$ Therefore $\exists i\in\Lambda$ s.t. $\beta\in\aleph((m_j)_b).$ Then $(m_j)_b\not\subset\beta.$ Thus either $b>\beta(m_j)$ or b and $\beta(m_j)$ are noncomparable. However, $\beta(m_j)=\beta(\sum_{i=1}^{k}r_i m_i)\geq\wedge_{i=1}^{k}\beta(m_i)>b.$ This is a contradiction. Hence $\beta(m_j)=1.$ This implies that $M=\beta_*$. This is a contradiction. Therefore $V(\cup_{i=1}^{k}(m_i)_b)=\varnothing$. Consequently, $\{\aleph((m_i)_b) \mid i\in 1,\ldots,k\}$ is a subcover of $\aleph.$ This completes the proof.

Definition 3.33: A topological space T is called irreducible if for any decomposition $T=A_1\cup A_2$ where A_1, A_2 are closed subsets of T, then $T=A_1$ or $T=A_2$.

Theorem 3.34: Let Y be a closed subset of $\aleph.$ If $J(Y)$ is L-prime submodule of M, then Y is irreducible.

Proof: Suppose that $J(Y)$ is L-prime submodule of M. Suppose $Y=Y_1\cup Y_2$, where Y_1,Y_2 are closed subsets of $\aleph.$ Then $J(Y)\subseteq J(Y_1)$ and $J(Y)\subseteq J(Y_2)$. Also $J(Y)=J(Y_1\cup Y_2)=J(Y_1)\cap J(Y_2)$. Then $J(Y_1)oJ(Y_2)\subseteq J(Y_1)\cap J(Y_2)\subseteq J(Y).$ Since $J(Y)$ is L-prime submodule of M, then $J(Y_1)\subseteq J(Y)$ or $J(Y_2)\in(J(Y):1_M)$. If $J(Y_1)\subseteq J(Y)$, the proof is complete. If $J(Y_2)\in(J(Y):1_M)$, then $J(Y_2)\subseteq J(Y)$. This completes the proof.

Definition 3.35: Let Y be a closed subset of $\aleph$ and $\eta\in Y$. Then η is called a generic point of Y if $Y=\overline{\{\eta\}}$, the closure of $\eta.$

We know that ıf $Y\subseteq\aleph$ where $\aleph$ is a topological space, then Y is irreducible if and only if $\overline{Y}$ is irreducible.

Theorem 3.36: Let Y be a closed subset of $\aleph$ and $\eta\in Y$ be a generic point of Y Then Y is irreducible.

Proof: Since η is a generic point of Y, then $Y=\overline{\{\eta\}}$. Since $\{\eta\}$ is irreducible, $\overline{\{\eta\}}$ is irreducible. Therefore Y is irreducible

4 Conclusion

Letting $L-\xi(M)=\{V(\mu.1_M) \mid \mu\in LI(R)\}$. It is easy to prove that this set always induces a topology T on $L-spec(M)$. R. Ameri and R Mahjoob showed that $L-\xi(M)$ induces a topology which is called Zariski topology if and only if M is a top module. By following them we show that a topology T on $L-spec(M)$ exists if and only if $L(M)$ is L_T_p module. Under this condition topology T on $L-spec(M)$ is T_0 space. Behind the existing T_1 space, if M is a Noetherian $R-$module, then the topological space $(\text{L-Spec(M)}=\aleph,T)$ is compact.

Acknowledgement

"Bayram Ali Ersoy's work was supported by the Scientific and Technological Research Council of Turkey (TUBITAK); Dan Ralescu's work was supported by a Taft Travel for Research Grant ".

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