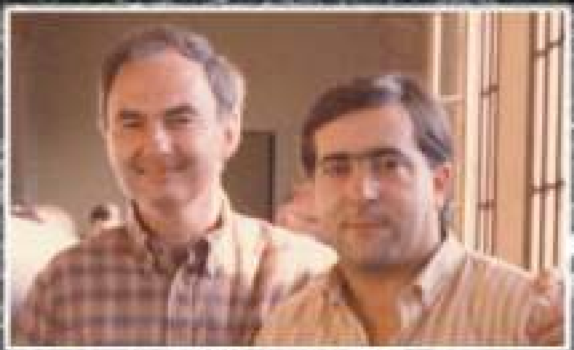
EUSFLAT

EUROPEAN SOCIETY FOR FUZZY LOGIC AND TECHNOLOGY

Mathware & Soft Computing

The magazine of the European Society for Fuzzy Logic and Technology



A conversation between J.L. Verdegay and J. Kacprzyk moderated by G. Pasi

BBVA Foundation Frontiers of Knowledge Award in the Information and Communication Technologies to Lofti A. Zadeh by L. Magdalena

The evolution of hybrid Soft Computing: A personal journey- P. Bonissone

Of machines and humans: The art of decision making- A. Figueiras-Vidal

Selected papers from second Brazilian Congress on Fuzzy Systems- R. Santiago and B. Bedregal

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Hot Topics

Scientific Reports

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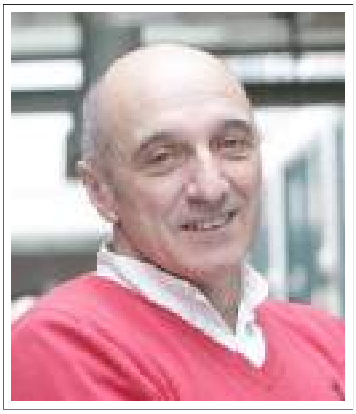
Scientific Reports

Eusflat Life

World Echoes

Message from the Editor-in-Chief (June 2013)

Humberto Bustince



Here it is the new issue of our Mathware&Soft Computing online magazine. And I want it to start by congratulating Prof. Zadeh for receiving the BBVA Frontiers of Knowledge award, which recognizes his many contributions to the advance of science. We have worked against time to include a brief account on the ceremony as well as some photos of this important distinction.

Also with this issue we start including works from conferences that are of interest for our community. So we include a selection of the works presented to the Second Brazilian Congress on Fuzzy Systems that was held at Natal on November 6-9, 2012. Brazilian fuzzy community is gaining influence in the scientific world due to the high quality of its work. The contributions included in this issue, selected by Benjamin Bedregal and Regivan Hugo Nunes Santiago are a living proof of such reality, and I hope they will help to make closer the ties that link of all us.

Apart from this news and novelties, our new issue opens with an interview of two outstanding figures of fuzzy theory in Europe. Janusz Kacprzyk and Jose Luis Verdegay share with all of us their personal views on fuzzy logic and science, among other things. I would like to thank particularly Gabriella Pasi, who has made her best to do this interview possible.

And following our series on those researchers awarded with the Cajastur Mamdani prize of Soft Computing, we include an interesting work on the evolution of hybrid soft computing written by Prof. Piero Bonissone, where he provides his deep insight in the topic.

We have also included in this issue a contribution by Anibal Figueiras which summarizes the successful talk he delivered to open the academic year in the Royal Spanish Academy of Engineering. And of course, we also have news, conference reports, thesis summaries ...

Once again, many thanks to all of you who are contributing to make this Mathware&Soft Computing online magazine possible and please ... enjoy the new issue!!

Humberto Bustince
Editor-in-chief

Message from the President (June 2013)

Javier Montero

Dear EUSFLAT member,

First of all, let me congratulate EUSFLAT for all the recent awards to some of its members. The following awards will be acknowledged in Edmonton, Canada, during the IFSA 2013 conference:

- Janusz Kacprzyk (EUSFLAT member) has been acknowledged with the IFSA Award, the most important acknowledgement given by the International Fuzzy Systems Association, IFSA.
- Krassimir Atanassov (EUSFLAT member), Miguel Delgado (EUSFLAT member), Antonio Di Nola (EUSFLAT member), János Fodor, Lluis Godo (EUSFLAT member), Francisco Herrera (EUSFLAT member), Donald Kraft and Chin-Teng Lin have been acknowledged as new IFSA Fellows.
- Gregory Smits (EUSFLAT member) has been acknowledged, together with Patrick Bosc and Olivier Pivert, with the IFSA 213 Best Paper Award, as authors of the paper "An Approach to Database Preference Queries Based on an Outranking Relation" published in our International Journal of Computational Intelligence Systems (IJCIS 5:789-804, 2012). This "Zadeh Price" is being selected from FSS, IJAR, IJCIS, IJUFKBS, JACIII, or any official journal published by institutional IFSA, among those papers published before the IFSA congress within the two previous calendar years (2011 and 2012 in this case). The authors should be members of EUSFLAT or any other IFSA institutional member. This is the second edition of this IFSA award, and as you might remember, the IFSA 2011 Best Paper Award was also acknowledged to EUSFLAT members.

I think we should be proud of the presence of EUSFLAT members within IFSA. The above acknowledgements show our scientific weight in the fuzzy community, despite being EUSFLAT a small association within IFSA.

An important Award was also acknowledged to Lotfi Zadeh, the BBVA Foundation Frontiers of Knowledge Award, being nominated by the European Centre for Soft Computing in Mieres, Spain, whose Director is Luis Magdalena (EUSFLAT member and past EUSFLAT President). This nomination counted with the support of this President of EUSFLAT, the ceremony being held in Madrid, June 20, 2013.

But more EUSFLAT members have been acknowledged with awards. For example, Irina Perfilieva (EUSFLAT member) was acknowledged, during the FLINS 2012 conference at Istambul, with the first Memorial Da Ruan Award. The paper presented in the same conference "Construction of strong equality index from implication operators", by Humberto Bustince (EUSFLAT Board member), Javier Fernández (EUSFLAT member), José Antonio Sanz (EUSFLAT member), Daniel Paternain (EUSFLAT member), Michal Baczynski (EUSFLAT member) and Radko Mesiar (EUSFLAT Board member), together with Gleb Beliakov, was also acknowledged with the FLINS 2012 Best Paper Award.

About our EUSFLAT Scientific Excellence Award, its Committee constituted by all past EUSFLAT Presidents (Francesc Esteva, Luis Magdalena, Ulrich Bodenhofer) plus me current President, has recently decided to acknowledge Petr Hájek as the 2013 awarded. Petr Hájek is a very well recognized logician, focussing his research in hard problems, and not only from a theoretical point of view, but looking for applications. From the nineties he became interested in Fuzzy logic. He worked in the foundations of fuzzy logic and can be considered as the father of what is called Mathematical fuzzy logic. He defined the logic BL, the Hilbert style calculus proved to be the logic of a continuous t-norms and their residua, he define its first order calculus and has published papers dealing with all theoretical problems about this logic and many of its expansions. Also he has dedicated a deep work trying to interpret many of Zadeh program (like "If then rules", fuzzy quantifiers, linguistic variables and modifiers, etc.), in the setting of Mathematical Fuzzy Logic. Petr Hájek is a very well recognized logician. He has mainly worked in hard theoretical problems, but he has always been concerned with applications as well. From the nineties he became interested in Fuzzy Logic, he worked in the foundations of fuzzy logic and specially is the father of what is called Mathematical fuzzy logic. He define the logic BL, the Hilbert style calculus proved to be the logic of a continuous t-norms and their residua, he define its first order calculus and has published papers dealing with all theoretical problems about this logic and many of its expansions. Also he has dedicated a deep work trying to interpret many of Zadeh program (like "If Then rules", fuzzy quantifiers, linguistic variables and modifiers, etc.) in the setting of Mathematical Fuzzy Logic.

In addition, I am notified by the President of the EUSFLAT 2013 Best Ph.D. Award Committee, Ulrich Bodenhofer (EUSFLAT Honorary member and past EUSFLAT member), that the EUSFLAT 2011 Best Ph.D. Award goes to Nicolas Madrid (currently at the University of Ostrava, Czech Republic) for his work "Measures of Inconsistency and Existence of Fuzzy Stable Models in a Residuated Framework" (University of Malaga, Spain), being Manuel Ojeda-Aciego his Ph.D. Advisor. The EUSFLAT 2012 Best Ph.D. Award goes to Marco Cerami (currently at Palacky University of Olomouc, Czech Republic), for his work "Fuzzy description logics from a mathematical fuzzy logic point of view" (University of Barcelona, Spain), being his Ph.D. Advisors Francesc Esteva, Felix Bou and Lluis Godo. We should thank both committees for selecting these two Ph.D. thesis among 10 applications: beside the President of both committees and Eulalia Szmidt (EUSFLAT Board member in charge of all EUSFLAT awards, acting as Secretary of both committees), Bernard de Baets (EUSFLAT Board member), Petr Cintula (EUSFLAT member) and Francesc Esteva (EUSFLAT Honorary member and past EUSFLAT President) served for the 2011 award, and Enric Trillas (EUSFLAT Honorary member), Radko Mesiar (EUSFLAT Board member) and Eyke Hüllemeier (EUSFLAT Board member) helped for the 2012 award. Congratulations for both awardees and their Ph.D. advisors!

Anyway, during the next EUSFLAT General Assembly, to be held in Milano during the EUSFLAT 2013 conference, a new EUSFLAT Board must be elected. Therefore, this message will be my last "Letter from the President".

It has been a great honor to serve you all as President of our scientific association. Thanks to you all for trusting in this board. Indeed it has been for me an intense but grateful duty, mainly thanks to the support of the whole EUSFLAT Board. I hope that our work will be useful for the next board, as the tremendous work of past boards has determined what this board has been able to do.

Let me shortly remind some main achievements of this EUSFLAT Board during these four years:

- A new on-line open access Mathware & Soft Computing project was launched in order to keep the title and history of our past official journal, but moving it into a live EUSFLAT Magazine. Thanks to the dedication of Humberto Bustince (EUSFLAT Board member) and his team at the Public University of Navarra, Spain, the new magazine was launched with unforgettable interviews, wonderful reports provided by key researchers, illustrative works and reports coming from our community.

- An agreement with the International Journal of Intelligent Systems, edited by Atlantis Press and now distributed by Taylor & Francias, was signed to become this journal into our EUSFLAT official journal, reducing the excessive cost in time and money that our past journal was causing, avoiding in this way an undesirable fee increase and providing in addition a long term EUSFLAT project. We should be aware that, since the journal had a lot of papers already accepted before this agreement, the devoted energy that Luis Martínez (EUSFLAT Board member) and his team are putting into this project needs some time to be seen. Anyway, the key support to this project should come from EUSFLAT members, submitting good papers to our official journal.

- Enric Trillas and Ulrich Bodenhofer were acknowledged as our second and third EUSFLAT Honorary members (first EUSFLAT honorary member was Francesc Esteva, first EUSFLAT President).

- Statutes where updated to correct certain details and make them fit the current running of EUSFLAT.

- The EUSFLAT Scientific Excellence Award was launched, being Didier Dubois (EUSFLAT member) its first awardee.

- The EUSFLAT Best Ph.D. Thesis was launched, being Oscar Ibañez (EUSFLAT member) its first awardee with his thesis on "Forensic identification by craniofacial superimposition using soft computing". Oscar Cordon (EUSFLAT Board member) was his advisor, together with Sergio Damas.

- The EUSFLAT Best Student Paper at EUSFLAT Conference Award was launched, being Martin Vita (EUSFLAT member) its first awardee, with the paper "Filters in algebras of fuzzy logics", co-authored with Petr Cintula (EUSFLAT member).

- The influence of the EUSFLAT Student Grants Program has been spread between a number of conferences, thanks to the experience of Bernard de Baets (EUSFLAT Board member). In return, EUSFLAT members have been able to get interesting discounts in the registration fees of these EUSFLAT supported conferences (AGOP, ESTyLF, FSTA, IPMU, ISCAMI, LFA, SCBS, SMPS, SUM and WILF, among others).

- A network with more than 20 conference streams was created in order to help coordination between main conferences related to our field of interest within Europe.

- A number of new agreements were signed with several associations, offering discounts either for double membership or in their official conference fees. At this moment, the list of these associations offering some kind of agreement with EUSFLAT include the Hungarian Fuzzy Association (HFA), the North American Fuzzy Information Processing Society (NAFIPS), the North European Society for Adaptative and Intelligent Systems (NSAIS), the International Association for Fuzzy-Set Management and Economy (SIGEF), the Catalan Artificial Intelligence Association (ACIA), the International Rough Sets Society (IRSS) and the Brazilian Society of Automatica (SBA). More similar agreements with other close associations might still come.

- The Committer Membership was created (although not yet implemented) to allow the possibility of extra incomes from researchers leading funded projects.

- EUSFLAT Working Groups, coordinated by Eyke Hüllermeier, have been invited to edit a book on their field of interest, in a series arranged with Atlantis Press. The first volume, promoted by our EUSFLAT Fuzzy Control Working Group is already in progress, with Fernando Matia (EUSFLAT member) as its main editor.

- EUSFLAT 2011 was organized in Aix-les-Bains, jointly with LFA and with the great job of Sylvie Galichet (EUSFLAT member) and her team. More than 200 attendants enjoyed the conference, and 156 papers where accepted.

- A kind and personal recruiting campaign has been performed to invite key researchers to join EUSFLAT.

- A new EUSFLAT flyer, kindly designed by Carlos Lopez-Molina (EUSFLAT member), has been made available in a number of conferences, inviting colleagues to join EUSFLAT (flyer is available on the EUSFLAT Web site).

- Web management has been improved thanks to the dedication of Jorge Casillas (EUSFLAT Board member). Beside technical improvements, the new EUSFLAT Web includes payment facilities, a private area for each EUSFLAT member, and an easy way to check EUSFLAT membership for those conferences offering discounts. Life of future EUSFLAT Treasures will be much easier with this new tool.

- Martin Stepnicka (Secretary of EUSFLAT) has re-activated the EUSFLAT Facebook page, please visit `http://www.facebook.com/EUSFLAT`, and the European Society for Fuzzy Logic and Technology (EUSFLAT) Linkedin group, please visit `http://www.linkedin.com/groups/European-Society-Fuzzy-Logic-Technology-148845?_mSplash=1`, both launched by previous EUSFLAT Board.

- A repository of historical documents and legal tips in Spain, useful for future EUSFLAT boards, is being created. Edurne Barrenechea, beside her tremendous work as EUSFLAT Treasurer, and Martin Stepnicka,as EUSFLAT Secretary, are preparing with me a number of documents that will hopefully help the next EUSFLAT President, Secretary and Treasurer.

Still, some other proposals on study within the EUSFLAT Board might be ready to be presented before the next General Assembly. EUSFLAT has at this moment around 250 members from more than 30 countries. EUSFLAT has at this moment around 250 members from more than 30 countries.

I cannot finish this kind of farewell without some words to my friend Da Ruan, who sadly passed away while being a member of the EUSFLAT board. This was indeed an extremely sad moment for this EUSFLAT board, and particularly for me. Thanks to his generous work, EUSFLAT has inherited a world-recognized and respected journal that can grow together with EUSFLAT, being free for all EUSFLAT members and having EUSFLAT a significant influence on its Editorial Board.

EUSFLAT 2013 conference is our next objective. Gabriella Pasi (EUSFLAT Vice President) and her team at Milano have been working very hard to assure a great and productive meeting. More than 151 papers have been submitted to EUSFLAT 2013 conference.

Looking forward to meet you in Milano for our EUSFLAT General Assembly, and wishing the best for the next EUSFLAT Board,

Javier Montero
President of EUSFLAT

REPORT

Our lives with fuzzy logic, decision making and optimization: a history of science and friendship

An interview to Janusz Kacprzyk and José Luis ("Curro") Verdegay

Gabriella Pasi
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I am very happy for the opportunity to have this virtual chat with two pioneers of Fuzzy Decision Making, Decision Analysis and Optimization; the aim of this joint interview is to offer readers a short history of the birth and setting of the research related to Fuzzy Decision Making, Decision Analysis and Optimization in Europe, and of the development and future perspectives of this important research area, and of Fuzzy Logic in general.

Which your scientific background is?

Janusz Kacprzyk: My background is control theory and control engineering as I got M.Sc. in automatic control from the Department of Electronics, Warsaw University of Technology in 1970, and immediately after that started working for the Institute of Automation, Polish Academy of Sciences, that changed the name many times and is now Systems Research Institute, Polish Academy of Sciences in Warsaw. This background is very important for various reasons. First, control theory - that is by some people considered to be the highest achievement of applied mathematics - is a difficult area that needs a good mathematical background along with some understanding of what a system under control, control, optimal control, trajectory, etc. really mean. Basically, it is an optimization focused area which has had for me a certain advantage and disadvantage. First, why looking at all kinds of problems, I was immediately inclined to a formal analysis, search for an optimal solution, etc. Some disadvantage was, from the point of view of my further interests I will elaborate upon later, that I had to somehow get accustomed to a rather logic focused reasoning I have been using later, in particular concerning the so called fuzzy logic control, in particular discussed by people like logicians or computer scientists speaking about implications, etc.

in which I could not see control, to be frank. But, this impression has been shared by many people in the control community who, after an initial interest in fuzzy (logic) control, because of its usefulness, have become somehow apprehensive. A second problem in this respect was that I was accustomed to a differential/difference equation representations of systems that is dominant in control, and I had to somehow change my perspective as rule based representations have become dominant and most promising in fuzzy systems.

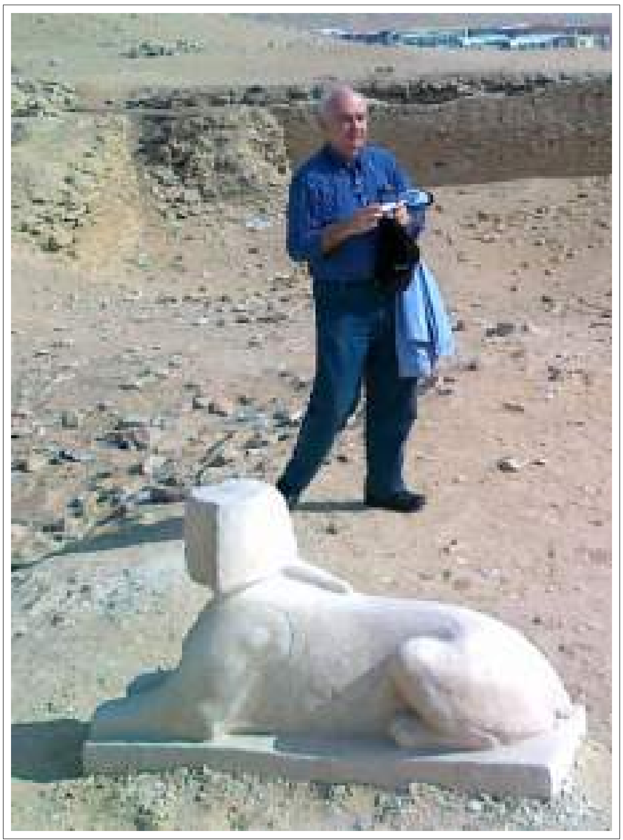
As usually, the mainstream of science changes over time, and at some time, maybe in the beginning of the 1970s or earlier, some new fields have become more relevant, better funded, etc. One of them was systems analysis which can be described as a consortium of tools and techniques from various areas like applied mathematics, systems theory, control, economics, decision sciences, etc. employed for solving relevant real world problems. In 1972 one of the most famous "think tank" institutes in Europe was founded, the International Institute of Applied Systems Analysis (IIASA) in Laxenburg, close to Vienna, in Austria. It turned out that this would considerably shape my work and career as I would explain later.

However, in 1973 I found Zadeh's paper "Outline of a new approach to the analysis of complex systems and decision processes" published in IEEE Transactions on Systems, Man and Cybernetics. This was for me like a thunder because I saw something that I could not imagine, i.e. that one can meaningfully represent various relationships, systems, dynamics, etc. in linguistic terms, and then use linguistic rules. It was quite a strange thing for a control theorist like me. I immediately found quite a natural application, to the optimization of organizational structures, and my paper was accepted for presentation and publications at the 1974 IFAC Conference on Identification in Udine, Italy. This was a prestigious, predominantly automatic control oriented (organized by the International Federation of Automatic Control, IFAC) conference but the reviewers have shown a vision. Of course, one of the reason was that though almost nobody knew about fuzzy sets, everybody knew Lotfi Zadeh who had already been famous for years.

However, I had to put those works on the shelf because I started a close collaboration with IIASA in ca. the mid-1970s, in the regional development project, to develop and implement a regional agricultural planning model for a specific region in Poland. This was a huge linear programming model (about 100000 variables and 10000 or 20000 constraints, if I remember well) that had to be solved at a supercomputer (as meant at that time) in the atomic research center in Pisa, Italy, via a telex line (yes, there were telexes at that time) from Austria. This was a very good experience, nothing fuzzy, but it concerned the solution of a real problem, interaction with real domain (agricultural) experts, and this had given me very much as that work was continued well into the 1980s. For the first time I encountered a strange, at least to me, phenomenon that a "wonderfully optimal" solution obtained by a long optimization process, I was so proud of, was immediately discarded by human domain experts as unrealistic, "strange", unacceptable, etc. At that time I realized for the first time that the strict optimization I was accustomed to may be not the one and only one, and best way to go and some "softness" in the problem specification and the very concept of what the solution we were looking for would be desirable. This had tremendously helped me to easier understand the first paper by Zimmermann on fuzzy linear programming, and also the first papers on some more general optimization problems by Tanaka, Asai and other people, and finally - great papers on fuzzy optimization and mathematical programming by Miguel Delgado, Curro Verdegay, and their associates.

Then, at the end of the 1970s I was more and more involved in fuzzy logic, and attended very relevant conferences, mostly organized by EURO (European Federation of Operational Research Societies) at which there were regular session or tracks on fuzzy logic. Luckily enough, we had at that time Hans - Jüergen Zimmermann, who was President of EURO, and Christer Carlsson, also one of top people in EURO, and both were very active in fuzzy logic.

At that time I started working on fuzzy multistage optimal control, notably fuzzy dynamic programming. For me, a control theorist, this was much more intuitive and natural than "fuzzy logic control" in which I could not see much control. This work had resulted in a series of papers, 3 books and a IEEE Computational Intelligence Society Fuzzy Pioneer Award in 2006. Moreover, about that time in the very end of the 1970s, I started at IIASA works on the use of fuzzy dynamic programming for the modeling and planning of the so called integrated regional development (now, maybe it would have been called sustainable. . .). This was a dynamic model based on 7 life quality indicators in which an optimal development policy (scenario) was determined by using fuzzy dynamic programming. A series of paper on this topic culminated in J. Kacprzyk and A. Straszak, "Determination of stable trajectories for integrated regional development using fuzzy decision models", IEEE Transactions on Systems, Man and Cybernetics, Vol. SMC 14, 310 313, 1984. This paper was cited as one of the best and most convincing examples of fuzzy modeling reported in the literature in a special book on the 50th anniversary of operational research: L.C. Thomas (Ed.) "Golden Developments in OR", Pergamon Press, Oxford, 1987.



At the same time a close collaboration with Mario Fedrizzi began. Basically, it was my second contact with the Italian fuzzy researchers because in 1977 I visited Aldo di Luca in the CNR institute in Arco Felice close to Naples, and also met some other people, notably Settimo Termini. Then, I met Mario Fedrizzi, and our first paper appeared in the very end of the 1970s. This was a great collaboration that had been continued until now, which had resulted in tens of joint papers on mostly group decision making, consensus, voting theory, etc. We were joined later by Hannu Nurmi and Sławek Zadrożny, and as a result we published quite a high number of all kind of publications, articles, edited volumes, etc.

This was all a kind of decision theoretic and analytic direction I used to work in the area of fuzzy logic at that time, and in 1981 I went to Ron Yager as a visiting professor, and stayed with him, and also at some other universities a couple of years, in particular because martial law was introduced in Poland and it was impossible to come back to join my family. This was a very fruitful period for me. First, I could collaborate with Ron, one of the most innovative and visionary scientists in our area. I could continue my work on all kinds of decision making, notably fuzzy multicriteria models, but I have (re)discovered another aspect of fuzzy logic that was not related to optimization, decision theory and analysis, etc. Namely, some human consistent representations of information, knowledge, reasoning, etc. Most important for my work were two things I learned at that time: Zadeh's first work on linguistically quantified propositions, and Ron Yager's use of linguistically quantified propositions to model aggregation in multicriteria decision making, and above all, to propose a new concept of a linguistic data summary.

Those works have inspired a multitude of my works in the future, notably the concept of a fuzzy majority in group decision making proposed in 1985, a new fuzzy majority based concept of a degree of consensus proposed with Mario Fedrizzi in 1987, new models of multistage fuzzy control, machine learning, etc. During that stay of mine in the USA in the beginning of the 1980s I had a chance to frequently visit Lotfi Zadeh in Berkeley who had constantly supported me and provided much inspiration - I will say more about this later. Moreover, while in Berkeley, I had a chance to meet Richard Bellman, the founder of dynamic programming, considered by some people to be the greatest applied mathematician of all times, and a close friend of Lotfi Zadeh. Bellman had greatly supported my works on fuzzy dynamic programming and fuzzy multistage control until his death. After his death in 1984, I was active in the Bellman Continuum conferences organized all over the world by his students and followers. In the mid-1980s I came back to Poland and immediately was invited to IIASA, at that time to work in the group of Sergei Orlovski from Russia (then Soviet Union), one of top people in fuzzy optimization who introduced some fundamental concepts based on fuzzy preferences and fuzzy utilities to decision making models. However, my work concerned a decision support system for local authorities from different countries along the Danube River. As a result of that work we proposed with my close collaborators Sławek Zadrożny and Andrzej Ziółkowski a new type of a fuzzy database query including a fuzzy linguistic quantifier that made it possible to find records such that, for instance, most or important conditions were satisfied. That type of a fuzzy query was, first of all, implemented at IIASA, then used in a hotel reservation Web site in Asia and had been part of a commercial database querying system sold by a spin-off company in the USA. This is probably one of very few commercial applications of fuzzy logic in the area of databases and querying. Moreover, a special article on this new query was published in Computer World China, with a huge circulation, and - from the scientific point of view - many articles have been published.

This work was in fact the beginning of my later involvement in more IT/ICT oriented research. In the next years, this was my main area of interest, and among other things which I will mention later on, I would say here that around the mid-1980s I started working on a new idea of computing with words initiated by Lotfi Zadeh. I published with him already in 1999 a huge two volume collection of works on the foundations and applications of this new idea, and I had been continuing this works along many dimensions until now.

I also have to tell that, though my works for many years, maybe decades, have been more in computer science (maybe better to say IT/ICT), I am still active as a "real" control theorist and I am also supervising PhD dissertations in mobile robotics though using evolutionary computing not fuzzy logic but the latter will be included in next dissertations that are planned.

José Luis ("Curro") Verdegay: My first contact with fuzzy set theory and fuzzy systems dates back to the late 70s. By then, the scientific level of research carried in the Department to which I belonged was quite low, with teaching and classic mathematics being the only topics on which we got encouragement. Totally by chance, I ran into the book "Introduction à la théorie des sous-ensembles flous à l'usage des ingénieurs, Vol. 1 Éléments théoriques de base" by Arnold Kaufmann. For me, it was like discovering a new world, which I immediately started to explore with the aid of my close friend Miguel Delgado, who also became my PhD supervisor.

Those were very intense years. We belonged to the Department of Statistics and Operational Research, where everybody disliked everything related to Fuzzy Logic. Little by little, we created a group of researchers on this topic, which included Amparo Vila, María Teresa Lamata, Serafín Moral, Antonio González, Luis de Campos and some others. We started gaining visibility through our papers and participation in conferences, meetings and workshop, within Spain and abroad. We strengthened links with Fuzzy Logic researchers from Catalonia, such as Enric Trillas, Llorenç Valverde, Claudi Alsina, Teresa Riera, Francesc Esteva and Ramón López de Mántaras. As a consequence, we won international recognition, and our research topic quickly moved away from the classical research of our Department of Statistics and Operational Research. Finally, supported by the Rector of the University of Granada and by Enric Trillas (at that time, President of the Spanish Council for Scientific Research - CSIC), we separated from our original Department and created the Department of Computer Science and Artificial Intelligence (DECSAI) which became in charge of the Grade on Computer Science as well as of the corresponding post-grade studies.

The first years were nothing but easy: little funding for research, lots of lectures, few offices and laboratories... But, step by step, we incorporated young teachers who we trained. Nowadays some of them are widely known all around the world. The first ones were Francisco (Paco) Herrera and Juan Luis Castro. Later, supported by the earliest research projects we got, professors Olga Pons, Juan Huete, Enrique Herrera and Oscar Cordón, among others, entered the department and eventually formed the Granada Team. I must acknowledge the generous support of people such as Enric Trillas, Llorenç Valverde, Claudi Alsina and Ramón López de Mántaras, who were always behind us to encourage us and show us the way to well-made Science. They were, and still are, close friends of us. To summarize, since them, with a little group of about 15 pioneers, we get at the current situation of DECSAI, whose 25th anniversary is being commemorated this year. About 100 people belong to DECSAI today, 23 of which are Full Professors, organized in 9 research groups headed by Miguel Delgado, Amparo Vila, María Tersa Lamata, Paco Herrera, Juan Luis Castro and others. We work in both theoretic and applied developments related to Fuzzy Logic, Soft Computing, and Intelligent Systems in general. Currently there are 3 spin-offs founded by DECSAI members.

Hence, as it may be obvious, my professional career has developed in parallel with the Department and the Granada Team. I received the Ph.D. degree in sciences from the University of Granada in 1981 and since 1991 I'm a full Professor at DECSAI and a member of the Models of Decision and Optimization (MODO) Research Group. Besides I'm Coordinator of the Master on Soft Computing and Intelligent Systems between University of Granada and University of Computer Sciences (Cuba). I've published seventeen books and more than 300 scientific and technical papers in leading scientific journals, and I've been Advisor of 19 Ph.D. dissertations. I have served on many international program committees and have attended numerous national and international conferences, congresses, and workshops. I've been Principal Researcher in a variety of national and international research and educational projects, and currently I lead a research project on "Applicability of the Soft Computing in Advanced Technology Environments: Sustainability". I've a large experience in the evaluation of the quality of academic institutions. I have been member and President of a number of committees with the European Training Foundation and the Spanish Ministry of Education. I'm also a member of the Editorial Board of several international leading journals in the area of Fuzzy Sets and Systems. I served as Chairman of DECSAI (1990-1994), President (founder) of the Spanish Association for Fuzzy Logic and Technologies (1990-1996), Advisor for Intelligent Technologies of the Spanish Science Inter-Ministry Commission (1995-1996) and Director of International Affairs at the University of Granada (1996-2000). I'm an IFSA fellow, IEEE Senior member, and Honorary Member of the Cuban Academy of Mathematics and Computation. Besides I've the Featured Possition of "Invited Professor" at the "Instituto Superior Politécnico José Antonio Echevarría" (Havana, Cuba), University of Holguín (Holguín, Cuba) and Central University "Marta Abreu" of "Las Villas" (Santa Clara, Cuba). Along all this time my research has focused on different aspects of decision making and optimization methods in fuzzy environments, with contributions to the resolution of real problems in a variety of contexts, such as Underground stations location in Kinshasa, design of beef cattle diets in Argentinean farms, control of tobacco leaf drying in Cuba, and some more. Currently, I serve as Delegate of the Rector for ICT at University of Granada. My current research interests are around solution methods and models of Dynamic Optimization Problems.

When did you encounter "fuzzy logic" for the first time, and when did you meet Lofti Zadeh for the first time?

Janusz Kacprzyk: This is a very good, yet a slightly complicated question. First, in my case, as somebody from control theory, systems analysis, etc., the fact is that I have known Lotfi Zadeh "for ever", even if not personally. One thing that many people do not know is that he has already been famous while introducing the concept of a fuzzy set in the mid 1960s. First of all, if I read that Lotfi Zadeh is famous as the inventor a fuzzy sets, I am starting somehow smiling because, for instance, if you just look at how many people use his ideas, you can easily see that his ideas related to the state space approach have been employed since the early 1960s (after his book with Charles Desoer) everywhere, explicitly and implicitly, by everybody who is doing mathematical modeling of any kind, systems modeling and optimization, etc. etc. The number of these people is far larger than the number of people in fuzzy logic!

The signs of Zadeh's extremely high stature in science were so many, and let me just tell about a couple of them. In the early/mid 1970s, when I was working at IIASA, the first Director was Howard Raiffa from Harvard University, a good friend of Zadeh, and he always mentioned Zadeh when we were meeting. IIASA was a very special think tank of a new formula (international, a "bridge" between the West and East), that have had always attracted most famous scientists, notably Nobel Prize winners. As a young person at that time I was often requested to accompany them to Vienna or somewhere else in Austria, and we had a chance to talk. All of them, both form the USA like Koopmans or the USSR, like Kantorovich, knew Zadeh and fully appreciated his stature and works, but none of them were mentioning his works on fuzzy sets ...

Finally, last year my PhD student from the famous Lincoln Lab at MIT defended his dissertation. I had a chance to often visit him at MIT and talk to him and many professors there, even younger ones. What was striking is that even if they were not proponents of fuzzy logic, as it often happens at such top universities, they fully appreciated Zadeh's greatness (but again not only due to fuzzy!) and were proud that he was an MIT graduate.

I met Lotfi Zadeh in person probably in the early/mid 1970s at some conference and has since then enjoyed his full support and inspiration. This has been so important to me and shaped not only my research but my attitude to people and science. But, as I mentioned above, my "meeting" with Zadeh dated back to many years before. I was telling the above because we, the fuzzy logic people, should remember that one of the main reasons that fuzzy logic had survived in spite of apprehension and criticism, and had been flourishing, is due to Lotfi Zadeh's stature in science. It is extremely difficult to launch a new theory and attract many people when a founder is not already known in science. Examples are numerous.

From my personal point of view Lotfi's support was extremely important because at that time we did not have anybody powerful enough in the Polish scientific establishment who was supporting fuzzy logic. The situation in Spain was different because of Enric Trillas who was both a great scientist supporting fuzzy logic, and a powerful person in the Spanish research structure.

Curro Verdegay: As I mentioned above, mi first encounter with fuzzy logic was motivated by Kaufmann's book, which I want to emphasize as the best book on the basics of fuzzy sets, and which I still recommend to my students to come into this topic. This happened around 1978-79. It was already very difficult to carry research on something that was outside the common topics, but it was even more to get financial support to go out of Spain. It was through Miguel Delgado that I met Enric Trillas and, with his continuous initiatives, support and encouragement, we eventually managed to organize a sort of "Spanish Working Group on Fuzzy Sets" which, to some extent, served us to "protect" against the defeat of the classical Spanish academia and, at the same time, set us free to investigate what we were really interested in. After hard work, and of course with our own funding, we finally got to participate in an international conference where I could contact Lotfi and meet him in person. It was in 1983 and I still remember it with emotion. His casualness, closeness and above all, the depth of his lectures were for me an example of behavior which I have always tried to follow. I believe that such conference, organized by Elie Sanchez in Marseille, has been the one with the highest scientific level in which I have ever taken part. In there I also had the opportunity to meet Arnold Kaufmann himself, as well as Elie Sanchez, Hans Jurgen Zimmermann, Ron Yager, Stefan Chanas, Didier Dubois and Henri Prade, Michio Sugeno, ...

How much was the meeting with Lotfi influencial to your subsequent research?

Janusz Kacprzyk: This has very much to do with what I told before but maybe I will rephrase this a little bit and put in another perspective. First, the influence of a person on research is a multiaspect issue that does not only concern research as such, and I will try to elaborate on that.

Let me start with somehow repeating what I said before that my meeting with Lotfi had a "virtual" and "real" dimension. That virtual dimension is related to the fact that Lotfi was famous in the control and systems research community, so that by necessity I was using his ideas, and in my pre-fuzzy period this was related to his state space approach I had been extensively using in my mathematical modeling works. My "real", personal meeting, or - better to say - numerous meeting - with Lotfi have had a tremendous impact on my research and career. One should take into account that when I met him for the first time I was very young, just in the beginning of my career, and it is in general very important at such a stage to have a mentor of a real stature, friendly and with a positive attitude. Lotfi was like this. I think that he could easily recognize that, as any other person, I had some stronger and weaker points, and he tried to inspire me taking that into account . . . But, I can see this now, with my experience.

To be more specific, in the first period of my work, I was mostly inspired by his works on linguistic modeling and he had been ingenious enough to show its potential to a person of an strict, control theoretic background. But he could understand limitations of such a person as his experience was similar. Then, what has resulted in some of my most relevant works, he inspired me through his paper with Richard Bellman on a fuzzy approach to decision making, notably fuzzy dynamic programming. He wrote that paper in 1970 and when we met for the first time he was no longer very much interested in that topic but, noticing my interest, he supported it and gave me very relevant suggestions that I had included in my further works.

In last years what I did mostly appreciate is that while he noticed my interest in computing with words, he was always supporting and inspiring me to do research in that area though we had been disagreeing a little bit because I was always stressing a need to relate computing with words to huge research efforts in computational linguistics and natural language understanding, processing and generation. To summarize, I would say that in our life we need an inspiration, boost and support by somebody who is not only a professional authority but a person with vision, and also friendly and full of positive energy, and of great personal qualities. When we meet such a person, and take advantage of such an encounter, then all great things can happen to us. This is exactly the role Lotfi has always played in my life, even beyond my professional life.

Curro Verdegay: All the Works by Lotfi have had a great influence in my scientific career. There is no work by him that, after having read it, has not stimulated my interest and subsequently my acknowledgment for what I believe is a prodigious mind. In those years, what had a greatest influence on my later research was the introduction and development of fuzzy logic from its basis and foundations (the concept of linguistic variable, and everything related to Approximate reasoning), as well as the conversations I could hold with him. Because, after that first meeting in Marseille, subsequent meetings with Lotfi became more and more often, as well as his visits to Spain. Therefore, there were a lot of opportunities to hear his wise thoughts, his simple, almost trivial conceptualizations (as if we were dealing with a fairy tale) of truly difficult problems that were unsolvable within the context of classic mathematics and probability.

Where and how did you meet each other the first time, and at what stage of your career were you?

Janusz Kacprzyk: This is both a simple and tricky question, as I will try to elaborate upon. First, as in the case of Lotfi, I will divide my meeting with Curro into the "virtual" and "real" one.

That "virtual" meeting with Curro has to do with a far wider issue. Namely, in the early mid 1980s, when I came back from the USA, I was close to the late Professor Helena Rasiowa, a famous mathematician and logician whose books like, for instance, "Mathematics of metamathematics", and many works have enjoyed a worldwide popularity. I used to visit her as she was living not far. We talked about mostly mathematics. She knew very well Lotfi, and was never against fuzzy logic. In a Wiley volume we edited with Lotfi in 1992, she wrote a very special article mentioning fuzzy logic in a positive sense, and this was probably the only example when a world class "non-fuzzy mathematician" wrote an article to a volume on fuzzy logic.

Professor Rasiowa told me once that she knew many well-known Spanish logicians, mostly from Catalonia, and one of them was Enric Trillas. I had known him for some years, and I even met him in New York in the early-1980s. I started looking more carefully at the Spanish fuzzy community who had already then been very active and prominent. Another very important aspect was that Enric Trillas had a very high position in the Spanish research administration structure which was the only example in the world as in virtually all countries people who were active in fuzzy logic did not have high level positions in science and hence could not support fuzzy research. This was the case in Poland, for instance, where the support for fuzzy logic was mediocre but step by step we had changed this. As I had always been inclined towards optimization, decision theory and analysis, and the like, I had naturally got interested in the works of the Granada group on optimization and related topics, notably by Miguel Delgado and Curro Verdegay. Even those early works of them, and their collaborators (and wonderful wifes!), have become real events in the area of fuzzy optimization that have been widely cited as mile stones.

So, similarly as in the case of Lotfi, I have known "virtually" Curro for a long time, but only from his works which is important but not enough . . . I think that our first personal, i.e. "real", meeting occurred at the first IFSA World Congress in 1985 in Palma de Mallorca in 1985. This was a great event that, first of all, showed a potential of the field of fuzzy logic, and the emergence of a really large and strong community. Moreover, a multitude of personal meetings, like mine with Curro, have inspired people, and often triggered joint works.

With Curro, and his Granada associates, we have immediately found a common language and came up with an idea to organize a joint Spanish - Polish meeting. Such initiatives were supported by authorities at both the Polish Academy of Sciences and the Consejo Superior de Investigaciones Cientificas. This meeting was held in 1988, and then next meetings followed in Poland and Spain. It was a very interesting experience because Poland was at the very end of the Communist system and Spain was for us a very special country which had some bad experience with totalitarism but after its end had catched up rapidly with an extraordinary progress. That is why we had always considered that the Spanish way was for us to follow.

That meeting was for me extremely important because I think that we have immediately found that our friendship is more than just a professional acquaintance, that we can always can rely on ourselves, and that this will last. I am honoured and extremely happy that, as it has always been, Curro remains a special person to me, a real friend. I can say the same about his spouse, Maite. This concerns in fact the entire Granada group. Such a relation is getting more and more rare nowadays when people are too busy to stop and think about things that are more important than an immediate loss or gain, and which last longer. Friendship is clearly one of the most important things of such a type.

Curro Verdegay: Yes, it was in the IFSA World Congress that was held in Mallorca in 1985. Actually I knew Janusz from long before, but only for his works. By then I was very interested in everything related to Multi-stage Decision, and Janusz had published a lot of works about it, which I knew because he had sent them to me by mail. Since the first time he answered my requests, I knew he had to be an exceptional person: he always replied very fast, including some kind manuscript words and answering every question I had made. Then in 1983 he published his book "Multistage Decision Making under Fuzziness (Verlag TÜV Rheinland, Cologne)", which I consider a must-have book for anyone interested in this topic, and soon after that, I met him personally. It was him who recognized me and came to chat with me after Enric Trillas had sent him to me for organizing a Spanish-Polish meeting. So in that first meeting we dealt with the organization of the first Spanish-Polish conference on "Computer Science and Systems Analysis", which was held in 1988 in Granada, in the middle of the first general strike held in Spain after Civil War. Janusz and myself immediately got along very well, and that first contact was the beginning of our friendship, which trascends the professional field, as I think of him as a member of my family.

You both worked on Fuzzy Optimization: can you shortly report your joint contributions in this area?

Janusz Kacprzyk: And, again, the response here may be viewed from a "real" and "virtual" perspective. The "real" perspective is related to our really joint work, an edited volume on fuzzy optimization published by Springer (Physica) in 1995. This was an important volume, some 7 years after my first volume on fuzzy optimization edited with Sergei Orlovski, which had satisfied a real need at its time. Then, Amparo, Curro, Miguel and myself, became convinced that there was a real need for putting together again a large book with a broad survey of what has been done in fuzzy optimization, its theory and applications in the years after that first volume. That 1995 project was a success from a scientific point of view, but more than that, our collaboration had shown that our friendship had survived very well, and had even been stronger, which is not always the case when people undertake joint works. . .

The "virtual" perspective is slightly more complicated. Let me repeat what I wrote before about my experience with IIASA where I was working years on large linear programming models for agricultural production planning. After a huge effort just to solve those huge models, and understand the results obtained that was even more important, I was more and more convinced that one of the reasons for a moderate success we had had with the agricultural domain specialists was the rigidness of the traditional linear programming models, and some fuzziness would be good. I tried to use Zimmermann's version of fuzzy linear programming which, from my point of view, was easier to implement, because it needed only the specification of some imprecisely specified bounds for the objective function value and constraints. I knew very well that the use of Verdegay and Delgado's approach with fuzzily specified parameters would be a far better choice. Unfortunately, this would need much money to obtain new, imprecisely specified (even as triangular or trapezoid numbers) parameters, right hand sides, etc. I could not do this for financial reasons, maybe also because of the size of the model and the fact that the use of that approach would increase too much the size of my, already huge model. However, Curro's work has clearly inspired mine.

Curro Verdegay: Since we met, we have organized many joint activities. The first were the First Spanish-Polish Conference on Computer Science and Systems Analysis (Granada , 1988) and the Second Polish-Spanish Conference on Systems Analysis (Rozalin, 1990). Then the TEMPUS program in "Computer Science and Analysis of Systems Training" (COAST) was developed and we, SRI-PAS and DECSAI, participated together with the University of Trento (Mario Fedrizzi), Technical University of Wroclaw (Stefan Chanas) and Lorand Eotvos University (Margit Kovacs and Robert Fuller). COAST was a good example of an international collaboration that contributed to enforce the theoretical basis of several topics: Fuzzy optimization, of course, but also fuzzy decision making (consensus reaching), Decision support systems in fuzzy environments, and so on. An important part of those contributions were collected in a book that constitutes, in my opinion, a reference work for anyone who wants to know the foundations of this topic: "Fuzzy Optimization: Recent Advances" (Physica-Verlag, 1994) in which, in addition to Janusz and myself, professors Miguel Delgado and Amparo Vila served as editors. After that first work, joint contributions have been very often and, in almost all cases, we have been accompanied in them by Mario Fedrizzi, whom I consider another excellent scientific and close friend of mine.

How your meeting had an influence on the development of Fuzzy Decision Making in Europe?

Janusz Kacprzyk: It is a difficult question that should be answered from various perspectives. First, from a more global, world scale perspective there has not been, and still is not something that might be described as a coherent theory of "fuzzy decision making" in the sense what you can find - to just name a few - in the schools of Raiffa, Kahneman and Tversky, Slovic, Roy, etc. In the fuzzy field there are many scattered works that address some particular problems but not from a general point of view, and they are rarely cited in works summarizing the state of the art in decision making, decision theory, decision analysis, etc. Maybe the only serious work in that fuzzy direction following the traditional, coherent approach is a series of books by K.K. Dompere that have appeared over the years in my Springer book series "Studies in Fuzziness and Soft Computing" in which he analyzes deeply utility theory under fuzziness, choice theory, etc. He is a real economist so that when he says an utility function, he knows what he means, and in most works on "fuzzy decision making" they have a mere valuation function, call it an utility function, with all kind of problems as mathematical assumptions cannot be omitted. The situation is better with respect to fuzzy preference based models due to works by, for instance, Ovchinnikov, De Baets, Fodor, etc. But, again, this is not a coherent theory.

First, for sure our meeting has resulted in an immediate decision to do something together, starting with the organization of some network. It is difficult to list everything but let me just name a few. First, we tried to gather a group of friends, notably Mario Fedrizzi, with whom I had collaborated for years, then the group was enlarged by - on the one hand - Hannu Nurmi and Sławek Zadrożny, who had been involved in joint works on group decision making, consensus formation, social choice and voting, and - on the other hand - by people like Ron Yager who, though being from outside of Europe, had been part of that group working together with us.

Curro Verdegay: To be honest, I think it has been very influent. The first Spanish-Polish meetings I commented before originated the TEMPUS programme which, apart from providing last-generation computers to the research groups involved in Poland and Hungary, enabled the birth of a working group that has been very productive in the field of fuzzy optimization and decision making. Concerning fuzzy optimization, important progress (now considered "classics") were made, only slowed down by the passing of my friend Stefan Chanas. Robert Fuller also played a prominent role in such advances. Regarding fuzzy decision making, the involvement of Mario Fedrizzi and Janusz opened a new research line which, more recently, has been intensely developed by my close friend, and disciple, Enrique Herrera-Viedma. This trend was initiated in the Seminars organized by Mario at University of Trento, which served as a catalyst of results, projects and publications with international academic recognition. Anyway, I should stress that the first meeting between Janusz and myself in Palma de Mallorca was due to Enric Trillas, who is in my opinion the true creator of all subsequent collaboration, and to whom we should be grateful for it.

Which are your research outcomes that you feel more proud of?

Janusz Kacprzyk: Though it is difficult to speak about own works, maybe I would try to give my brief, clearly biased opinion. I will list in this list both some theoretical works and practical applications.

First, just because this work has been probably best assessed by the community since I received for this the

2006 IEEE CIS Fuzzy Pioneer Award, these are my works on fuzzy multistage optimal control, notably fuzzy dynamic programming. I am particularly glad that I was able in this respect to propose some new models, like with a fuzzy termination time, and - what was probably the most interesting and challenging part of that work - a new class of models with an infinite termination times which was mathematically very challenging. I was happy that this work had provided me with a multistage planning model that was used in to find optimal development scenarios in regional planning, and that work had been cited as one of the best and most convincing examples of fuzzy modeling reported in the literature in a special book on the 50th anniversary of operational research in L.C. Thomas (Ed.) "Golden Developments in OR", Pergamon Press, Oxford, 1987.

Then, I think that I am very happy with my works on fuzzy database queries, which were predominantly done with Sławek Zadrożny, notably on queries with fuzzy linguistic quantifiers. These works are widely cited by most people in the community, have implementations and are part of a commercial product.

I think that I can also be satisfied with my works, also done mostly with Sławek Zadrożny, on linguistic summaries of data, both in the static and dynamic (time series) context. The introduction of an analysis in terms of Zadeh's protoforms, and above all the proposal of how to generate linguistic summaries by using the linguistic quantifier driven fuzzy querying system is interesting both theoretically and practically, and has also been implemented in a decision support system in a small computer retailer.

And last but not least, I am satisfied with my involvement in the research on intuitionistic fuzzy sets. I think that I have been supporting the founder of this theory, Krassimir Atanassov, and help him and the community to advance the theory by taking into account opinions of the wider audience, and hence to shape this theory in a constructive manner, to fully take advantage of its strengths. This has been done in collaboration with Eulalia Szmidt, to whom I am grateful for a long time and fruitful collaboration.

And, in addition to fuzzy logic, I am particularly happy with the results of my 3 PhD students. First, for the work of my PhD student from MIT who has developed, implemented and tested a novel evo-devo (evolutionary developmental biology inspired) approach to satellite image classification, and of my 2 PhD students working on the use of memetic algorithms for the global and local path planning in non-holonomic mobile robots. This works have been done for a commercial robot and the algorithms developed will be offered as an option in the path planning software.

From a more formal point of view, I am glad that I have been elected Full Member of the Polish Academy of Sciences, a body of 360 professors from all fields, as a recognition for my works mostly on fuzzy logic which would not have been possible in the past, and even the greatest and most powerful opponent of fuzzy logic was not too strongly against me ....

I am also very proud of being elected a Foreign Member of the Spanish Royal Academy of Economic and Financial Sciences (RACEF) because, though I am not en economist, that prestigious group, to which many great scientists belong, including many Nobel Prize winners like Kahneman, Stiglitz and Aumann, has appreciated my works on the use of fuzzy logic for economic modeling and management.

Curro Verdegay: Every single result I have obtained involving Fuzzy Sets and Systems has been inspired in a previous concept or idea concerning a problem conceptualized by Zadeh, with no exceptions. Hence, what I feel more proud of is the fact that Lotfi considers me as one of his friends, and has allowed me to share his ideas and thoughts. But, together with this, I am also very proud of each of the 19 doctoral dissertations I have supervised, among which I can cite those by Francisco Herrera, Enrique Herrera-Viedma or David Pelta. In any case, all my main findings have been around Fuzzy Optimization, particularly around Fuzzy Mathematical Programming, and Fuzzy Decision Making, mainly multiperson decision making. As theoretic topics (alpha-cut based approach for solving Fuzzy Optimization problems, using fuzzy control rules as termination criterion of the algorithms or fuzzy sets based metaheuristics) as real practical applications developed worldwide (Argentina, Cuba, Brazil, R.D. Congo, ...) have an special feeling for me.

Apart from the valuable scientific outcome of your activity, which is the most important outcome of your work in this field?

Janusz Kacprzyk: If one takes my activity in the field of fuzzy logic but not related to research, I think that I have always tried to contribute to the fuzzy community mostly by working in EUSFLAT, IFSA, IEEE Computational Intelligence Society. To be slightly more specific, I have been active in EUSFLAT practically since its foundation, serving in various bodies.

Moreover, for many years I have been active in IFSA, being first a member of the Council, but I think that my service as Treasurer in the period of 2003 - 2007 was very difficult and important because during that time we had to transfer IFSA funds from Japan, and after a period of difficulties in financial operations, we had finally found a solution to transfer the IFSA account to Finland where it still is. My service to IFSA has culminated in the period of 2007 - 2009 when I served as President. I think that during my term we were doing well, in particular with respect to an extension of the program of supporting young researchers, mostly by subsidizing their participation at conferences and awarding their papers, dissertations, etc.

I also consider my activities at and for IEEE Computational Intelligence Society which practically started in 2007 after my elevation to the level of Fellow of IEEE and when I received Fuzzy Pioneer Award, both in 2006. I served in many bodies like the Award Committee, Fellows Committee, etc. and wrote references for many people who were then elevated to those high levels, received prestigious IEEE medals and award, etc. In the last three years I served as a member of the Administrative Committee (Adcom) of IEEE CIS, and also as a lecturer in the Distinguished Lecturer Program, promoting fuzzy logic all over the world. I think that I can also consider as my accomplishment the fact that for almost 20 years I have been the editor in chief of 5 book series at Springer that belong to the largest in the world, notably "Studies in Fuzziness and Soft Computing" and "Studies in Computational Intelligence", with hundreds of books and volumes. Tens of people from the fuzzy field have had a place where their works could have been published by one of the world's most reputable and second largest scientific publisher who could guarantee distribution and exposure as opposed to small and obscure publishers who can only promise this.

Of course, the biggest achievement and joy of any scientist and scholar are his or her disciples and followers, and I was happy to have many of them, about 10 PhD student from Poland, China and the USA, and many collaborators who I tried to properly direct and shape using my experience.

Curro Verdegay: I would like to emphasize several things. First, that by no means the achievements that can be imputed to mean are due to my merits only. In this sense I am convinced that I would have achieved nothing without the support, work and involvement of my Department, the Department of Computer Science and Artificial Intelligence of the University of Granada, generally known as GRANADA Team. As it is well-known, the GRANADA Team poses such members as Miguel Delgado, Amparo Vila, Francisco Herrera, Mª Teresa Lamata, Enrique Herrera-Viedma, Oscar Cordón, Juan Luis Castro and many others. I am very proud and satisfied with the recognition we have won for the research carried during the last two decades. But it also fills me with pride to have been the founder and first president of the "Fuzzy Logic and Technologies (FLAT) Spanish Association", which we later turned into the "European Society for Fuzzy Logic and Technology" (EUSFLAT). And last but not least, I am very satisfied as well with my university cooperation activities, first with universities of the old East Europe, then with some universities of Sub-Saharan Africa, mainly DR Congo, and in the last years with Cuban universities, where Fuzzy Sets and Systems have an important presence in the research activity carried out in the main universities of such country.

The scientific community flourished around Lotfi Zadeh and Fuzzy Logic has an important and invaluable characteristics: it is friendly and human relationships have been developed and gave birth t solid friendships. Can you shortly report on your experience?

Janusz Kacprzyk: This is a very important question and issue. First, if you look at Lotfi Zadeh, then - at the first glance - he may seem to be a lonely player because he has rarely been writing papers with somebody else, and also the number of his PhD student is not as high as one could expect. However, if you look from a wider perspective, you can clearly see that the number of his disciples is huge. By a disciple I mean a person who has been inspired by Lotfi, and whose career has been shaped and boosted by his or her encounter and interaction with Lotfi. In this sense this has been the case for many, maybe most, of people working in the fuzzy field who have gained a stature and recognition because they have met Lotfi and his novel ideas at a right time. Otherwise, they - or we, in general - would have had problems to get a similarly high position in traditional areas, notably in mathematics or computer science.

Moreover, looking at Lotfi's works and activities from a broader perspective, one can say that his friendly and constructive attitude, his famous saying "whatever they say take it as a compliment", and his great human qualities in general, have contributed to a large extent to both the development and proliferation of fuzzy logic, and the fuzzy community, and an overall friendly atmosphere in that community. As I always say, in science there is a need to fight for money, recognition, etc. but one should concentrate on that fight with other fields, and within the own field one should support each other to get a synergy and attain much more together. This is what I have clearly learned from Lotfi.

Curro Verdegay: Me personally (although I think I am not wrong if I speak on behalf of the whole GRANADA Team), I am very grateful to Lotfi for a lot of things, some of which I have already mentioned through this interview. There is one more that stands apart: he has always acted as an Ambassador of GRANADA Team. He was appointed Doctor Honoris Causa by the University of Granada in 1996. One could think it was from then that he started his role as "ambassador", but this is not true. Probably due to the close relationship he has always had with Enric Trillas, and also due to Enric's family links with Granada, Lotfi has always gifted the GRANADA Team with his friendship. As a consequence, we have a large group of friends all over the world, among which the first ones (just for being the first ones) pose a special place in our memoirs. I am referring to members such as the already mentioned Elie Sanchez, Enrique Ruspini, Takeshi Yamakawa, Ron Yager, Philippe Smets, Piero Bonisonne or Michio Sugeno. And, in a more personal aspect, friends such as Mario Fedrizzi, Bernadette Bouchon, Christer Carlsson, Hans Zimmermann, Robert Fuller or Fernando Gomide have been very important for me. And finally I have to recall the Spanish family of Fuzzy Sets and Systems, generated around Enric Trillas and impossible to enumerate as they are hundreds!

Since 1965, Fuzzy Set Theory, Fuzzy Logic, Possibility Theory and their applications had a big evolution. Which scientific perspective you see in this domain? And which are the big research challenges of next years?

Janusz Kacprzyk: This is an extremely important question for each field of science, and maybe is particularly true for fuzzy logic, for various reasons. First, one should bear in mind that fuzzy logic is predominantly within applied sciences, notably in applied mathematics. Of course, there are some questions in the so-called “fuzzy mathematics” that are more general and they should be seriously pursued, notably by attracting more top “traditional” mathematicians to that research.

The second aspect is related to the above mentioned fact that fuzzy logic is within applied science. This is a big advantage but one should bear in mind that we cannot stay in fuzzy logic control only because it has been proved to be useful. We should go further into control and introduce more sophisticated, more “control like” applications of fuzzy logic.

Another thing that we should be aware of is that if we assume the fact that has been known for years or decades, that (mathematical) modeling is the king, then a natural question is which type of models people mostly use in various fields of science and technology. Many people in fuzzy logic think in this context of rule based models only but this is not true. Most of mathematical modeling is done by using differential and difference equations, to an extent higher maybe by the order of magnitude. Yet, the progress in the fuzzification of differential/difference equations is not adequate. I spoke about this already in 1999 at the FUZZ-IEEE Conference in Seoul, got some explanations but the fact remains valid: people who are using differential/difference equations often express a need for some imprecise formulation but they cannot find tools and techniques that are effective and efficient enough.

A similar situation is with broadly perceived decision theory, more sophisticated types of optimization, for instance non-differentiable optimization, and many other fields in which people want us to propose serious and comprehensive models capable of handling fuzziness. But, one should know that models existing in those areas are sophisticated and well developed and a straightforward fuzzification that is just based on intuition, common sense and simple tricks will not be accepted by those communities.

In this respect I would also mention a need to reach out to other communities. We tend to somehow operate within a restricted community of “fuzzy people” who are mostly computer scientists, applied mathematicians or like this. We overlook very interesting developments and applications of fuzzy sets in, for instance, mathematical economics (viz. Dompere’s books I mentioned before) or

social sciences. We consider them too simple mathematically, forgetting of course that our works are probably judged in the same way by mathematicians. Those works do present real applications of fuzzy logic that provides new tools for analyses and solutions to those areas.

And finally, a big challenge is to find ways for making fuzzy logic a generally acceptable tools for real world implementation. Very much has been done in this respect for instance by Piero Bonissone and Dimitar Filev who have combined a high level science with real applications. We do need more such people as no area that is within applied sciences can survive without real applications. Of course, by applications I do not mean what comes out of even successful research projects, national or European, but an implementation, i.e. when somebody, but not the government, wants to pay money for what we do, and really needs our work.

These are just some challenges I can see which are somehow triggered by my long time involvement in fuzzy and not only fuzzy research, real applications, collaboration with famous think tanks and labs. They are serious but if I look at so many great people in our community, I am sure that they will find ways to face those challenges and overcome any difficulties we may have. I am sure that fuzzy logic will flourish in the future as it has done since its inception.

And also, I am sure that my friendship with Curro will continue, and we have not yet said the last word, in science and beyond.

Curro Verdegay: From my point of view there are two main research lines to be investigated. On the one hand, it is necessary to go on with what we call fundamental (or basic) science, which is, in my opinion, absolutely necessary to achieve new scientifically interesting results. Moreover I suggest re-visiting, and possibly reconsidering, some basic (classic) concepts concerning Fuzzy Sets and Systems. On the other hand, I think it is now time for innovation and transfer as well. We should work with that in mind, in the directions pointed out by the HORIZON 2020 European programme by strengthening research in Future and Emerging Technologies through research, technological development, demonstration and innovation. In my opinion, we, the diversity of research groups working in Fuzzy Sets and Systems, have a great opportunity within the HORIZON 2020 programme to work together, which I believe is a must if we want to continue our research in these topics. For that reason, and using EUSFLAT as a common platform for communicating and sharing our knowledge -which gives us a competitive advantage not found in other fields- I think it is necessary to organize ourselves soon and do a call, maybe by organizing a workshop, in order to coordinate our research lines, bring together the teams working in similar topics, and collect future trends to ellaborate a joint proposal so we manage to achieve support for the development of research projects within HORIZON 2020. Of special relevance may be to study the applications of fuzzy sets and systems to topics such as the cyber security, intelligent information management systems based on advanced data mining, adversarial decision making, machine learning, statistical analysis and visual computing technologies.

In any case, let me to say to finish from my side, that I'm sure whatever the project to develop in the near future and beyond, Janusz and me will be once again involved in new proposals and joint scientific ventures that will permit us to strengthen our friendship even more.

CURRO VERDEGAY

by Pedro Burillo

The Universitary Reform Law, published in Spain in September, 1983, opened the door for modernizing universities, which up to then were anchored in pre-democratic costumes. In its development, in 1984 knowledge areas were created, being defined as "those fields of knowledge characterized by the homogeneity of its purpose, a common historical tradition and the existence of a community of researchers, either national or from abroad". One of those areas, that of Computer Science and Artificial Intelligence, brought together teachers from different fields of Science and Informatics faculties (later known as Computer Engineering Schools), such as Algorithmics, Applied Mathematics, Logics, Automata, Computing, Artificial Intelligence, etc. Many of the original staff came from Sciences Faculties, since Informatics Faculties have just recently been created, in 1976. This fact allowed me to meet Prof. José Luis Verdegay Galdeano ("Curro" for his friends and almost everybody), as two of the few teachers originally in our area. Although both of us were researchers in the same topic, Fuzzy Sets Theory, we have not met before, so we could start a common journey that has finally derived into a sincere friendship of which I feel proud. The birth of our knowledge area, and specially its later development -which depended almost solely of the original staff-, came associated to many works in which the role of Prof. Verdegay has been, in my opinion, crucial. On the one hand, the usual participation in Committes for selecting professors in the area for the different universities required that we fixed general criteria for choosing the candidates, a very difficult point and where consensus was hardly obtained, but the good sense of Prof. Verdegay made agreements possible, although they were not always respected, as it could be expected. Moreover it was necessary to carry on collective actions that made the newly created knowledge area visible in the Spanish universities: all the professors in the area made great efforts to increase the number of young researchers, of Ph. D. Thesis and of conferences, and to boost publications in prestigious journals of the fuzzy world. And simultaneously to all this work there was also academic gestion tasks: probably one of the most important ones was that of the configuration and development of the deprtments which, in some cases, were created around the Computer Sciences and Artificial Intelligence knowledge area. This was the case in the University of Granada, were Prof. Verdegay land other colleagues laid tha basis of what would become an excellent department in an excellent university. The academic attraction of people like Prof. Verdegay has made that nowadays, the Department of Computer Sciences and Artificial Intelligence of the University of Granada is a worldwide reference on fuzzy research. At the end of the eighties, an idea that had been considered by many professors in the fuzzy world became a reality: the creation of a Spanish Society of Fuzzy Logic and Technology (FLAT), with Prof. Verdegay as a founder. Such society brought together almost every Spanish researcher in fuzzy theory and in 1991 FLAT started its highest opinion and reflection tool: The Spanish Conference on Fuzzy Logic and Technology (ESTYLF) whose first edition was held at the University of Granada. Later, in 1998, FLAT joined to researchers of the rest of Europe to become the European Society for Fuzzy Logic and Technology (EUSFLAT) in whose foundation Prof. Verdegay also took part. And many other things, since it is very difficult to resulme in a few lines the huge work of Prof. Verdegay in the Computer Sciences and Artificial Intelligence knowledge area and bot in the national and in the international fuzzy communities. Work that he has always acomplished with great profesionality and a fine sense of humour, and I believe thet the latter is one of the most remarkables notes in his personality. Let it be so for many years!

JANUSZ KACPRZYK

by Sławomir Zadrożny

Janusz is an institution, both in his home country and world-wide. It is very difficult to characterize his research work, his organizational achievements, even more difficult to characterize himself, in a brief note like this one. The picture will be always fuzzy, incomplete and imprecise. He is a man of the vision, a genuine leader and a very modest and friendly person. Janusz has graduated as a control engineer. No wonder that he was attracted very quickly by the ideas of Lotfi Zadeh, also an accomplished control engineer. Of course, Janusz himself is a best candidate to tell the story of his research interests, motivations and inspirations, and he does this in a very interesting way in the interview. Here are just a few impressions and facts concerning his numerous skills and achievements. Janusz is fluent in many languages. He can deliver a speech in French, Italian, Spanish, German, Russian, not to mention English and Polish. I had a chance to admire his remarkable ability to switch from one language to another without any difficulty. And to observe how important is a possibility to communicate with another person in her or his mother tongue. How much easier is to find a common understanding of a given problem, either scientific or related to everyday life. It surely made Janusz to put high on his research agenda the issue of natural language modeling. The modeling here is meant in the spirit of foundations of fuzzy logic rather than in natural language processing (NLP) domain. Both lines of research well complement each other and Janusz's contribution to the former is very important. In particular, he introduced and/or developed many concepts based on the application of the elements of natural language in decision making, data querying and mining or machine learning, to name just a few areas. Janusz Kacprzyk, born in 1947, graduated in 1970 from the Department of Electronic, Warsaw University of Technology in Warsaw, Poland with M.Sc. in automatic control. He received his Ph.D. in systems analysis, D.Sc. (habilitation) in computer science from the Polish Academy of Sciences, and Full Professor title in technical sciences, awarded by President of the Republic of Poland. He is a Chairman of the Council of Provosts, Division IV of Engineering Sciences of the Polish Academy of Sciences, supervising 13 research institutes and 23 national committees. He is Professor of Computer Science at the Systems Research Institute, Polish Academy of Sciences, and Professor of Automatic Control at: PIAP - Industrial Institute of Automation and Measurements, in Warsaw, Poland, and Department of Electrical and Computer Engineering, Cracow University of Technology, in Cracow, Poland. He is Honorary Foreign Professor at the Department of Mathematics, Yli Normal University, Xinjiang, China, and Visiting Scientist at the RIKEN Brain Research Institute in Tokyo, Japan. He is Full Member of the Polish Academy of Sciences and Foreign Member of the Spanish Royal Academy of Economic and Financial Sciences (RACEF). He is Fellow of IEEE and IFSA. He has been a frequent visiting professor in the USA, Italy, UK, Mexico and China. He is the author of 5 books, (co)editor of 60 volumes, (co)author of more than 400 papers. He is the editor in chief of 5 book series at Springer, and the editor in chief of 2 journals, co-editor and associate editor of 3 journals, and a member of editorial boards of more than 50 journals. In 2007-2009 he was President of IFSA

(International Fuzzy Systems Association), and currently he is President of the Polish Operational and Systems Research Society. He is a member of Award Committee of IEEE Computational Intelligence Society (CIS), Adcom (Administrative Committee) of IEEE CIS, and a Distinguished Lecturer of IEEE CIS. Formerly, he was an alternate member of IEEE Fellows Committee. He received many awards, notably: 2006 IEEE CIS Pioneer Award in Fuzzy Systems, 2006 Sixth Kaufmann Prize and Gold Medal for pioneering works on soft computing in economics and management, IFSA 2013 Award, 2007 Pioneer Award of the Silicon Valley Section of IEEE CIS for contribution in granular computing and computing in words, 2000 AutoSoft Journal Lifetime Achievement Award in recognition of pioneering and outstanding contributions to the field of soft computing, 2009 Jubilee Diploma of the Bulgarian Union of Scientists for a long-term collaboration with the informatics section, and the Award of the 2010 Polish Neural Network Society for exceptional contributions to the Polish computational intelligence community.

REPORT

Prof. Lotfi Zadeh received the BBVA Foundation Frontiers of Knowledge Award for enabling computers and machines to behave and decide like human beings

Luis Magdalena

According to the Jury, by equipping computers to tolerate real-world complexities and decide accordingly, fuzzy logic transforms them from mere calculating machines, and allows appliances and systems to operate autonomously. The award consist of 400,000 EUR, a diploma and a commemorative artwork. The Awards ceremony took place past June 20th 2013 in Madrid, and was preceded by a Gala Concert taking place on June 19 at Royal Theatre of Madrid.

The BBVA Foundation Frontiers of Knowledge Award in the Information and Communication Technologies (ICT) category has been granted in this fifth edition to the electrical engineer Lotfi A. Zadeh , "for the invention and development of fuzzy logic". This "revolutionary" breakthrough, affirms the jury in its citation, has enabled machines to work with imprecise concepts, in the same way humans do, and thus secure more efficient results more aligned with reality. In the last fifty years, this methodology has generated over 50,000 patents in Japan and the U.S. alone.

The award consists, in each of the eight categories, of 400,000 EUR, a diploma and a commemorative artwork.

Lotfi arrived in Madrid on June 19 and was received by part of the European fuzzy community which accompanied him during the Gala Concert taking place that evening. Rudolf Kruse, Ellie Sanchez, Rudolf Seissing, Javier Montero, Enric Trillas and Luis Magdalena, among others, attended the important event. The Gala Concert included works from Richard Wagner, Alban Berg, Pierre Boulez and Igor Stravinsky, performed in the magnificent environment of the Royal Theatre of Madrid.

After the concert, Prof. Zadeh jointly with Luis Magdalena, Director General of the European Centre for Soft Computing (promoter of his nomination) and Ramón López de Mántaras (secretary of the Jury), had dinner at the restaurant of Westin Palace Hotel in Madrid (where all awardees and members of the different juries were hosted).

All information including a video message by Prof. Zadeh can be found at Fundación BBVA (`http://www.fbbva.es/TLFU/tlfu/ing/microsites/premios/fronteras/galardonados/2012/informacion.jsp`).

Prof. Zadeh was nominated by Luis Magdalena, Director General of the European Centre for Soft Computing (`http://www.softcomputing.es`) and Secretary of the IFSA Board. The jury in this category was chaired by George Gottlob, Professor of Computer Science at the University of Oxford (United Kingdom), with Ramon Lopez de Mantaras, Director of the Artificial Intelligence Research Institute of the Spanish National Research Council (CSIC) acting as secretary. Remaining members were Oussama Khatib, Professor in the Artificial Intelligence Laboratory in the Computer Sciences Department of Stanford University (United States), Rudolf Kruse, Head of the Department of Knowledge Processing and Language Engineering at Otto-von-Guerike-Universitat Magdeburg (Germany), Mateo Valero, Director of the Barcelona Supercomputing Center (Spain) and Joos Vandewalle, Head of the SDC Division in the Department of Electrical Engineering at the Katholieke Universiteit Leuven (Belgium).

Dinner after the Gala Concert: Prof. Zadeh, Luis Magdalena and Ramón López de Mántaras.

Headquarter of the BBVA Foundation. Location of the Awards Ceremony.

The awardees in the eight categories were received by BBVA CEO, Mr. Francisco González, who chaired the ceremony jointly with Prof. Emilio Lora, President of the Spanish Research Council. (Left) Mr. Francisco González with the awardees.

Prof. López de Mántaras read the decision of the Jury and accompanied Prof. Zadeh while receiving the Diploma from Mr. González and Prof. Lora. (Right) Ramón López de Mántaras, Francisco González, Lotfi Zadeh and Emilio Lora.

Then Prof. Zadeh addressed the audience to thank the Award and offer a short view of fuzzy logic and some figures showing its successful application. His talk was enthusiastically received. (Left) Prof. Zadeh addressing the audience.

Finally, a cocktail was offered to all attendants, in the garden of the palace. (Right) The cocktail after the Awards Ceremony.

(Left) The Jury during the announcement of their decisión.

REPORT

The Evolution of Hybrid Soft Computing: A Personal Journey

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This paper presents the author's personal journey throughout the inception of Soft Computing (SC) and its evolution into Hybrid Soft Computing (HSC)[1]. In 1994, Zadeh defined the concept of Soft Computing as "a loose association or partnership of components". Since then, SC has undergone several transformational phases. In our journey, we provide a retrospective view of SC past four phases, its current view (fifth phase) and a prospective view, in which we envision the role of SC within the new trend of model ensembles and model fusion. The retrospective view starts with the *hybridization* phase, which is driven by the inherit ease of integration of SC components and leads to the concept of Hybrid Soft Computing. The second phase, inspired by traditional AI problem formulations, is a *two-level model characterization* based on object-level and meta-level reasoning. The third phase addresses the *knowledge and meta-knowledge representation* required by each of these reasoning levels using a linguistics analogy. The fourth phase is an *extension* of the heuristics used at the meta-level, i.e., Meta-heuristics (MH's), from evolutionary algorithms to other global search methods. The current view (fifth phase) *consolidates* the above phases and *separates* offline MH's, used for design and tuning, from online MH's, used for run-time selection or aggregation of object-level models. This view suggests a broader use of SC components, since it enables us to use hybrid SC techniques at each of the MH's levels as well as at the object level. Furthermore, this separation facilitates model lifecycle management, which is required to maintain the models' vitality and prevent their obsolescence over time. Finally, the prospective view analyzes the recent trend of using model ensembles rather than individual models: the elements of the ensemble are the object-level models, while the fusion mechanisms are the online MH's that select or aggregate them.

1. Introduction: The Origin of Soft Computing

Soft Computing (SC) is a concept originally proposed by Zadeh (1994) as: "*an association of computing methodologies that includes as its principal members fuzzy logic (FL), neuro-computing (NC), evolutionary computing (EC) and probabilistic computing (PC)*". In his definition, Zadeh contrasted this new concept with Hard Computing by highlighting SC ability to "*exploit the tolerance for imprecision, uncertainty, and partial truth to achieve tractability, robustness, low solution cost, and better rapport with reality*". Since the term "*association of components*" has rather loose semantics, this encouraged many researchers to provide their own interpretations, refining and evolving this term in a variety of ways.

In the early nineties we can also trace the origin of *Computational Intelligence* (CI), a concept very similar to SC. CI is based on three technical pillars: *Neural networks*, to create functional approximations from input-outputs training sets; *Fuzzy systems*, to represent imprecise knowledge and perform approximate deductions with it; and *Evolutionary systems*, to create efficient global search methods based on optimization through adaptation. Clearly, CI has a broad overlapping with Soft Computing. Based on the definition provided by the IEEE Computational Intelligence, CI covers "*biologically and linguistically motivated computational paradigms*". Its scope excludes probabilistic reasoning systems, and includes other nature-inspired methodologies, such as swarm computing, ant colony optimization, etc. For a deeper insight into the historical origins of the CI consult Bezdek (1992, 1994), Marks (1993), and Palaniswami and Fogel (1995).

In the remaining sections, we will focus on the evolution of Soft Computing. In section 2, we provide a retrospective description of SC, while in section 3 we describe its current state. In section 4 we offer a prospective view and describe the role of SC within the emerging trend of using model ensembles rather than individual models. Finally, in section 5 we draw some conclusions from this journey[2].

2. A Retrospective View of Soft Computing

2.1 Phase 1: Hybridization

In the late nineties, Bonissone (1997) noted that SC components were converging into a new concept: *Hybrid Soft Computing (HSC)*. HSC components were divided into *reasoning* and *search* techniques. The reasoning techniques were *knowledge-driven* tools used to translate domain knowledge into models. The search techniques were *data-driven* tools used to extract models from the data, rather than starting from subject matter experts. Figure 2.1 - adapted from Bonissone (1997) - illustrates this dichotomy.

The hybridization is a natural consequence of integrating domain knowledge with field data. In the upper part of Figure 1, we find the individual SC components. In the lower part of the Figure 1, the dashed box encloses various examples of HSC. Zadeh further revisited this concept in 1998.

[1] In 2008 the author received the *II Cajastur International Prize for Soft Computing*, for applications of Hybrid Soft Computing, which is the motivation for this retrospective and prospective view of the field.

[2] This paper extends an earlier view of the evolution of Soft Computing described in (Bonissone 2010).

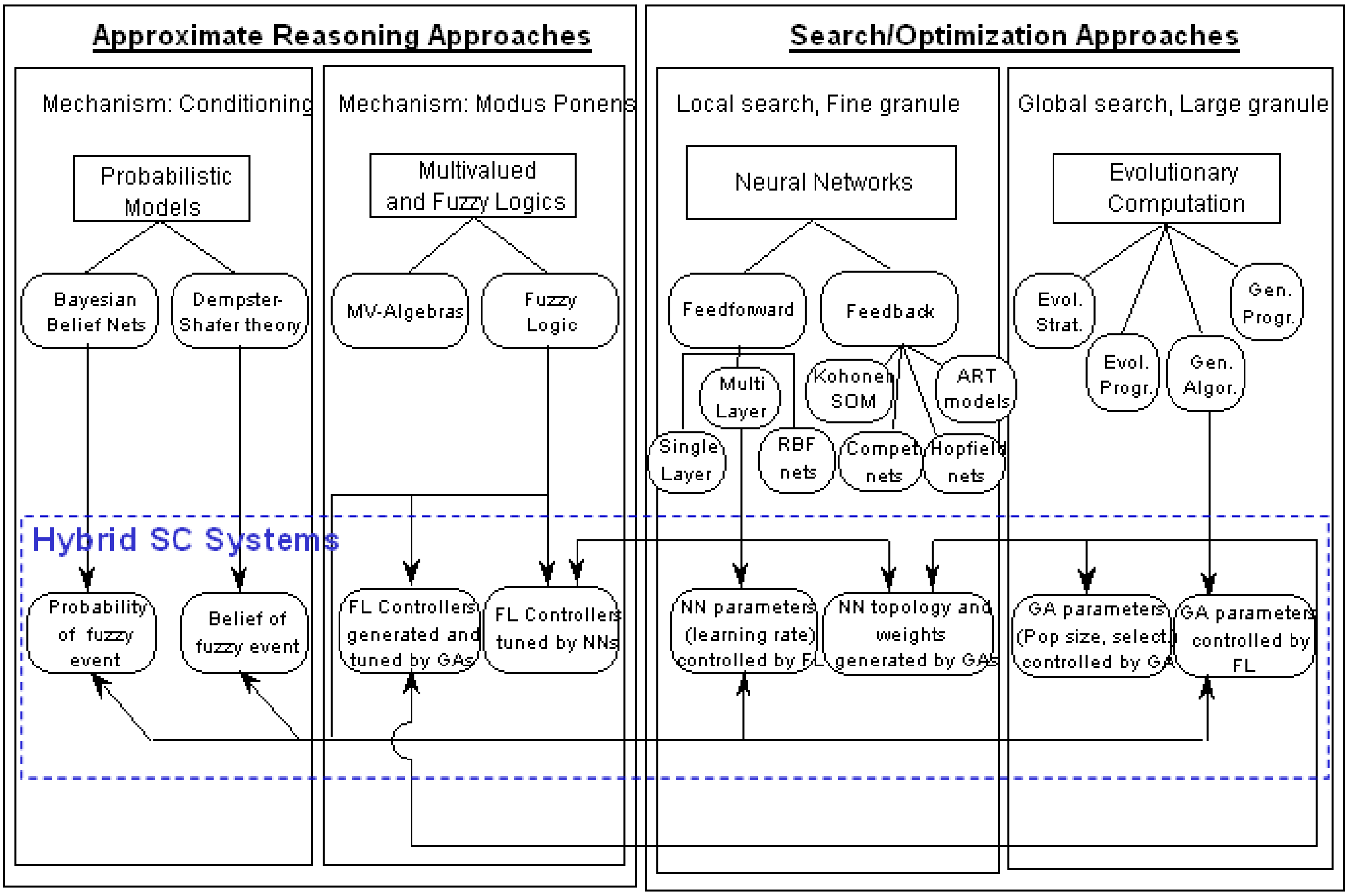

Figure 1 Soft Computing Overview - adapted from (Bonissone, 1997).

A few years later, in (Bonissone et al., 1999), we focused on the ease with which SC components could be integrated into Hybrid SC system, and stated: "*we have seen an increasing number of hybrid algorithms, in which two or more SC technologies have been integrated to leverage the advantages of individual approaches. By combining smoothness and embedded empirical qualitative knowledge with adaptability and general learning ability, these hybrid systems improve the overall algorithm performance.*" In the same reference, we emphasized the synergy generated by the use of *search* components to generate or tune *reasoning* components. The concept of Hybrid SC was further refined by other researchers, who explored the use of global search methods, such as evolutionary algorithms for generating probabilistic systems (Larrañaga et al., 2002), fuzzy systems (Cordon et al., 2004), and neural networks (Yao, 1999).

2.2 Phase 2: Two-level Modeling (Object- and Meta-Reasoning)

In 2003, we analyzed the modeling problem using a two-level approach common to many traditional AI problem formulations. The first level was the *object-level*, in which SC techniques were used to implement run-time models to solve domain-specific problems. The second one was the *meta-level*, in which SC techniques were used to generate, improve, update, and control the object-level models. This two-level decomposition suggested symmetry between reasoning and search methodologies, so that we could use knowledge and reasoning to control search and vice-versa. In the same reference (Bonissone 2003), and later in (Bonissone et al, 2006), we introduced a distinction between *offline* and *online Metaheuristics (MH's)*. *Offline MH's* dealt with the batch design of object-level models. Once the design was complete, run-time object-level models were generated and used to solve the problem without further modifications. This relationship is analogous to the compiler/run-time model relationship that we find in Machine Learning (ML), where ML algorithms extract relevant information from the training set to generate run-time models. *Online MH's* on the other hand, are used to monitor, guide, and control the resources of the run-time model. Figures 2a and 2b, adapted from Bonissone et al (2006), illustrate the use of offline MH's for model parameter design and on-line MH's for model parameter run-time control.

Figure 2a illustrates the use of offline MH's: a meta-level Problem Solver (PS) determines the best parameters to be used by the object-level PS, through a set of experiments against a set of representative problems. The parameters set remain constant throughout the execution of the object-level PS. An instance of these MH's is the meta-level GA proposed by Grefenstette (1986). Figure 2b illustrates the use of online MH's: a controller, driven by rules contained in a Knowledge based (KB), continuously evaluates the performance of an object-level PS and periodically modifies its parameter set. An instance of these MH's is the use of a fuzzy controller to modify population size and probability of mutation of a genetic algorithm (Bonissone et al, 2006).

2.3 Phase 3: Domain Knowledge Representation

Having established this two-level structure, we focused on the knowledge representation required by each structure

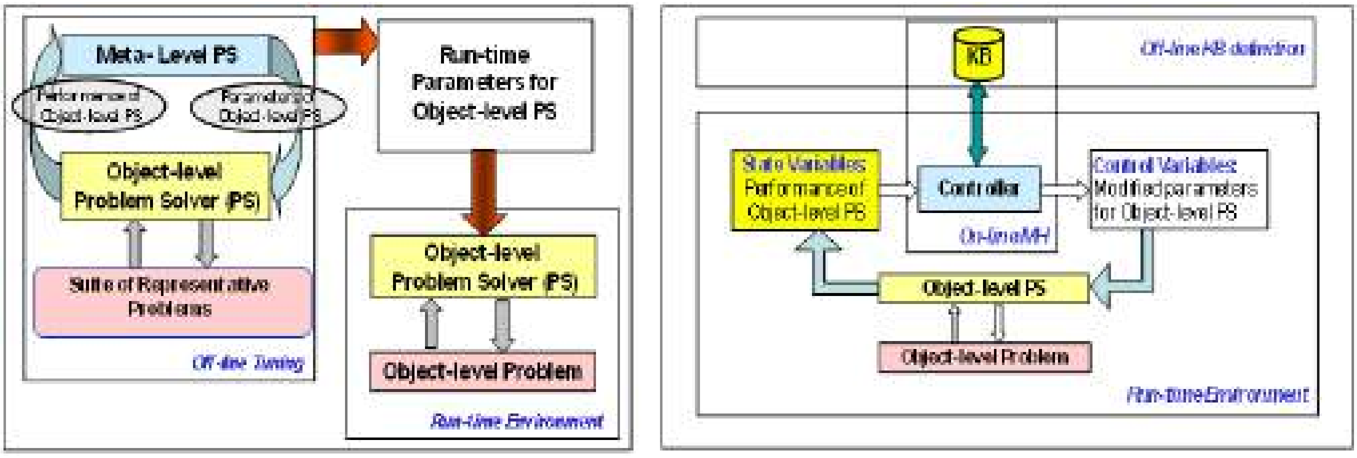

Figure 2. Fig. 2.a Offline Meta-Heuristics; Fig 2b Online Meta-Heuristics - adapted from (Bonissone, 2003).

(Bonissone, 2006). To measure the depth of such knowledge, we proposed a *linguistics analogy* in which the knowledge's depth ranges from lexical (e.g., event codes that appear in an equipment control log), to morphological (e.g., the *event code taxonomy*), syntactic (e.g., *signatures* derived from ordering of the vent codes), semantic, (e.g., *meaning of the codes*, usually based on domain knowledge) and pragmatic (e.g., context-dependent *model selection*).

With this ordering, we established a decision framework defined as the cross-product of the *time-horizon* over which a decision was needed (from single decision to short, medium, long and life-long term) and the *knowledge depth* required by the decision making model, as illustrated in Figure 3. By observing this figure, we observe that we can develop models based on shallow knowledge (from lexical to syntactic) only for short-term (tactical) horizon applications. In these cases, it is common to construct an ensemble of such models, ensuring their diversity (in the sense of low error correlations) and performing a fusion to increase the output's reliability.

As the time horizon increases, deeper domain knowledge is required to create the object-level models. Furthermore, the outputs of the models become more complex (schedules, plans, Multi-Criteria Decision Making) and less suitable for fusion. These sophisticated models require the use of semantics, since - as it in the case of system analysis - semantics allows us to decompose the meaning of a communication (model) into the meanings of its components and their relationships.

For instance, by incorporating engineering knowledge, we can identify key system variables, leverage their functional dependency to verify the correctness of other variables, extract the most informative features to create more compact representations, etc. Furthermore, the performance metrics associated with these tasks become less precise and more qualitative in nature. This characteristic is very suitable for the use of fuzzy system as possible fitness evaluator for the evolutionary algorithms that might be used to explore the models.

To build meta-level models, we resort to *pragmatics*, which use external, contextual knowledge to fully understand the meaning of our communication. While all prior levels dealt with information contained in the message itself (object-level), pragmatics requires higher-level knowledge (meta-level) to provide the contextual information needed to disambiguate a sentence, correctly interpret its meaning, etc. For model building, this means leveraging contextual information, such as operational regimes, environmental conditions, and health deterioration, to determine the degree of applicability of local models and select the best one (or their best mixture).

Additional information about SC approaches for different knowledge types and the applications to Prognostics and Health Management (PHM) can be found in (Bonissone, 2006).

2.4 Phase 4: Extending Offline Meta-Heuristics

This two-level approach was further refined in Verdegay et al. (2008). When dealing at the meta-level, we extended global search techniques from evolutionary algorithms to a variety of meta-heuristics, such as *relaxation* and *search MH's*. With this extension we emphasized the fact that the meta-heuristics were used to perform a search in the object-model design space. As such, we should be able to use a variety of search methods, such as taboo search, scatter search, hill climbing, greedy-like, multi-start, variable neighborhood, simulated annealing, evolutionary search, etc. This is illustrated in Figure 4, adapted from Verdegay et al. (2008).

3. Current View of Soft Computing

The SC phases described in the previous sections have converged into a framework that represents the current view of Hybrid SC (Bonissone 2010). In this framework, hybridization has been structured as a three-layer approach, in which each layer has a specific purpose:

- Layer 1: *Offline Metaheuristics (MH's)*. They are used in batch mode, during the model creation phase, to design, tune, and optimize run-time model architectures for deployment. Then they are used to adapt them and maintain them over time. Examples of offline MH's are global search methods, such as evolutionary algorithms, scatter search, tabu search, swarm optimization, etc.

- Layer 2: *Online MH's*. They are part of the run-time model architecture, and they are designed by offline MH's. The *online MH's* are used to integrate/interpolate among multiple object-models, manage their complexity, and improve their overall performance. Examples of online MH's are fuzzy supervisory systems, fusion modules, etc.

- Layer 3: *Object-level Models*. They are also part of the run-time architecture, and they are designed by offline MH's to solve object-level problems. For simpler cases we use *single object-level models* that provide an individual SC functionality (functional approximation, optimization, or reasoning with imperfect data). For complex cases we use *multiple object-level models* in parallel configuration (ensemble) or sequential configuration (cascade, loop), to integrate functional approximation with optimization and reasoning with imperfect data.

The underlying goal is to reduce or eliminate manual intervention in any of these layers, while leveraging CI capabilities at every level. We can manage complexity by finding the best model architecture to support problem decomposition, create high-performance local models with limited competence regions, allow for smooth interpolations among them, and promote robustness to imperfect data by aggregating diverse models. Let us examine some case studies that further illustrate this concept.

3.1 SC Applied to the Development of Offline MH, On-line MH, and Object Models

In Table 1, we list a sample of SC applications developed according to the separation between object- and meta-level. In these applications, the object-level models were based on different technologies such as *Machine Learning* (Support Vector Machines, Random Forest), *Statistics* (Multivariate Adaptive Regression Splines - MARS), Classification Analysis and Regression Trees - CART), Hotelling's T^2), *Neural Networks* (Feed-forward, Self Organizing Maps - SOM), *Fuzzy Systems, Evolutionary Systems, Case-based reasoning*, etc. The on-line Meta-Heuristics was mostly based on fuzzy aggregation of complementary local models or fusion of competing models. The off-line Meta-Heuristics was mostly implemented by evolutionary search in the model design space. A synopsis of these applications can be found in (Bonissone 2013). Detailed descriptions of the applications can be found in the references listed in the last column of Table 1.

4. A Prospective View of SC

Over the past few years, we have experienced an emerging trend favoring the use of model ensembles over individual models. The elements of these ensembles are object-level models, the fusion mechanism is an *online MH's*, and their overall design is still guided by *offline MH's*, as discussed in section 3. This trend is driven by the improved performance obtained by the ensembles. By fusing an ensemble of *diverse* predictive models, we boost the overall prediction accuracy while reducing its variance. Fumera and Roli (2005) confirmed theoretically the claims of Dietterich (2000a). They proved that averaging of classifiers outputs guarantees a better test set performance than the worst classifier of the ensemble. Moreover, under specific hypotheses, such as linear combiners of individual classifiers with unbiased and uncorrelated errors, the fusion of multiple classifiers can *improve the performance of the best individual classifiers*. Under ideal circumstances (e.g., with an infinite number of classifiers) the fusion can provide the optimal Bayes classifier (Tume & Ghosh, 1996).

4.1 Construction of Model Ensembles The ensemble construction requires the creation of *base models*, an *ensemble topology*, and a *fusion mechanism*. Let's briefly review these concepts.

Base Models. Base models are the elements to be fused - the object-level models discussed in the previous sections.

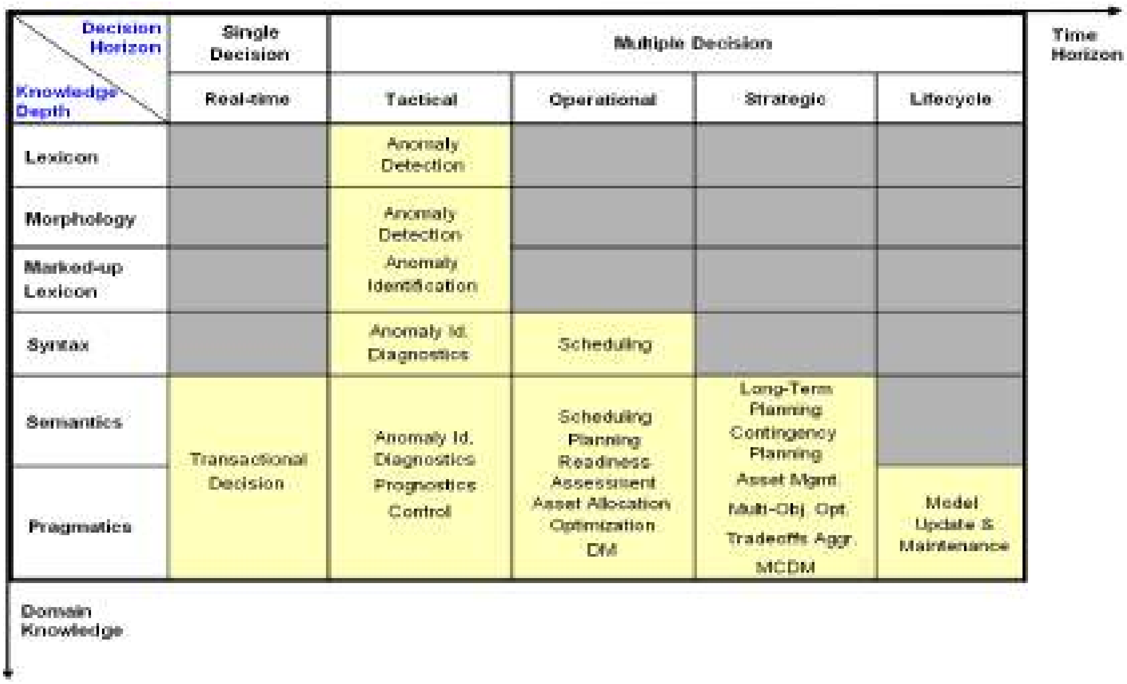

Figure 3. Framework for SC applications - adapted from (Bonissone, 2006).

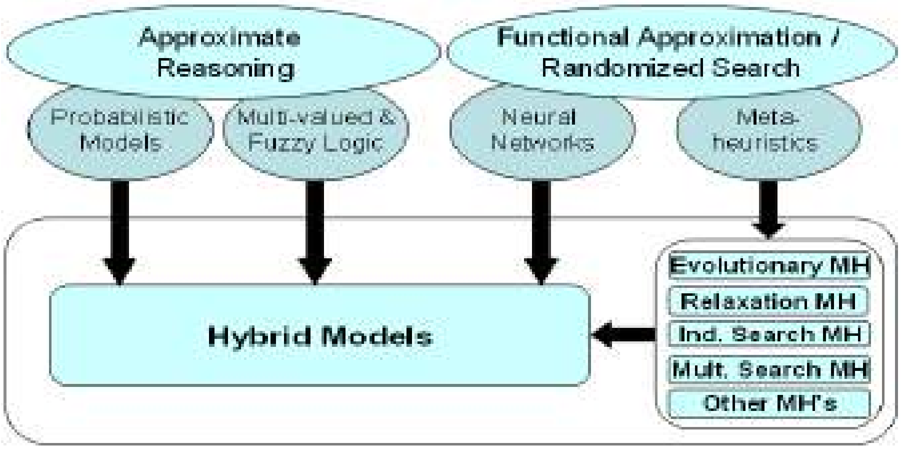

Figure 4. Example of On-Line Meta-Heuristics - adapted from (Verdegay, et al., 2008).

These models need to be *diverse*, e.g. they need to have low error-correlations. They could differ in their parameters, and/or in their structure, and/or in the SC techniques used to create them. Additional information about the process for injecting diversity in their design can be found in (Kuncheva and Whitaker, 2001; Kuncheva, 2004; Bonissone 2013).

Topology. The ensemble can be constructed by following a *parallel* or *serial* topology (or in some cases, a hybrid one). The most common topology is the parallel one, in which multiple models are fed the same inputs and their outputs are merged by the fusion mechanism. In the serial topology the models are applied sequentially (as in the case when we first use a primary model, and in case of it failing to accept a pattern, a secondary model is used to attempt a classification).

Fusion Mechanism (online MH's). These mechanisms can be divided based on two criteria: (a) their type of aggregation; and (b) their dependency on the base model inputs.

The first criterion is concerned with the regions of competence of the base models to be aggregated and subdivides the fusion mechanisms into:

(a1) *Selection* - a mechanism to fuse *disjoint, complementary* models. The base models are trained on disjoint regions of the feature space and, for each pattern only one model is responsible for the final decision. Selection uses a *binary relevance weight* for each complementary model (where all but one of the weights are zero). A typical example of this type of fusion is the use of decision trees, in which the leaf node reached by the input determines the selected model (Breiman et all, 1984).

(a2) *Interpolation* - a mechanism to fuse *overlapping complementary* models. The base models are trained on different, overlapping regions of the feature space and for each state a subset of models is responsible for the final decision. Interpolation determines a *fuzzy relevance weight* for each complementary model. Usually, the weights values are normalized to the [0,1] interval and add up to one. A typical example of this type of fusion is the use of hierarchical fuzzy systems, commonly found in control applications (Jang 1993).

(a3) *Integration* - a mechanism to fuse *competitive* models. All base models are trained on the same feature space and for each input all models contribute to the final decision according to their *relevance weight*. Integration determines the relevance weight for each competitive model. A typical example of this type of fusion is the use of kernel-based weighted models (Atkeson et all, 1997).

The second criterion is concerned with the dependency of the meta-model (fusion) on the inputs to the base models and subdivides the fusion mechanisms into:

(b1) *Static* mechanisms, in which the relevance weights assigned to the base models are determined in *batch mode* (prior to using the models), and are applied uniformly to the outputs of all object-level models. This mechanism is typical of algebraic expressions used to compute relevance weights (Polycar, 2009).

(b2) *Dynamic* mechanisms, in which the relevance weights are determined at *run time*, and their values are a function of the base models inputs, internal states, and outputs. This mechanism is typical of dynamic models used to compute the relevance weights. In this case, fusion is truly performed by meta-models (Bonissone 2011, 2012).

We want to emphasize that the key characteristic for a successful ensemble is ***models diversity***, i.e., the low correlation among the errors of the object-level models: models should be as accurate as possible while avoiding coincident errors. This concept is described in Kuncheva and Whitaker (2001) where the authors propose four *pairwise* and six *non-pairwise* diversity measures to determine the models difference. A complete treatment of this topic can be found in (Kuncheva 2004). Among the many approaches for injecting diversity in the creation of an ensemble of models, we find *bagging* (Breiman 1996), *boosting* (Freud & Schapire, 1997), *random subspace* (Ho 1998), *randomization* (Dietterich, 2000b), and *random forest* (Breiman, 2001).

5. Conclusions

We have described the evolution of Soft Computing along the various phases of its development. In its current

Problem Instance	Problem Type	Model Design (Offline MH's)	Model Controller (Online MH's)	Object-level models	References
Insurance Underwriting: Risk Mgmt.	Classification	EA	Fusion	Multiple Models: NN, Fuzzy, MARS	(Bonissone et al., 2002)
Anomaly Detection (System)	Classification	Model T-norm tuning	Fuzzy Aggregation	Multiple Models: SVM, NN, Case-Based, MARS	(Bonissone et al., 2004)
Anomaly Detection (System)	Classification	Manual design	Fusion	Multiple Models: Kolmogorov Complexity, SOM. Random Forest, Hotteling T2, AANN	(Bonissone & Iyer, 2007)
Anomaly Detection (Model)	Classification & Regression	EA-base tuning of fuzzy supervisory termset	Fuzzy Supervisory	Multiple Models: Ensemble of AANN's	(Bonissone et al., 2009)
Best Units Selection	Ranking	EA-base tuning of similarity function	None	Single Model: Fuzzy Instance Based Models (Lazy Learning)	(Bonissone et al., 2006)
Mortgage Collateral Valuation	Regression	Manual design	Fusion	Multiple Models: ANFIS, Fuzzy CBR, RBF	(Bonissone et al., 1998)
Load, HR, NOx Forecast	Regression	Multiple CART trees	Fusion	Multiple Models: Ensemble of NN's	(Bonissone et al., 2011)
Aircraft Engine Fault Recovery	Control/Fault Accommodation	EA tuning of linear control gains	Crisp supervisory	Multiple Models (Loop): SVM + linear control	(Goebel et al., 2006)
Portfolio Rebalancing	Multi-objective Optimization	Seq. LP	None	Single Model: MOEA (SPEA)	(Subbu et al., 2005)
Power Plant Optimization	Multi-objective Optimization	Manual design	Fusion	Multiple Models (Loop): MOEA + NN's	(Subbu et al., 2006)
Flexible Mfg. Optimization	Optimization	Manual design	Fuzzy supervisory	Single Model: Genetic Algorithms	(Subbu & Bonissone 2003)

Table1 Examples of CI Applications at Meta-level and Object-level

phase, we have proposed a design methodology, based on three levels, to separate design and run-time issues. The first level leverages *offline MH's* to search for the most appropriate models. The design of these models is based on a problem decomposition strategy to manage problem complexity and create manageable components that can be modified and maintained over time. This complexity is resolved by using a hierarchical architecture. The second level addresses the top of the architecture, with *online MH's* that act as a supervisor, fusion, or resource controllers to integrate multiple object-level models. The third level addresses the bottom of the architecture, with (individual or multiple) *object-level models* that focus on the performance/complexity tradeoff. In the case of multiple models, they can be complementary models (requiring selection or interpolation performed by the online MH's) or competing models (requiring aggregation performed by the online MH's). Finally, we analyzed the role of SC within the recent trend of using model ensembles rather than individual models: the element of the ensemble are the object-level models, while the fusion mechanism is the online MH's that selects or aggregates them. The intrinsic ability of SC techniques to be integrated with other sibling techniques, such as Statistics or AI, allows us to leverage SC at all three levels.

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REPORT

Of machines and humans. The art of decision making

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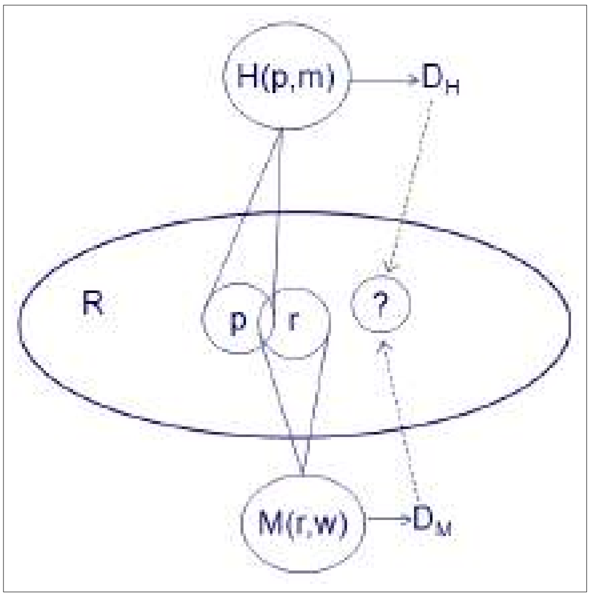
Decision making is the process that leads to select one alternative among a number of them. It is the most frequent activity of human beings, specially at unconscious levels. We decide considering sensorial and memory data, and internal feelings, i.e., considering reality and a system of cost values.

Yet decision making is not simply a very frequent -practically continuos- human task, but it also modifies our state and, consequently, influences our lives. So, to pay attention to how we decide, and even to decision problems and models, is a must.

After the studies of Herbert Simon, two antagonic schools appeared in the field of cognitive psychology. On the one hand, the descriptive school, which supports the theory that we decide in a practically optimal manner, at least at the unconscious level or when we are familiar with the problem to be addressed. This seems to be true when the available information is not complete. On the other hand, the researchers who defend the normative position present many evidences of wrong human decisions, ranging from perceptual and memory to processing mistakes, including logical and probabilistic errors.

At the present time, theories that encompass both perspectives -supported not only by evidences in decision making experiments, but also by neuroscience discoveries- indicate that our brains include two decision making systems -composed by a great variety of elements-, plus a control system. Following the denominations of Keith Stanovich, there are an "autonomous" mind which works according to intuitive processes, and an "algorithmic" mind, which applies logical and statistical principles. The control system is the "reflective" mind, which calls to the "algorithmic" part when not satisfied by the "autonomous" decision. In short, two diverse mechanisms under a selection-based aggregation scheme.

Along the last seven decades, scientists and engineers have conceived, implemented, and applied algorithms that, after a training period considering instances of a given problem and adapting the values of some internal parameters, are able to solve decision problems by processing registered values of relevant variables. It is evident that they use in an efficient manner some kinds of information that are unfamiliar for human beings, and viceversa. It is also clear that they process the data in a purely analytical way; i.e., in a different form that the mode we apply. This means that these "learning machines" are useful as autonomous agents to act in environments that are impossible or dangerous for humans. Yet the learning machines can also serve as decision support tools for experts, or for common people, to suggest them alternative elections. This means that learning machines are a sort of "cognitive tools", and that to adopt them will provide us benefits not less important that those coming from the use of mechanical tools.



A human being perceives (p) reality (R) and takes a decision DH = H(p,m) using memory (m). A learning machine uses registered data (r) in R and its learned parameters (w) to construct DM =M(r,w). These are similar processes, but there is diversity: Inputs and processing modes (H, M) are different.

The social character of our species has produced a new type of intelligence to solve decision problems, the so-called "collective intelligence". Under adequate conditions -the individuals are distributed, i.e., they get different portions of information, and the individuals act independently, i.e., they process the information in different forms-, simple aggregations of individual decisions are better than those individual decisions. These processes appear by themselves if appropriate social rules exist. Consequently, the social diversity is also a source of general benefit.

The same ideas are being applied to the design of learning machines, giving rise to the appearance of machine ensembles. As expectable, diversity and aggregation are the essential aspects of these ensembles. Today, our knowledge of both is extremely limited, and more research is much needed to extend the possibilities of these combined structures. Reformulating the equations of some traditional ensembles can originate new designs, such as using a gate to generate functional weights to combine observed variables starting from the mixture-of-expert architectures. These new schemes can be efficiently trained for decision making, and offer a natural form of aggregating human decisions and, even more, human and machine decisions, without the limitations of other traditionally used aggregation models or other human-machine integration. An important remark: Human and machine diversity is so important, and interaction can be so productive, that this is probably one of the most attractive research lines in the open field of decision making. It is also relevant to explore how different kinds of machine learning diversity can be used to improve machine ensembles and human-machine aggregation. And, finally, it must be remarked that the analysis of the possibilities of cross-fertilization among ensemble design and collective intelligence promotion are also promising.

Decision making is the process that leads to select one alternative among a number of them. It is the most frequent activity of human beings, specially at unconscious levels. We decide considering sensorial and memory data, and internal feelings, i.e., considering reality and a system of cost values.

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At the present time, theories that encompass both perspectives -supported not only by evidences in decision making experiments, but also by neuroscience discoveries- indicate that our brains include two decision making systems -composed by a great variety of elements-, plus a control system. Following the denominations of Keith Stanovich, there are an autonomous mind which works according to intuitive processes, and an algorithmic mind, which applies logical and statistical principles. The control system is the reflective mind, which calls to the algorithmic part when not satisfied by the autonomous decision. In short, two diverse mechanisms under a selection-based aggregation scheme.

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REPORT

Selected Papers from the Second Brazilian Congress on Fuzzy Systems

Regivan Santiago and Benjamín Bedregal, Editors

In 2006 the first step toward the organization of a brazilian research community around the theme: "Fuzzy" was taken. It was done through the organization of the **First Symposium on Applications of Fuzzy Logic** (Primeiro Simpósio de Aplicações de Lógica Fuzzy - SALF - in Portuguese). The idea was the establishment of a meeting with a regional character. The second edition of SALT in 2008 revealed that we were in front of an area with a great demand, since the number of participants overcame the expectations.

On September 8-11th of 2009, the **Brazilian Applied Mathematical Society (SBMAC)** promoted the first Mini-symposium on the "Foundations and Applications of Fuzzy Logic" as a satellite event of the National Congress on Applied Mathematics, which was held in Cuiab/'a - MT. At that time about 2.000 brazilian CNPq Lattes curricula involving the keyword "Fuzzy" were found. With this scenario, the **Thematic Committee on Fuzzy Systems** was created in that society, which induced the organization of an event with a national character around the theme "Fuzzy". This event was called: **First Brazilian Congress on Fuzzy Systems** (I Congresso Brasileiro em Sistemas Fuzzy - I CBSF; in Portuguese), substituting the III SALF which was already been planned.

The I CBSF expanded the scope of SALF, which focused mostly on applications. The word "System" assumed a broad sense, in order to capture both the investigation on Computational and Logical systems. With this scenario, the congress became the first event in Latin America on this theme and was supported by the following scientific societies:

- Brazilian Society of Automatica (SBA),
- Brazilian Society of Computational Intelligence (SBIC)
- Brazilian Society on Applied and Computational Mathematics (SBMAC).
- International Fuzzy Systems Association - IFSA
- North American Fuzzy Information Processing Society (NAFIPS)

Within this context from 06th to 09th november 2012 the II CBSF was held in Natal-RN, organized by the Federal University of Rio Grande do Norte and also with the support of two more scientific societies:

- European Society for Fuzzy Logic and Technology (EUSFLAT)
- Brazilian Society of Computation (SBC)

The congress was organized in 7 plenary sessions, 3 minicourses and 21 technical sessions. The topics of the meeting were divided into two categories Theoretical and Applied aspects of Fuzzy Systems.

The proceedings was published by the **Brazilian Society for Applied and Computational Mathematics (SBMAC)** and can be downloaded from `http://www.dimap.ufrn.br/~cbsf/pub/anais/2012/CBSF-Proceedings.pdf`. The auhors of the best papers were invited to submit a revised version to the present issue of **Mathware & Soft Computing Magazine (MSCM)**, they are:

- *Brouwerian Autoassociative Morphological Memories and Their Relation to Traditional and Sparsely Connected Autoassociative Morphological Memories.*

  This paper introduces the notion of Brouwerian autoassociative morphological memories (BAMMs) defined on a complete Brouwerian lattice. The autoassociative fuzzy implicative memory of Gödel is an example of BAMM. Some theoretical results concerning the storage capacity, noise tolerance, and a characterization of the patterns recalled by novel memories are given.

- *Training strategies for a fuzzy CBR cluster-based approach.*

  It presents an approach to Case-Based Reasoning which is founded on fuzzy relations and residuated implications operators. The authors propose the creation of clusters by using fuzzy gradual rules. The application of Fuzzy ART neural network to create such clusters is investigated.

- *Projections of solutions for fuzzy differential equations.*

  In this paper the authors propose the concept of fuzzy projections on subspaces of F(Rn) obtained from Zadeh's extension of canonical projections in Rn. Some of the main properties of such projections are investigated and some properties of fuzzy projection solutions of fuzzy differential equations are reviewed.

- *FuzzyDT- A Fuzzy Decision Tree Algorithm Based on C4.5.*

  Decision trees are popular models in machine learning due to the fact that they produce graphical models, as well as text rules, that end users can easily understand. Their combination with Fuzzy paradigm has produced Fuzzy Decision Tree Models. In this paper, the authors expand previous experiments and present more details of the FuzzyDT algorithm, a fuzzy decision tree based on the classic C4.5 decision tree algorithm.

- *Performance Analysis of Evolving Fuzzy Neural Networks for Pattern Recognition.*

  The Evolving Fuzzy Neural Networks (EFuNNs), recently proposed by Kasabov are dynamic connectionist feed forward networks with five layers of neurons and they are adaptive rule-based systems. This work assesses the accuracy of EFuNNs) for pattern recognition tasks using seven different statistical distributions data. Results of assessment are provided and diferent accuracy according to the statistical distribution of data are shown.

- *Fuzzy Differential Equations with Arithmetic and Derivative via Zadeh's Extension.*

  The authors propose the notion of derivatives for fuzzy functions obtained via Zadeh's extension of the classical derivative operator and implement it in dynamical systems. Particularly, solutions to fuzzy initial value problems (FIVPs) that preserve the main properties and characteristics of functions of the base space, as periodicity and stability are found.

- *A Total Order for Symmetric Triangular (Interval) Fuzzy Numbers.*

  Interval-valued fuzzy numbers are specified by interval-valued membership functions. In the modeling of some applications (e.g, decision making, game theory), it is enough to consider symmetric triangular interval fuzzy numbers. In this paper the authors introduce a total order for symmetric triangular (interval) fuzzy numbers.

- *Fuzzy Computing from Quantum Computing - Case Study in Reichenbach Implication Class.*

  The authors show that quantum computing extends the class of fuzzy sets, taking advantage of properties such as quantum parallelism. The central idea lies in the association of quantum states with membership functions of fuzzy subsets and the rules for the processes of fuzzyfication are performed by unitary quantum transformations.

- *Actions of Automorphisms on Some Classes of Fuzzy Bi-implications*

  In a previous paper the authors have studied two classes of fuzzy bi-implications based on t-norms and r-implications, and shown that they constitute increasingly weaker subclasses of the Fodor-Roubens bi-implication. Now they prove that each of these three classes of bi- implications is closed under automorphisms.

We hope that this issue could make the reader to percieve a little of what is made in Brazil in terms of Fuzzy Systems. The next meeting is programmed to 2014 in João Pessoa-PB, we hope to see you there.

REPORT

Brouwerian Autoassociative Morphological Memories and Their Relation to Traditional and Sparsely Connected Autoassociative Morphological Memories

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Autoassociative morphological memories (AMM) are memory models that use operations of mathematical morphology for the storage and recall of pattern associations. These models can be very well defined in a mathematical structure called complete lattice. In this paper, we introduce the Brouwerian autoassociative morphological memories (BAMMs) that are defined on a complete Brouwerian lattice. The autoassociative fuzzy implicative memory of Gödel is an example of BAMM. The sparsely connected autoassociative morphological memory is also an example of BAMM. Some theoretical results concerning the storage capacity, noise tolerance, and a characterization of the patterns recalled by the novel memories are given in this paper.

1. Introduction

Mathematical morphology (MM) is a theory used in the processing and analysis of images or objects [1, 2]. It have been developed initially in the 1960s by Matheron and Serra as a set of tools to extract geometrical information in the study of binary images. In the 1980s, several approaches have been proposed to extend MM from binary to gray-scale images. The umbra approach, developed by Sternberg, consists in applying the elementary operations of binary MM on the set of points on and below the graph of a gray-scale image [2]. An approach based on the concept of threshold or level-sets have been proposed by Serra [1, 3]. From a theoretical viewpoint, the binary as well as the two aforementioned gray-scale approaches to MM can be very well conducted in a mathematical structure called complete lattice [3]. Specifically, the umbra approach is defined on a complete lattice-ordered group extension which is obtained by endowing a complete lattice with a group operation [4]. The level set approach can be defined in a complete Brouwerian lattice [5].

Besides the many applications in image processing, MM have been successfully used for the development of lattice computing models including neural networks [6, 7, 8]. Neural networks are mathematical models inspired by the human brain. An essential attribute of the brain, called associative memory, refers to the ability to recall information by association [9, 10].

The first autoassociative memory (AM) models based on MM, referred to as the traditional autoassociative morphological memories (TAMMs), were introduced by Ritter and Sussner in the 1990s [8]. In general terms, TAMMs are very similar to the correlation-based linear associative memory given that the usual matrix multiplication is replaced by lattice-based operations [9, 10]. Nevertheless, TAMMs are non-linear models that exhibit attractive features including optimal absolute storage capacity and convergence in a single iteration when employed with feedback [8]. TAMMs have been applied successfully for the reconstruction of corrupted or incomplete patterns, analysis of hyperspectral images, classification, and times series prediction [11, 7, 12, 13]. On the downside, TAMMs are computationally too expensive for dealing with multivalued, large-scale patterns because their weight matrices are fully connected.

In contrast to TAMMs, the sparsely connected autoassociative morphological memories (SCAMMs) are attractive models for the storage and recall of large-scale patterns because they require less computational resources than many other memory models [14, 15]. In fact, the recording phase of SCAMMs require only pn^2 comparisons, where p is the number of fundamental patterns and n represents their length. Also, the retrieval phase requires only calculations of maxima and minima. Notwithstanding, SCAMMs share many attractive properties with TAMMs, including optimal absolute storage capacity.

In the context of MM, the neurons of a TAMM perform an elementary operation of the umbra approach to gray-scale MM. Similarly, the neurons of SCAMMs perform elementary operations of the threshold approach. In this paper, we introduce the class of Brouwerian autoassociative morphological memories (BAMMs) which are endowed with neurons that perform operations of the level-set approach. From an algebraic point of view, BAMMs are more complex than SCAMMs because the complete lattice must be endowed with a relative pseudo-complement operation. However, since a group operation is not required, BAMMs are simpler than TAMMs.

The paper is organized as follows. Section 2 provides the mathematical background and presents the most important approaches to gray-scale MM. TAMMs and SCAMMs are briefly reviewed in Sections 3 and 4, respectively. Besides introducing BAMMs, Section 5 contains some theoretical results concerning the storage capacity and noise tolerance of these memories. We also establish a relationship between BAMMs and SCAMMs in this section. The paper finishes with the concluding remarks in Section 6.

2. A Brief Review on Mathematical Morphology

Mathematical morphology (MM) can be very well conducted in a mathematical structure called complete lattice [3]. A partially ordered set $\mathbb{L}$ is a complete lattice if every subset $X \subseteq \mathbb{L}$ has a supremum $\bigvee X$ and an infimum $\bigwedge X$ [5]. If $X = \{x_1, \ldots, x_n\}$ is a finite subset of $\mathbb{L}$, we also denote $\bigvee X$ and $\bigwedge X$ by $\bigvee_{i=1}^{n} x_i = x_1 \vee \ldots \vee x_n$ and $\bigwedge_{i=1}^{n} x_i = x_1 \wedge \ldots \wedge x_n$, respectively.

A complete lattice $\mathbb{B}$ is Brouwerian if and only if the equation

$$a \wedge \left(\bigvee X\right) = \bigvee_{x \in X} (a \wedge x), \tag{1}$$

holds true for any $a \in \mathbb{B}$ and $X \subseteq \mathbb{B}$ [5]. In this case, for any given elements $a, b \in \mathbb{B}$, the set $S = \{x \in \mathbb{B} : a \wedge x \leq b\}$ has a maximal element, called relative pseudo-complement of a in b and denoted by b/a. The infimum (or meet operation) and the relative pseudo-complement satisfy the following equivalence for $a, b, x \in \mathbb{B}$:

$$a \wedge x \leq b \iff x \leq b/a. \tag{2}$$

Moreover, the following equalities hold true for any $x \in \mathbb{B}$ where $\top = \bigvee \mathbb{B}$ and $\bot = \bigwedge \mathbb{B}$ denote respectively the largest and the least elements of $\mathbb{B}$:

$$\top/x = \top, \quad \bot/x = \bot, \quad x/\top = x, \quad \text{and} \quad x/\bot = \top. \tag{3}$$

In addition, we have

$$a \wedge (b/a) = a, \quad \forall\, a, b \in \mathbb{B}. \tag{4}$$

Any totally ordered complete lattice is an example of complete Brouwerian lattice [5]. Hence, the extended real numbers $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ and the extended integers $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$ represent complete Brouwerian lattices. Brouwerian lattices have been studied by Brouwer and Heyting as a generalization of the Boolean algebra. They also constitute a fruitful modification of the two-valued logic in which the validity of proof by contraction is not assumed [5].

A complete lattice $\mathbb{G}$ endowed with two order-preserving binary operations, denoted by the symbols "$+$" and "$+'$", is called a complete lattice-ordered group extension if "$+$" and "$+'$" coincide and form a group on the set of finite elements $\mathbb{F} = \mathbb{G} \setminus \{\top, \bot\}$ [4]. Furthermore, these two binary operations are extended as follows for the infinities:

$$\bot + \top = \top + \bot = \bot \quad \text{and} \quad \bot +' \top = \top +' \bot = \top. \tag{5}$$

In this paper, we refer to a complete lattice-ordered group extension as a cloge. Both $\bar{\mathbb{R}}$ and $\bar{\mathbb{Z}}$ represent cloges with the usual addition.

Erosion and dilation are two elementary operations of MM [1]. Any other operation in MM is derived by combining the elementary operations [16]. In the context of complete lattices, an erosion is any operator that commutes with the infimum. Dually, an operator that commutes with the supremum is a dilation [3, 17]. Formally, given two complete lattices $\mathbb{L}$ and $\mathbb{M}$, the operators $\delta : \mathbb{L} \to \mathbb{M}$ and $\varepsilon : \mathbb{M} \to \mathbb{L}$ represent respectively a dilation and an erosion if the following hold for any $X \subseteq \mathbb{L}$ and $Y \subseteq \mathbb{M}$:

$$\delta\Big(\bigvee X\Big) = \bigvee_{x \in X} \delta(x) \quad \text{and} \quad \varepsilon\Big(\bigwedge Y\Big) = \bigwedge_{y \in Y} \varepsilon(y). \tag{6}$$

We say that two operators $\delta : \mathbb{L} \to \mathbb{M}$ and $\varepsilon : \mathbb{M} \to \mathbb{L}$ form an adjunction (ε, δ) if the following equivalence holds true for $x \in \mathbb{M}$ and $y \in \mathbb{L}$:

$$\delta(x) \leq y \iff x \leq \varepsilon(y). \tag{7}$$

Besides being close related to the concepts of Galois connection and residuum of an operator [18], the notion of adjunction plays an important role in MM [3, 17]. In particular, if (ε, δ) forms an adjunction, then δ is a dilation and ε is an erosion.

Example 1 (Binary MM) *The binary approach toward MM is defined on the power set $\mathcal{P}(X)$, where X denotes the Euclidean space $\mathbb{R}^d$ or the digital space $\mathbb{Z}^d$, ordered by inclusion. Given an arbitrary but fixed set $S \in \mathcal{P}(X)$, called structuring element, the binary erosion $\varepsilon_S^{\mathcal{B}}(A)$ and the binary dilation $\delta_S^{\mathcal{B}}(A)$ of an image $A \in \mathcal{P}(X)$ by S are given by*

$$\varepsilon_S^{\mathcal{B}}(A) = \bigcap_{x \in \bar{S}} A_x \quad \text{and} \quad \delta_S^{\mathcal{B}}(A) = \bigcup_{x \in S} A_x, \tag{8}$$

where $\bar{S} = \{-x : x \in S\}$ and $S_x = \{y + x : y \in S\}$ denote respectively the reflection of S around the origin and the translation of S by $x \in X$.

Broadly speaking, a gray-scale image corresponds to a function $f : X \to \mathbb{V}$, where the set of values $\mathbb{V}$ usually represents a totally ordered complete lattice such as $\bar{\mathbb{R}}$ or $\bar{\mathbb{Z}}$ [19]. The following examples reviews briefly three approaches toward gray-scale MM. For details, we refer the reader to [3, 20].

Example 2 (Umbra Approach) *Let $\mathbb{G}$ denote a cloge such as $\bar{\mathbb{R}}$ or $\bar{\mathbb{Z}}$ endowed with the usual addition. The umbra of a gray-scale image $f : X \to \mathbb{G}$ is the set $U = \{(x,t) : t \leq f(x), x \in X\}$. Conversely, the top of an umbra U is the gray-scale image given by $f(x) = \bigvee\{t \in \mathbb{G} : (x,t) \in U\}, \quad \forall\, x \in X$. The umbra approach to gray-scale MM, introduced by Sternberg [2], is derived by taking the top of the set obtained by applying the elementary operations of binary MM on the umbras of the image and structuring element. Equivalently, the elementary operations of the umbra approach to MM can be defined as follows: Given a gray-scale image f and a fixed structuring element $s : X \to \mathbb{G}$, the umbra erosion and the umbra dilation of f by s are the images $\varepsilon_s^{\mathcal{U}}(f)$ and $\delta_s^{\mathcal{U}}(f)$ given respectively by the following equations for any $x \in X$:*

$$\begin{aligned} \varepsilon_s^{\mathcal{U}}(f)(x) &= \bigwedge_{y \in X} f(y) +' \big(\neg s(y - x)\big) \quad \text{and} \\ \delta_s^{\mathcal{U}}(f)(x) &= \bigvee_{y \in X} f(y) + s(x - y). \end{aligned} \tag{9}$$

The notation $\neg x$ represents the conjugate of an element $x \in \mathbb{G}$ and is defined as follows:

$$\neg x = \begin{cases} \bot, & \text{if } x = \top, \\ \top, & \text{if } x = \bot, \\ -x, & \text{otherwise.} \end{cases} \tag{10}$$

Example 3 (Level Set Approach) *Consider a complete Brouwerian lattice $\mathbb{B}$. A gray-scale image $f : X \to \mathbb{B}$ can be decomposed into a family of level sets $\{F_t\}_{t\in\mathbb{B}}$, where each $F_t = \{x \in X : f(x) \geq t\} \subseteq X$. Conversely, a family of level sets $\{F_t\}_{t\in\mathbb{B}}$ yields a gray-scale image by means of the equation $f(x) = \bigvee\{t \in \mathbb{B} : x \in F_t\}, \quad \forall x \in X$. Hence, the level set erosion $\varepsilon_s^{\mathcal{L}}(f)$ and the level set dilation $\delta_s^{\mathcal{L}}(f)$ of an image f by an structuring element s are defined as the gray-scale image obtained from the family of level sets $\{\varepsilon_{S_t}^{\mathcal{B}}(F_t)\}_{t\in\mathbb{B}}$ and $\{\delta_{S_t}^{\mathcal{B}}(F_t)\}_{t\in\mathbb{B}}$, where F_t and S_t denote respectively the level sets of f and s. Alternatively, $\varepsilon_s^{\mathcal{L}}(f)(x)$ and $\delta_s^{\mathcal{L}}(f)(x)$ are given by the following equations for any $x \in X$, where $s(y-x)/f(y)$ is the relative pseudo-complement of $f(y)$ in $s(y-x)$:*

$$\begin{aligned}\varepsilon_s^{\mathcal{L}}(f)(x) &= \bigwedge_{y\in X} f(y)/s(y-x) \quad \textit{and}\\ \delta_s^{\mathcal{L}}(f)(x) &= \bigvee_{y\in X} f(y)\wedge s(x-y),\end{aligned} \tag{11}$$

Example 4 (Threshold or Flat Approach) *In contrast to the last two examples, the threshold approach probes a gray-scale image $f : X \to \mathbb{V}$ by a binary structuring element $S \subseteq X$. Precisely, the threshold erosion $\varepsilon_S^{\mathcal{T}}(f)$ and the threshold dilation $\delta_S^{\mathcal{T}}(f)$ of f by S are given respectively by the following equations for any $x \in X$:*

$$\varepsilon_S^{\mathcal{T}}(f)(x) = \bigwedge_{y\in S_x} f(y) \quad \textit{and} \quad \delta_S^{\mathcal{T}}(f)(x) = \bigvee_{y\in \bar{S}_x} f(y). \tag{12}$$

Note that the threshold approach to gray-scale MM requires only a complete lattice structure of the set of values $\mathbb{V}$. Moreover, it can be viewed as a particular instance of the level set and umbra approaches. Precisely, (12) can be derived from (9) and (11) by considering respectively the structuring elements $s_{\mathcal{U}}, s_{\mathcal{L}} : X \to \mathbb{V}$, where $\mathbb{V}$ stands either a cloge $\mathbb{G}$ or a complete Brouwerian lattice $\mathbb{B}$, defined as follows for all $x \in X$:

$$s_{\mathcal{U}}(x) = \begin{cases} 0, & x \in S,\\ \bot, & x \notin S,\end{cases} \qquad \textit{and} \qquad s_{\mathcal{L}}(x) = \begin{cases} \top, & x \in S,\\ \bot, & x \notin S.\end{cases} \tag{13}$$

3. Traditional Morphological Autoassociative Memories

The first autoassociative morphological memories, referred in this paper as the traditional autoassociative morphological memories (TAMMs), were introduced by Ritter and Sussner in the 1990s [8, 21]. In a broad sense, an autoassociative morphological memory is defined as a type of associative memory that performs an elementary operation of MM at every node, possibly followed by the application of an activation function [4]. In particular, the nodes of a TAMM perform either dilations or erosions inspired by the umbra approach to gray-scale MM. Precisely, these memories are defined as follows.

Let $\mathbb{G}$ denote a cloge and consider an autoassociative fundamental memory set $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$, where each pattern $\mathbf{x}^\xi = [x_1^\xi, \ldots, x_n^\xi]^T$ corresponds to a column vector in $\mathbb{G}^n$. First, the fundamental memories are used to synthesize the synaptic weight matrix $W^{\mathcal{U}} = [w_{ij}^{\mathcal{U}}] \in \mathbb{G}^{n\times n}$ as follows for all $i, j = 1, \ldots, n$:

$$w_{ij}^{\mathcal{U}} = \bigwedge_{\xi=1}^{p} \left(x_i^\xi +' (\neg x_j^\xi)\right). \tag{14}$$

Then, given an input $\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{G}^n$, the patterns $\mathbf{y} = [y_1, \ldots, y_n]^T \in \mathbb{G}^n$ and $\mathbf{z} = [z_1, \ldots, z_n]^T \in \mathbb{G}^n$ recalled respectively by the TAMMs $\mathcal{W}^{\mathcal{U}}$ and $\mathcal{M}^{\mathcal{U}}$ are given by the following equations for all $i = 1, \ldots, n$:

$$y_i = \bigvee_{j=1}^{n} \left(x_j + w_{ij}^{\mathcal{U}}\right) \quad \text{and} \quad z_i = \bigwedge_{j=1}^{n} \left(x_j +' (\neg w_{ji}^{\mathcal{U}})\right). \tag{15}$$

Note the similarity between (15) and (9) by identifying the component x_j with the value $f(y)$ and the entries $w_{ij}^{\mathcal{U}}$ with the values of the structuring element s. Such as the umbra approach in Example 2, TAMMs require a cloge structure of $\mathbb{G}$.

In view of the lattice-based operations, TAMMs differ drastically from "classical" associative memory models such as the famous Hopfield network and its generalizations [10, 22]. In particular, we can list the following properties of the TAMMs $\mathcal{W}^{\mathcal{U}}$ and $\mathcal{M}^{\mathcal{U}}$ [8, 11]:

1. Both TAMMs $\mathcal{W}^{\mathcal{U}}$ and $\mathcal{M}^{\mathcal{U}}$ exhibit optimal absolute storage capacity. Thus, one can store as many patterns as desired in these memories.

2. Every output remains stable under repeated application of the TAMMs. In other words, they exhibit one-step convergence.

3. The pattern recalled by $\mathcal{W}^{\mathcal{U}}$ represents the smallest fixed point of the memory greater than or equal to the input $\mathbf{x}$. Hence, this TAMM recalls an original pattern $\mathbf{x}^\xi$ exactly only if $x_i \leq x_i^\xi$ for every $i = 1, \ldots, n$.

   Dually, the $\mathcal{M}^{\mathcal{U}}$ yields the greatest fixed point smaller than or equal to the input pattern. Consequently, an original pattern $\mathbf{x}^\xi$ is recalled correctly by this memory model only if $x_i^\xi \leq x_i$ for all $i = 1, \ldots, n$.

4. The set of fixed points of the TAMMs include the original patterns in the fundamental memory set as well as a large number of spurious patterns. Recall that a spurious pattern is a memory that was unintentionally stored in the model.

Let us conclude this section by recalling that the TAMMs $\mathcal{W}^{\mathcal{U}}$ and $\mathcal{M}^{\mathcal{U}}$ are isomorphic to a class of fuzzy morphological associative memories (FMAMs) [23]. Furthermore, $\mathcal{W}^{\mathcal{U}}$ is close related to the Lukasiewicz autoassociative fuzzy implicative memory, which also belongs to the broad class of FMAMs [24, 25].

4. Sparsely Connected Morphological Autoassociative Memories

In contrast to TAMMs, sparsely connected morphological autoassociative memories (SCAMMs) [14], also known as sparsely connected lattice autoassociative memories [15], are defined on an arbitrary complete lattice and do not require a group operation apart from the lattice operations. Specifically, let $\mathbb{L}$ denote a complete lattice.

Given a fundamental memory set $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\} \subseteq \mathbb{L}^n$, we synthesize the set

$$\mathcal{S} = \{(i,j) : x_j^\xi \leq x_i^\xi, \forall \xi = 1, \ldots, p\}, \quad (16)$$

referred to as the set of synaptic junctions[3].

After the presentation of an input pattern $\mathbf{x}$, the supremum and infimum operations are used to define the patterns $\mathbf{y}$ and $\mathbf{z}$ recalled respectively by the SCAMMs $\mathcal{W}^{\mathcal{T}}$ and $\mathcal{M}^{\mathcal{T}}$ as follows:

$$y_i = \bigvee_{(i,j)\in\mathcal{S}} x_j \quad \text{and} \quad z_i = \bigwedge_{(j,i)\in\mathcal{S}} x_j, \quad \forall\, i = 1, \ldots, n. \quad (17)$$

These equations reveal that the SCAMMs are equipped with nodes that perform morphological operations inspired by the threshold approach. Indeed, we can identify the set of synaptic junctions $\mathcal{S}$ in (17) with the structuring element in (12).

As expected from Example 4, the SCAMMs arise from the TAMMs via application of a threshold operation analogous to the ones given by (13). As pointed out in [15], this remark explains why $\mathcal{W}^{\mathcal{U}}$ and $\mathcal{M}^{\mathcal{U}}$ exhibit a better error correction capability than $\mathcal{W}^{\mathcal{T}}$ and $\mathcal{M}^{\mathcal{T}}$ in certain applications concerning the storage and recall of gray-scale images [14]. Nevertheless, the TAMM models, whose weight matrices are fully connected, are computationally too expensive for dealing with large-scale patterns. In contrast, the construction of the set of synaptic junctions uses only pn^2 comparisons, where n is the length of the patterns and p is the number of fundamental memories. Also, the cardinality of $\mathcal{S}$ is usually much less than n^2, the number of synaptic weights of a fully connected network. Despite the low computational effort, SCAMMs share with TAMMs the five properties listed in Section 3. Therefore, they are attractive models for storing and reconstructing multivalued, large-scale patterns such as color images [14].

5. Brouwerian Autoassociative Morphological Memories

In the preceding sections, we pointed out that TAMMs are close related to the umbra approach while the SCAMMs resemble the threshold approach. In this section, we introduce a class of autoassociative morphological memories whose neurons perform either dilations or erosions similar to the level-set approach to gray-scale MM. The novel memories are referred to as Brouwerian autoassociative morphological memories (BAMMs) in view of the mathematical structure used to define them.

Let $\mathbb{B}$ be a complete Browerian lattice and consider a fundamental memory set $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\} \subseteq \mathbb{B}^n$. Since the number of synaptic weights of a fully connected network may grow impractical in applications with large n, let us permit BAMM models with a sparse structure. To this end, a network architecture is defined as a subset $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$, where $\mathcal{N} = \{1, 2, \ldots, n\}$. The i-th output neuron is connected to the j-th input node if and only if $(i,j) \in \mathcal{A}$. As a consequence, we only use the synaptic weights $w_{ij}^{\mathcal{L}}$ such that $(i,j) \in \mathcal{A}$. Moreover, these weights are computed as follows for all $(i,j) \in \mathcal{A}$, where the symbol "/" denotes the relative pseudo-complement operation:

$$w_{ij}^{\mathcal{L}} = \bigwedge_{\xi=1}^{p} (x_i^\xi / x_j^\xi). \quad (18)$$

In contrast to TAMMs and SCAMMs, BAMMs are recurrent networks that feedback upon themselves using a synchronously update mode. In other words, the retrieval phase of a BAMM is described by a dynamic rule such as the discrete-time Hopfield network [22]. Formally, given an input pattern $\mathbf{x} \in \mathbb{B}^n$, we first set $\mathbf{y}(0) = \mathbf{z}(0) = \mathbf{x}$ and define the sequences $\{\mathbf{y}(t)\}_t$ and $\{\mathbf{z}(t)\}_t$ recursively as follows for any positive integer t and $i = 1, \ldots, n$:

$$\begin{aligned} y_i(t+1) &= \bigvee_{(i,j)\in\mathcal{A}} \left(y_j(t) \wedge w_{ij}^{\mathcal{L}}\right) \quad \text{and} \\ z_i(t+1) &= \bigwedge_{(j,i)\in\mathcal{A}} \left(z_j(t) / w_{ji}^{\mathcal{L}}\right). \end{aligned} \quad (19)$$

Then, the t-step BAMMs $\mathcal{W}_t^{\mathcal{L}}, \mathcal{M}_t^{\mathcal{L}} : \mathbb{B}^n \to \mathbb{B}^n$ are the mappings given by

$$\mathcal{W}_t^{\mathcal{L}}(\mathbf{x}) = \mathbf{y}(t) \quad \text{and} \quad \mathcal{M}_t^{\mathcal{L}}(\mathbf{x}) = \mathbf{z}(t). \quad (20)$$

Note that $\mathbf{y}(1)$ and $\mathbf{z}(1)$ correspond respectively to the patterns recalled by the single-step memories $\mathcal{W}_1^{\mathcal{L}}$ and $\mathcal{M}_1^{\mathcal{L}}$ evaluated at the input pattern $\mathbf{x}$. Also, we have

$$\begin{aligned} \mathcal{M}_{t+1}^{\mathcal{L}}(\mathbf{x}) &= \mathcal{W}_1^{\mathcal{L}}\left(\mathcal{W}_t^{\mathcal{L}}(\mathbf{x})\right) \quad \text{and} \\ \mathcal{M}_{t+1}^{\mathcal{L}}(\mathbf{x}) &= \mathcal{M}_1^{\mathcal{L}}\left(\mathcal{M}_t^{\mathcal{L}}(\mathbf{x})\right), \end{aligned} \quad (21)$$

for any positive integer t and $\mathbf{x} \in \mathbb{B}^n$. The following theorem, whose proof have been omitted due to page constraints but follows almost immediately from (2), asserts that $\mathcal{M}_1^{\mathcal{L}}$ and $\mathcal{W}_1^{\mathcal{L}}$ form an adjunction in $\mathbb{B}^n$.

Theorem 1 *For any $\mathbf{u}, \mathbf{v} \in \mathbb{B}^n$, we have $\mathcal{W}_1^{\mathcal{L}}(\mathbf{u}) \leq \mathbf{v}$ if and only if $\mathbf{u} \leq \mathcal{M}_1^{\mathcal{L}}(\mathbf{v})$.*

Proof. Given $\mathbf{u}, \mathbf{v} \in \mathbb{B}^n$, the equivalences below follow from the definition of the supremum and infimum operations as well as the adjunction relationship given by (2):

$$\begin{aligned} \mathcal{W}_1^{\mathcal{L}}(\mathbf{u}) \leq \mathbf{v} &\iff \left[\mathcal{W}_1^{\mathcal{L}}(\mathbf{u})\right]_i \leq v_i, \forall i \in \mathcal{N} \\ &\iff \bigvee_{(i,j)\in\mathcal{A}} \left(u_j \wedge w_{ij}^{\mathcal{L}}\right) \leq v_i, \forall i \\ &\iff \left(u_j \wedge w_{ij}^{\mathcal{L}}\right) \leq v_i, \forall (i,j) \in \mathcal{A} \\ &\iff u_j \leq \left(v_i / w_{ij}^{\mathcal{L}}\right), \forall (i,j) \in \mathcal{A} \\ &\iff u_j \leq \bigwedge_{(i,j)\in\mathcal{A}} \left(v_i / w_{ij}^{\mathcal{L}}\right), \forall j \in \mathcal{N} \\ &\iff \mathbf{u} \leq \mathcal{M}_1^{\mathcal{L}}(\mathbf{v}). \square \end{aligned}$$

Theorem 1 confirms that $\mathcal{M}_1^{\mathcal{L}}$ is an erosion while $\mathcal{W}_1^{\mathcal{L}}$ is a dilation. As a consequence, BAMMs indeed belong to the class of morphological neural networks [4]. The following example shows that some FMAM models belong to the class of BAMMs.

[3] We would like to point out that, in order to simplify our presentation, the set $\mathcal{S}$ given by (16) differs slight from the set of synaptic junctions defined in our previous paper [15].

Example 5 *Note that the fuzzy implication of Gödel corresponds to the relative pseudo-complement in the complete Brouwerian lattice* $[0,1]$. *Thus, the autoassociative fuzzy implicative memory (AFIM) of Gödel is an example of the single-step BAMM* $\mathcal{W}_1^{\mathcal{L}}$ *with the fully connected architecture* $\mathcal{A} = \mathcal{N} \times \mathcal{N}$ *[24]. Dually, the BAMM* $\mathcal{M}_1^{\mathcal{L}}$ *corresponds to the FMAM adjoint of the Gödel AFIM [25].*

Recall that the Gödel AFIM exhibit one-step convergence. In other words, $\mathbf{y}(t) = \mathbf{y}(1)$ *for any input pattern* $\mathbf{y}(0)$ *and positive integer* t. *Hence, the sequence* $\{\mathbf{y}(t)\}_t$ *produced recursively by the Gödel AFIM is convergent.*

In general, we define the $*$-step BAMM model as the mapping $\mathcal{W}_*^{\mathcal{L}} : \mathbb{B}^n \to \mathbb{B}^n$ that yields the limit $\lim_t \mathbf{y}(t)$ whenever the sequence $\{\mathbf{y}(t)\}_t$ converges for any $\mathbf{y}(0) \in \mathbb{B}^n$. Dually, if $\{\mathbf{z}(t)\}_t$ converges for all $\mathbf{x} = \mathbf{z}(0) \in \mathbb{B}^n$, the $*$-step BAMM $\mathcal{M}_*^{\mathcal{L}} : \mathbb{B}^n \to \mathbb{B}^n$ is given by the equation $\mathcal{M}_*^{\mathcal{L}}(\mathbf{x}) = \lim_t \mathbf{z}(t)$. We would like to point out that the convergence of the sequences $\{\mathbf{y}(t)\}_t$ and $\{\mathbf{z}(t)\}_t$ depend on the network architecture $\mathcal{A}$ of the BAMMs.

The following theorem provides a sufficient condition for the existence of the limits $\mathbf{y}_* = \lim_t \mathbf{y}(t)$ and $\mathbf{z}_* = \lim_t \mathbf{z}(t)$. Furthermore, it shows that both $\mathbf{z}_*$ and $\mathbf{y}_*$ belong to the set $\mathcal{F}$ of all fixed points of the single-step memories $\mathcal{W}_1^{\mathcal{L}}$ and $\mathcal{M}_1^{\mathcal{L}}$:

$$\mathcal{F} = \left\{\mathbf{v} \in \mathbb{B}^n : \mathcal{M}_1^{\mathcal{L}}(\mathbf{v}) = \mathbf{v}\right\} = \left\{\mathbf{u} \in \mathbb{B}^n : \mathcal{W}_1^{\mathcal{L}}(\mathbf{u}) = \mathbf{u}\right\}. \tag{22}$$

Theorem 2 *Given a fundamental memory set* $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\} \subseteq \mathbb{B}^n$, *define* $W^{\mathcal{L}}$ *by means of* (18) *using a network architecture* $\mathcal{A}$ *such that* $(i,i) \in \mathcal{A}$ *for all* $i \in \mathcal{N}$. *Then, the BAMMs* $\mathcal{W}_*^{\mathcal{L}}$ *and* $\mathcal{M}_*^{\mathcal{L}}$ *given by the limit of the sequences in* (19) *are well defined and satisfy the following for any input pattern* $\mathbf{x} \in \mathbb{B}^n$ *and for all* $t \geq 1$:

$$\bigvee\{\mathbf{v} \in \mathcal{F} : \mathbf{v} \leq \mathbf{x}\} = \mathcal{M}_*^{\mathcal{L}}(\mathbf{x}) \leq \mathcal{M}_{t+1}^{\mathcal{L}}(\mathbf{x}) \leq \mathcal{M}_t^{\mathcal{L}}(\mathbf{x}) \leq \mathbf{x}, \tag{23}$$

and

$$\mathbf{x} \leq \mathcal{W}_t^{\mathcal{L}}(\mathbf{x}) \leq \mathcal{W}_{t+1}^{\mathcal{L}}(\mathbf{x}) \leq \mathcal{W}_*^{\mathcal{L}}(\mathbf{x}) = \bigwedge\{\mathbf{u} \in \mathcal{F} : \mathbf{x} \leq \mathbf{u}\}. \tag{24}$$

Moreover, the inclusion relationship $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p\} \subseteq \mathcal{F}$ *holds true. Therefore,*

$$\mathcal{W}_t^{\mathcal{L}}(\mathbf{x}^\xi) = \mathbf{x}^\xi \qquad \text{and} \qquad \mathcal{M}_t^{\mathcal{L}}(\mathbf{x}^\xi) = \mathbf{x}^\xi, \tag{25}$$

for all $\xi = 1, \ldots, p$ *and for any positive integer* t.

Proof. Since (23) is very similar to (24), we will only deduce the former.

First of all, note that $(i,i) \in \mathcal{A}$ and $w_{ii}^{\mathcal{L}} = \top$ for all $i = 1, \ldots, n$. Hence, for any input pattern $\mathbf{x} \in \mathbb{B}^n$, we have

$$\left[\mathcal{M}_1^{\mathcal{L}}(\mathbf{x})\right]_i = \bigwedge_{(j,i)\in\mathcal{A}} (x_j / w_{ji}^{\mathcal{L}}) \leq (x_i / w_{ii}^{\mathcal{L}}) = x_i, \quad \forall i = 1, \ldots, n.$$

In other words, the inequality $\mathcal{M}_1^{\mathcal{L}}(\mathbf{x}) \leq \mathbf{x}$ holds true. By (21), we conclude that

$$\mathcal{M}_{t+1}^{\mathcal{L}}(\mathbf{x}) = \mathcal{M}_1^{\mathcal{L}}\left(\mathcal{M}_t^{\mathcal{L}}(\mathbf{x})\right) \leq \mathcal{M}_t^{\mathcal{L}}(\mathbf{x}), \forall t = 1, 2, \ldots$$

which implies the inequalities in (23).

Since $\mathbf{z}(t) = \mathcal{M}_t^{\mathcal{L}}(\mathbf{x})$, the sequence $\{\mathbf{z}(t)\}_t$ is decreasing. By Proposition 13.3 in [3], it converges to $\mathbf{z}_* = \bigwedge\{\mathbf{z}(0), \mathbf{z}(1), \ldots\}$. Thus, the memory $\mathcal{M}_*^{\mathcal{L}}$ is well defined. Furthermore, by taking the limit in both sides of (21), we conclude that $\mathcal{M}_*^{\mathcal{L}}(\mathbf{x}) = \mathbf{z}_*$ is a fixed point of $\mathcal{M}_1^{\mathcal{L}}$, i.e., $\mathcal{M}_1^{\mathcal{L}}(\mathbf{z}_*) = \mathbf{z}_*$.

Let $\mathcal{F} = \{\mathbf{v} \in \mathbb{B}^n : \mathcal{M}_1^{\mathcal{L}}(\mathbf{v}) = \mathbf{v}\}$ denote the set of all fixed points of $\mathcal{M}_1^{\mathcal{L}}$. We will show that $\mathbf{z}_*$ is the greatest element of $\mathcal{F}$ less than or equal to $\mathbf{x}$. The inequality $\mathbf{z}_* = \bigwedge\{\mathbf{z}(0), \mathbf{z}(1), \ldots\} \leq \mathbf{z}(0) = \mathbf{x}$ is trivial. Now, suppose that there is $\mathbf{v} \in \mathcal{F}$ such that $\mathbf{z}_* \leq \mathbf{v} \leq \mathbf{x}$. Note that $\mathcal{M}_t^{\mathcal{L}}(\mathbf{z}_*) = \mathbf{z}_*$ and $\mathcal{M}_t^{\mathcal{L}}(\mathbf{v}) = \mathbf{v}$ for all t because $\mathbf{z}_*, \mathbf{v} \in \mathcal{F}$. Also, since $\mathcal{M}_t^{\mathcal{L}}$ is order preserving for all t, the inequalities $\mathcal{M}_t^{\mathcal{L}}(\mathbf{z}_*) \leq \mathcal{M}_t^{\mathcal{L}}(\mathbf{v}) \leq \mathcal{M}_t^{\mathcal{L}}(\mathbf{x})$ hold true. By taking the limit, we obtain $\mathbf{z}_* \leq \mathbf{u} \leq \mathbf{z}_*$, which implies the equation $\mathbf{z}_* = \bigvee\{\mathbf{v} \in \mathcal{F} : \mathbf{v} \leq \mathbf{x}\}$.

Let us show that the sets of fixed points of $\mathcal{M}_1^{\mathcal{L}}$ and $\mathcal{W}_1^{\mathcal{L}}$ coincide. On one hand, we know from (23) that $\mathcal{M}_1^{\mathcal{L}}(\mathbf{x}) \leq \mathbf{x}$ for all $\mathbf{x} \in \mathbb{B}^n$. Also, if $\mathcal{W}_1^{\mathcal{L}}(\mathbf{x}) = \mathbf{x}$, then $\mathbf{x} \leq \mathcal{M}_1^{\mathcal{L}}(\mathbf{x})$ by the adjunction relationship given by Theorem 1. Therefore, $\mathcal{W}_1^{\mathcal{L}}(\mathbf{x}) = \mathbf{x}$ implies $\mathcal{M}_1^{\mathcal{L}}(\mathbf{x}) = \mathbf{x}$. The converse follows similarly.

Finally, the following reveals that the set of fundamental memories $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$ is included in $\mathcal{F}$. Precisely, we show that $\mathbf{x}^\xi = \mathcal{W}_1^{\mathcal{L}}(\mathbf{x}^\xi)$ holds for a given $\xi \in \{1, \ldots, p\}$. By (18), we obviously have $w_{ij}^{\mathcal{L}} \leq x_i^\xi / x_j^\xi$. Since $a \wedge (b/a) = a$ for any $a, b \in \mathbb{B}$, we conclude that $x_j^\xi \wedge w_{ij}^{\mathcal{L}} \leq x_j^\xi \wedge (x_i^\xi / x_j^\xi) = x_i^\xi$. Taking the maximum in the left side of this expression, we deduce

$$\bigvee_{(i,j)\in\mathcal{A}} \left(x_j^\xi \wedge w_{ij}^{\mathcal{L}}\right) \leq x_i^\xi, \quad \forall i = 1, \ldots, n.$$

In other words, $\mathcal{W}_1^{\mathcal{L}}(\mathbf{x}^\xi) \leq \mathbf{x}^\xi$. But, from (24), we also have $\mathbf{x}^\xi \leq \mathcal{W}_1^{\mathcal{L}}(\mathbf{x}^\xi)$, which implies the equation $\mathcal{W}_1^{\mathcal{L}}(\mathbf{x}^\xi) = \mathbf{x}^\xi$. □

Theorem 2 reveals that the memories $\mathcal{W}_*^{\mathcal{L}}$ and $\mathcal{M}_*^{\mathcal{L}}$ share with the TAMMs and SCAMMs the five properties listed in Section 3. In particular, $\mathcal{W}_*^{\mathcal{L}}$ and $\mathcal{M}_*^{\mathcal{L}}$ exhibit optimal absolute storage capacity. Also, they are completely characterized by their fixed points. In the context of MM, $\mathcal{W}_*^{\mathcal{L}}$ and $\mathcal{M}_*^{\mathcal{L}}$ correspond respectively to a closing and an opening [3].

Let us conclude this section by establishing a relationship between BAMMs and SCAMMs. Precisely, we know from Example 4 that the threshold approach can be viewed as a particular instance of the level-set approach to gray-scale MM. Analogously, the following theorem shows that BAMMs generalize the SCAMM models.

Theorem 3 *Given a fundamental memory set* $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\} \subseteq \mathbb{B}^n$, *where* $\mathbb{B}$ *is a complete Brouwerian lattice, define the SCAMMs* $\mathcal{W}^{\mathcal{T}}$ *and* $\mathcal{M}^{\mathcal{T}}$ *by means of* (16) *and* (17). *Then, there are single-step BAMMs* $\mathcal{W}_1^{\mathcal{L}}$ *and* $\mathcal{M}_1^{\mathcal{L}}$ *such that* $\mathcal{W}_1^{\mathcal{L}}(\mathbf{x}) = \mathcal{W}^{\mathcal{T}}(\mathbf{x})$ *and* $\mathcal{M}_1^{\mathcal{L}}(\mathbf{x}) = \mathcal{M}^{\mathcal{T}}(\mathbf{x})$ *for all input pattern* $\mathbf{x} \in \mathbb{B}^n$. *Furthermore, the network architecture* $\mathcal{A}$ *of the BAMMs coincide with the set of synaptic junctions* $\mathcal{S}$ *of the SCAMMs.*

Proof. Given a fundamental memory set $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\} \in \mathbb{B}$, define the set of synaptic junctions $\mathcal{S} \subseteq \mathcal{N} \times \mathcal{N}$ by means of (16) and let $\mathcal{A} = \mathcal{S}$ be the network architecture of the BAMMs. In this case, the following equivalences hold true:

$$\begin{aligned} (i,j) \in \mathcal{A} \quad &\Longleftrightarrow \quad x_j^\xi \leq x_i^\xi, \; \forall \xi = 1, \ldots, p, \\ &\Longleftrightarrow \quad x_i^\xi / x_j^\xi = \top, \; \forall \xi = 1, \ldots, p, \\ &\Longleftrightarrow \quad \bigwedge_{\xi=1}^{p} x_i^\xi / x_j^\xi = \top, \\ &\Longleftrightarrow \quad w_{ij}^{\mathcal{L}} = \top. \end{aligned}$$

Therefore, for any input pattern $\mathbf{x} \in \mathbb{B}^n$, the first terms of the recursive sequences given by (19) with $\mathbf{y}(0) = \mathbf{x} = \mathbf{z}(0)$ are:

$$y_i(1) = \bigvee_{(i,j)\in\mathcal{A}} \left(y_j(0) \wedge w_{ij}^{\mathcal{L}}\right) = \bigvee_{(i,j)\in\mathcal{S}} x_j, \quad \forall i = 1, \ldots, n,$$

and

$$z_i(1) = \bigwedge_{(j,i)\in\mathcal{A}} \left(z_j(0) / w_{ij}^{\mathcal{L}}\right) = \bigwedge_{(i,j)\in\mathcal{S}} x_j, \quad \forall i = 1, \ldots, n.$$

Comparing these two equations with (17), we conclude that the memories $\mathcal{W}_1^{\mathcal{L}}$ and $\mathcal{W}^{\mathcal{T}}$, as well as $\mathcal{M}_1^{\mathcal{L}}$ and $\mathcal{M}^{\mathcal{T}}$, coincide.

Theorem 3 shows that SCAMMs can be obtained from BAMMs by considering an specific network architecture. Since SCAMMs exhibit one-step convergence, we conclude that BAMMs also exhibit one-step convergence for certain network architectures. Nevertheless, it is not hard to devise an example in which more than one iteration of (20) are required for the convergence of the BAMMs $\mathcal{W}_t^{\mathcal{L}}$ and $\mathcal{M}_t^{\mathcal{L}}$.

Finally, Theorem 3 also shows that BAMMs are potentially better than SCAMMs. We confirmed this remark by means of computational experiments concerning the retrieval of corrupted color images [26]. Moreover, by adopting an appropriate network architecture, BAMMs consumed less computational resources than the SCAMMs. Unfortunately, we are not able to formalize the relationship between TAMMs and BAMMs apart from the fact that the former is defined in a richer mathematical structure.

6. Final Remarks

This article introduces the class of Brouwerian autoassociative morphological memories (BAMMs), which are morphological neural networks defined in a complete Brouwerian lattice. Broadly speaking, BAMMs are equipped with neurons that perform either a dilation or an erosion of the level-set approach to gray-scale mathematical morphology. The novel class of morphological memories include the autoassociative fuzzy implicative memories (AFIMs) of Gödel, its adjoint memory, and the sparsely connected autoassociative morphological memories (SCAMMs) [14, 15, 23, 24, 25].

In contrast to many other fully connected morphological memories such as the traditional autoassociative morphological memories (TAMMs) and the broad class of fuzzy morphological associative memories [25, 23], we permit BAMMs with a network architecture $\mathcal{A} \subseteq \mathcal{N} \times \mathcal{N}$, where $\mathcal{N} = \{1, \ldots, n\}$. By adopting an appropriate set $\mathcal{A}$, BAMMs demand few computational resources even for the storage and retrieval of large patterns. On the down side, the recall phase of BAMMs are defined recursively according to (19) and (20). Nevertheless, Theorem 2 showed that the sequences produced by BAMMs converge to a fixed point if $(i,i) \in \mathcal{A}$ for all $i \in \mathcal{N}$. Therefore, we defined $\mathcal{M}_*^{\mathcal{L}}$ and $\mathcal{W}_*^{\mathcal{L}}$ as the memories that yield the limit of the sequences as the recalled patterns. Theorem 2 also asserts that both $\mathcal{M}_*^{\mathcal{L}}$ and $\mathcal{W}_*^{\mathcal{L}}$ exhibit properties similar to TAMMs and SCAMMs, including optimal absolute storage capacity. Furthermore, the retrieval phase of $\mathcal{M}_*^{\mathcal{L}}$ and $\mathcal{W}_*^{\mathcal{L}}$ are completely characterized by the set of fixed points of the two memories.

In the future, we plan to investigate the performance of BAMMs for the storage and retrieval of multivalued, large patterns such as color images. In particular, we intent to study the tolerance of the novel memories with respect to Gaussian and impulsive noise that are usually introduced due to sensor defect, environmental interference, and failure to store or transfer a pattern.

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REPORT

Training strategies for a fuzzy CBR cluster-based approach

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We present an approach to Case-Based Reasoning grounded on fuzzy relations and residuated implications operators. We propose to create clusters of cases in the base using fuzzy gradual rules, modelling the principle "the more similar the problem descriptions are, the more similar the solution descriptions are". We also study the use of the Fuzzy ART neural network to create the clusters. We study a set of strategies to obtain weights for cases in the training base, considering the existence of not of clusters, and how to calculate the solution to a new problem. The obtained results for a real-world application are superior than those from the literature.

1. Introduction

Case-Based Reasoning (CBR) ([1], [7]) solves problems using a principle that can be stated as "similar problems have similar solutions" [1]. A case base consisting of solved problems, modelled as pairs (problem, solution), is used to determine the solution to a new problem. The first step of this procedure consists in retrieving problems in the base that are similar to the problem at hand: it determines the cases in the base that are relevant to solving that particular problem. The second step consists in reusing the solutions of these relevant problems, so as to produce a solution to the problem at hand.

Weights can be attached to cases, so those considered more relevant for a given application have higher weights than the less relevant ones. Weight vectors can also be assigned to description variables: one can use either the same weight vector for all cases, or assign individual weight vectors to each case, so that more significant attributes inside a case receive higher weights.

In [2] and [8], fuzzy similarity relations associated to each description and solution variables spaces were used to derive individual vector weights through the learning algorithm proposed in [12]. It was shown that weighting the attributes in each case in the training set tends to lead to better results than non-weighted counterparts.

The inconvenience with weighting is that the learning process is usually computationally expensive, which may make it impossible to be used in large case bases. An approach to allow learning weights in large case bases consists in extracting fragments of the case base and obtaining weights for cases inside those fragments. Each fragment is then considered as a case base itself. When a new problem is presented to the base, first a solution is calculated acording to each fragment using the individual vectors associated to that fragment. Then a final solution is either chosen from one of the fragments or aggregated from various ones.

An approach to create case base fragments was proposed in [6]. The proposed method is based on creating a classic relation between cases, which takes into account the resemblance between cases in the problem description space but also in the solution description space. Then based on this relation, clusters of similar cases from the base are extracted; the clusters group cases that have both similar problem descriptions and similar solutions. To compute a solution for a new problem, the authors propose to adopt the solution from the cluster with which the problem at hand has the highest overall problem resemblance. Only the clusters that are maximal are considered in the process.

This approach was generalized to the case where the relation between cases is not dichotomic but takes values in the scale, leading to a fuzzy relation instead of a crisp one [11]. The problem then is to extract clusters from this fuzzy relation. The authors propose to first extract the relevant level-cuts from the fuzzy relation, thus creating a set of crisp relations, for which the procedure devised in [6] can be applied in a straightforward manner. Alternatively, in [18], a Fuzzy ART neural network [15] was used to compute clusters of similar cases. However, in this case it is not garanteed that the cases in cluster are somewhat similar in both the problem description and solution spaces.

Another important issue is how to create the training base to obtain weight vectors for the cases in each cluster. Two reasonable options are: i) the cases inside the cluster itself and ii) a larger set containing the cases in the cluster. Contrary to option i), option ii) means that negative information is taken into account to compute the weights in a cluster, which intuitively does not lead to optimal vector weights. A set of strategies based on these options was proposed and tested in [17] and further exploited in [18].

The main goal of this paper is to address the determination of training bases for the clusters and to compare the fuzzy approach and the neural network approach to obtain the clusters, based on [17], [11] and [18].

This paper is organized as follows. In Section 2 we give some basic definitions and the notation used throughout the text. In Section 3 we describe the fuzzy approach to CBR outlined above. The Fuzzy ART neural network is described in Section 4. Section 5 we study training strategies for our approach and in Section 6 we describe a real-world application and use it compare our results to those from the literature. Section 7 finally brings the conclusions.

2. Definitions and Notations

In this section, we present the definitions and notations used throughout this paper. The definitions involving fuzzy concepts (triangular norms, residuated operators and proximity relations), come from well-established literature (see, e.g., [4] [5]). The definitions related to graph theory are the consecrated ones.The concept of imprecise partitions is new (up to our knowledge).

An operator $\top : [0,1]^2 \to [0,1]$ is called a t-norm if it is commutative, associative, monotonic and has 1 as neutral element. An operator $\perp : [0,1]^2 \to [0,1]$ is called a t-conorm if it is commutative, associative, monotonic and has 0 as neutral element. Examples of t-norms are the minimum and the product; examples of t-conorms are the maximum and the bounded sum.

Given a left-continuous t-norm $\top$, a residuated implication operator $\to_\top$ is defined as $\forall x, y \in [0,1], x \to_\top y = \sup_{z\in[0,1]} \top(x,z) \leq y$. Some well-known operators are

- the Gödel implication, residuum of $\top = \min$, defined as $x \to_{\top G} y = 1$, if $x \leq y$, and $x \to_{\top G} y = y$, otherwise;
- the Goguen implication, defined as $x \to_{\top \Pi} y = 1$, if $x \leq y$, and $x \to_{\top \Pi} y = y/x$, otherwise.

Also noteworthy is the Rescher-Gaines implication operator, defined as $x \to_{\top RG} y = 1$, if $x \leq y$, and $x \to_{\top RG} y = 0$, otherwise. $x \to_{\top RG}$ is not a residuated operator itself but is the point-wise infimum of all residuated implications.

A proximity relation S on a domain U is a binary fuzzy relation described by the mapping $S : U \times U \to [0,1]$, that is both reflexive ($\forall x \in U, S(x,x) = 1$) and symmetric ($\forall x, y \in U, S(x,y) = S(y,x)$). When a proximity relation also satisfies the t-norm transitivity property ($\forall x, y, z \in U, \top(S(x,y), S(y,z)) \leq S(x,z)$ for some t-norm $\top$), it is called a similarity relation. The set of proximity relations on a given domain U forms a lattice (not linearly ordered) with respect to the point-wise ordering (or fuzzy-set inclusion) relationship. The top of the lattice is S_{top} which makes all the elements in the domain maximally similar: $S_{top}(x,y) = 1$ for all $x, y \in U$. The bottom of the lattice S_{bot} is the classical equality relation: $S_{bot}(x,y) = 1$, if $x = y$, and $S_{bot}(x,y) = 0$, otherwise. In this context, particularly useful are families of fuzzy relations $\mathcal{S} = \{S_0, S_{+\infty}\} \cup \{S_\beta\}_{\beta \in I \subseteq (0,+\infty)}$ that are such that: (i) $S_0 = S_{bot}$, (ii) $S_{+\infty} = S_{top}$, and (iii) $\beta < \beta'$, then $S_\beta \prec S_{\beta'}$, where $S \prec S'$ means $\forall x, y \in U, S(x,y) \leq S'(x,y)$ and $\exists x_0, y_0 \in U, S(x_0,y_0) < S'(x_0,y_0)$.

A hypergraph is a generalization of a non-directed graph, where edges can connect any number of vertices. Formally, it can be represented as a pair, $H = (N, E)$, where N is a set containing the nodes (or vertices) and E is a set of non-empty subsets of N, called hyperedges. The set of hyperedges E is thus a subset of $2^N \backslash \emptyset$, where 2^N is the power set of N. An "ordinary graph" is then a hypergraph in which all hyperedges have at most two elements. Each graph can be associated to a corresponding hypergraph, whose hyperedges are the cliques of the initial graph. Given a hypergraph $H = (N, E)$, a hyperedge $A \in E$ is said to be maximal when $\nexists B \in E$, such that $A \subseteq B$ and $A \neq B$. Each hyperedge in E is a clique, therefore, the set of maximal hyperedges is the set of maximal cliques of E.

Let B be a subset of a domain U. A set $B' = \{B_1, ..., B_z\}, B_i \subseteq B, B_i \neq \emptyset$, is an imprecise partition of B when $\bigcup_{i=1,z} B_i = B$ and $\forall i, j \in 1, z, i \neq j, \nexists B_i, B_j \in B$, such that $B_i \subseteq B_j$. Each $B_i \in B$ is called an imprecise class. An imprecise partition does not allow one class to be contained inside another one but, contrary to classic partitions, it does allow non-empty intersections between classes. Let B' and B'' be two imprecise partitions of B. B' is said to be finer than B'' (denoted by $B' \preccurlyeq B''$) when $\forall h' \in B', \exists h'' \in B'', h' \subseteq h''$. Reciprocally, B'' is said to be coarser than B'.

3. Fuzzy Cluster-Based CBR Approach

In the following, we describe a fuzzy cluster-based approach to Case-Based Reasoning [6] and [11], that can be synthesized as follows:

- A binary relation is created in the base, reflecting how close are the problem descriptions and the solutions of two cases.
- Clusters of similar cases are extracted from the base, using the binary relation.
- Weight vectors are learned for each case inside a given cluster.
- When a new problem is presented to the base, its overall similarity (strength) to the cases in a cluster is calculated.
- The solution to the new problem is calculated using the cases in the cluster with the highest strength.

Let a case c be defined as an ordered pair $c = (p, o) \in P \times O$ where p is the description of the case problem and o the description of its solution. $P = \{P_1 \times ... \times P_n\}$ and O are respectively the (n-ary) problem description and the (unary) solution spaces. Let $S_{in} : P^2 \to [0,1]$ and $S_{out} : O^2 \to [0,1]$ respectively denote the proximity relations used on the problem and solution spaces. S_{in} may obtained by using a suitable aggregation function applied on the set of proximity relations $\{S_1, ..., S_n\}$, each of which corresponding to a description variable. For example, using the arithmetic means, we have: $S_{means}(p_i, p_j) = \frac{1}{n}\sum_{k=1,n} S_k(p_{ik}, p_{jk})$.

3.1 Fuzzy Case Resemblance Relations

We define a Fuzzy Case Resemblance Relation (FCRR) as the mapping $F_\phi : C^2 \to [0,1]$, defined as

$$F_\phi(c_a, c_b) = \begin{cases} 0, \text{if } S_{in}(p_a, p_b) = 0 \\ \phi(S_{in}(p_a, p_b), S_{out}(o_a, o_b)), \text{otherwise,} \end{cases} \quad (26)$$

where $\phi = \to_\top$ is a residuated implication operator obtained from a T-norm $\top$ (see Section 2). This relation models a gradual formalization of the basic CBR principle the more similar the problem descriptions are, the more similar the solution descriptions are. Note that two cases are considered dissimilar whenever their problem descriptions and/or their solutions are completely dissimilar.

3.2 Obtaining Crisp Case Resemblance Relations

A fuzzy case resemblance relation F_ϕ is not necessarily crisp. To be able to obtain clusters from the case base, we can first take a relevant level-cut from F_ϕ, and then extract clusters from the resulting crisp relation. A set of hypergraphs are then derived from the crisp relations

$$\forall\alpha \in (0,1], F_{\phi,\alpha}(c_i,c_j) = \begin{cases} 1, \text{if } F_\phi(c_i,c_j) \geq \alpha \\ 0, \text{otherwise} \end{cases} \quad (27)$$

Each $F_{\phi,\alpha}$ is called a Crisp Case Resemblance Relation (CCRR). The number of cases is finite and we thus obtain a finite number of CCRRs from a FCRR F_ϕ, one for each distinct value greater than 0 in F_ϕ.

3.3 Using Attribute Weights

The solution to a cluster and the cluster strength can both be determined using weighted operators. Weights can be attached to cases, so that the sum of weights attached to the cases in the base is 1. Weights can also be attached to description variables, so that the sum of weights attached to the variables is 1. Finally, individual weight vectors can be attached to each case, so that the sum of weights attached to the variables inside each case is 1. In [12] we find an algorithm for the learning of individual weight vectors, based on fuzzy relations. Many aggregation functions have weighted counterparts, as for instance the means, t-norms and t-conorms operators, which can be used to derive a weighted S_{in} relation. For example, using the individual attribute weighting approach, the weighted means as aggregation operator yields $S^w_{means}(p_i,p_j) = \sum_k w_k \times S_k(p_{ik},p_{jk})$, where w is a weight vector such that $\forall k$, $w_k \in [0,1]$ and $\sum_k w_k = 1$. A weighted version of S_{in} can also be used to compute the clusters themselves. If individual weight vectors are used, the resulting relation is possibly asymmetric and one has to make it symmetric before applying the rest of the formalism (see [6]).

3.4 Obtaining Clusters from a Case Base

Let $c_a = (p_a, o_a)$ and $c_b = (p_b, o_b)$ denote two cases in C. Let $R = F_{\phi,\alpha}$ be a CCRR obtained as a level cut from a given FCRR F_ϕ and $\alpha > 0$. Based on CCRR R, the case set can be organized through a decomposition in clusters based on this crisp resemblance relation. Several (possibly intersecting) clusters of cases can be obtained from R. The latter can in turn be represented as a hypergraph. More precisely, a hypergraph $H = (C,E), E \subseteq C^2$ is said to be compatible with CCRR R iff it obeys the following conditions:

- $\forall c_a, c_b \in C$, if $R(c_a,c_b) = 1$, then $\exists h \in E$, such that $\{c_a,c_b\} \subseteq h$.
- $\forall c_a, c_b \in C$, if $R(c_a,c_b) = 0$, then $\nexists h \in E$, such that $\{c_a,c_b\} \subseteq h$.

A notable hypergraph compatible with R is the one containing the maximal cliques of E. Another way to calculate the clusters for a base case is the use of artificial neural networks (see section 4).

3.5 Computing a Solution to a New Problem according to a Cluster

Given a case base C, similarity measures S_j for each description variable v_j, global similarity measures S_{in} and S_{out}, and a hypergraph $H = (C,E)$ compatible with $R = F_{\phi,\alpha}$, for a given a residuated operator ϕ and a value $\alpha \in (0,1]$, we have to derive an appropriate solution o^* for a new problem description, denoted p^* in the following. We propose to derive solution o^* from the clusters containing cases whose problem descriptions are somewhat similar to p^*, denoted $E^* = \{h \in E \mid \forall c_i = (p_i,o_i) \in h, S_{in}(p_i,p^*) > 0\}$. For each $h = \{c_1,...,c_r\} \in E^*$, we compute its corresponding solution for p^*, denoted by o^*_h, using a suitable aggregation function that takes into account the set of solutions o_i as well as the similarity between each p_i and p^*, considering the cases (p_i,o_i) in h. For example, using the weighted means as aggregation function we obtain $o^*_h = \sum_{i=1,r} \frac{S_{in}(p_i,p^*) \times o_i}{\sum_{i=1,r} S_{in}(p_i,p^*)}$. S_{in} is substituted by its counterpart in weighted frameworks (see below). Also, in case non-numerical description variables occur, the aggregation method above cannot be applied but can be replaced with a weighted voting method.

3.6 Determining Cluster Strength in Relation to a New Problem

Let O^* be the set of solutions for p^* from the clusters in E^*. To select a final solution o^* from O^*, one can aggregate the solutions produced by the clusters, or simply choose a cluster $h \in E^*$ and make $o^* = o^*_h$. To select a cluster in the latter option, we propose to take the cluster whose overall cases problem description resembles most p^*. We define the cluster strength of each cluster $h = \{c_1,...,c_r\} \in E^*$, $c_i = (p_i,o_i)$, in relation to a problem p^* as

$$str_f(h,p^*) = f(S_{in}(p_1,p^*),...,S_{in}(p_n,p^*)).$$

where f is a suitable aggregation function, such as the arithmetic means, a t-norm, a t-conorm, etc.

4. Fuzzy ART

Neural network can also be used to obtain a set of clusters of similar cases from a case base. Adaptive Resonance Theory (ART) [15] is a class of neurally inspired models of how the brain performs clustering and classification of sensory data, and associations between the data and representation of concepts. ART models solve the so-called stability plasticity dilemma where new patterns are learned without forgetting those learned previously.

The Fuzzy Adaptive Resonance Theory neural network model (Fuzzy ART) is a kind of ART neural network that accepts analog inputs in the real interval [0,1] [16]. Familiar inputs activate the category, whereas unfamiliar inputs trigger either adaptive learning by an existing category or a commitment of a new category. Fuzzy ART performs unsupervised learning of categories under continuous presentation of inputs, through a process of adaptive resonance in which the learned patterns adapt only to relevant inputs, but remain stable under irrelevant or insignificant ones.

The behavior of Fuzzy ARTs lends itself well to simple geometrical interpretation of category prototypes as hyper-rectangles in the input space. These rectangles are allowed to overlap each other. Although Fuzzy ART always responds the same way to a familiar input, the overlaps are inconvenient if categories are mutually exclusive.

Each neuron in the input layer of the Fuzzy ART neural network is connected to all the neurons on the output layer. In the present work, each neuron on the input layer corresponds to problem description variable and each neuron of the output layer of the network represents a cluster. The values on the connections to an output neuron can be seem as a prototype for that neuron. The problem description of each case in the training base is compared to these prototypes, using a chosen metric, e.g. the Euclidean distance. If, for a given case, the similarity found between the problem description and the prototype of a cluster is greater or equal than a certain threshold (vigilance), the case is inserted into the cluster, and the cluster prototype is recalculated, taking into account the inserted case. If none of the existing prototypes is sufficiently similar to the case presented to the network, a new neuron representing a new cluster is created. In the implementation adopted here, the neurons are visited sequentially and a case will only belong to a single cluster. Moreover, the class of the case inn the training base is not taken into account in the process.

Self-organizing maps (SOM) could also be used to calculated a set of clusters from a case base. However, its training process is slower than the one presented by the Fuzzy ART neural network. Another obstacle is how to determine the number of clusters. In Fuzzy ART each neuron represents itself a cluster [19]. Using SOM, however, we would have to cluster the neurons in the network.

5. Training strategies

We are interested in what should be the training base to derive a weight vector for a case inside a cluster. Two reasonable options are i) the cases inside the cluster itself and ii) a larger set containing the cases in the cluster. Contrary to option ii), option i) means that negative information (i.e. cases that have similar problem descriptions but dissimilar solutions) is not taken into account to compute the weights in a cluster, which intuitively should not lead to optimal vector weights.

We have devised a series of types of experiments to test the most effective combination of sets of cases used for training and for the calculation of results. We use the letters "W" and "R" in order to specify the type of training and the type of calculation of results, respectively. We may either not use weights (W-) in an experiment, or obtain them considering the whole base as a single cluster (W), or yet obtain them considering several clusters (W+). The results may be calculated using either a single cluster (R) or several ones (R+). These strategies have been employed in [17], [18], [19] e [20].

Moreover, W++ denotes the training strategy in which when a set of cases larger than the cases in a cluster itself are used for obtaining weights for the cases in the cluster[4].

Here we address six types of experiments

- W-R (no weights, results calculated for single cluster),
- W-R+ (no weights, results calculated for several clusters),
- WR (weights learned for a single cluster, results calculated for single cluster),
- WR+ (weights learned for a single cluster, results calculated for several clusters),
- W+R+ (weights learned for several clusters using the cluster itself as training base, results calculated for several clusters), and
- W++R+ (weights learned for several clusters using enlarged cluster as training base, results calculated for several clusters).

The missing combinations W+R and W++R are not address in this paper. They both mean that as many weight vectors are obtained for a given case as the number of clusters to which it belongs. On the other hand, a single cluster is used for the calculation of results. Thus, all weight vectors calculated should be aggregated into one using, for example, the arithmetic mean of weight vectors.

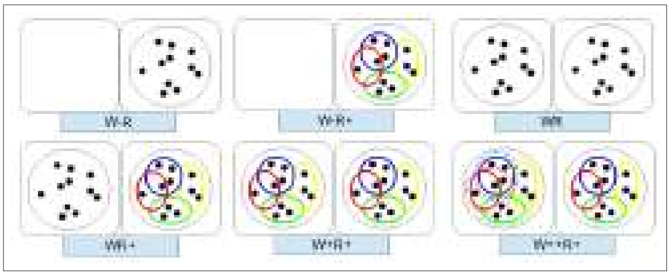

Fig 1. Illustration of experiment typology

In the weighted strategies, we need to construct a training base for each cluster $h \in E$ derived from case base C, denoted as CT_h. For W+ we simply make $CT_h = h$.

For W++, an enlarged cluster serves as the training base for the calculation of weight vectors to be associated to the cases in that cluster. Let $c' \in C$ be a case, such that $c' \notin h$. Two reasonable methods to allow case c' to belong to CT_h are:

- c' is somewhat related to all cases in h: $c' \in CT_h$ iff $\inf_{c \in h} S_{in}(c, c') > 0$.
- c' is somewhat related to some cases in h: $c' \in CT_h$ iff $\sup_{c \in h} S_{in}(c, c') > 0$;

Approach i) is more restrictive and produces training bases smaller than ii). In our experiments, we have used approach i), since approach ii) tends to make the enlargement of the cluster CT_h become the whole base C itself.

Other functions can be devised. An interesting possibility is to relax the conditions above, setting another value than 0 as threshold for the similarities between a case and a cluster.

[4]Note that the use of R++, in which enlarged clusters would be used to calculate the result for a new case could also be envisaged.

6. Classification of Schistosomiasis prevalence

Schistosomiasis mansoni is a disease with social and behavioral characteristics. Snails of the Biomphalaria species, the disease intermediate host, uses water as a vehicle to infect man, the disease main host. In Brazil, six million people are infected by it, mainly in poor regions of the country [14]. According to the data presented at the Brazilian Information System for Notifiable Diseases (SINAN) of the Ministry of Health, from 1995 to 2005, more than a million positive cases were reported, 27% of them in the State of Minas Gerais. In [10], the authors present a classification Schistosomiasis prevalence for the State of Minas Gerais, using remote sensing, climate, socioeconomic and neighborhood related variables. Two approaches were used, a global and a regional one. In the first approach, a unique regression model was generated and used to estimate the disease risk for the entire state. In the second approach, the state was divided in four regions, and a model was generated for each one of them. Imprecise classifications were also generated for both approaches, using the estimated standard deviation and several reliability levels as basis [10]. In [8] the authors compare those experiments with a similarity based approach, weighting the cases individually.

In the original experiments presented in [10], the disease prevalence data was provided by the Health Secretary of the State of Minas Gerais state. The prevalence is known for 197 municipalities out of the 853 composing the state (see Figure 2.a).

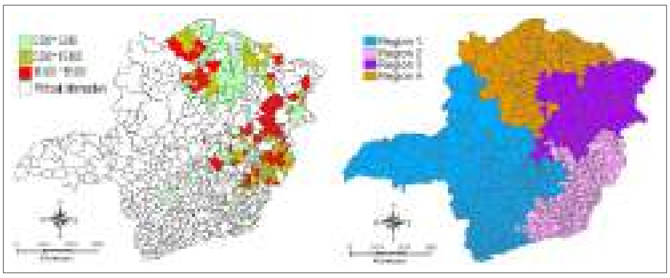

Fig 2. State of Minas Gerais in Brazil: a) known prevalence of Schistosomiasis and b) regionalization obtained with SKATER (source: [10])

From the original 86 variables, a smaller set was selected, according to tests using multiple linear regression [10]; the independent variables chosen were those that had high correlation with the dependent variable and low correlation with other independent variables.

Method	W-R	W-R+	WR	WR+	W+R+	W++R+
Fuzzy similarity	42.86%	57.14%	50.00%	71.42%	14.29%	57.14%
Fuzzy ART(T_1)	42.86%	42.86%	42.86%	64.29%	42.86%	42.86%
Fuzzy ART(T_2)	42.86%	64.29%	42.86%	35.71%	50.00%	42.86%
Fuzzy ART(T_3)	42.86%	57.14%	42.86%	21.43%	35.71%	42.86%

Table 2. Classification accuracy: test set, regional approach for R3. [17][18]

Two main approaches were used: i) a global one, in which all the municipalities with known disease prevalence were used, for either constructing or validating a linear regression model, and ii) a regional one, in which the state was divided in four homogeneous regions and a linear regression model was created for each one of them. In this work we're interested in the regional approach.

The number of independent variables used in the experiment varied; in the regional approach 4 variables were used for region 3 (see details in [10]).Approximately 2/3 of the samples were used as training set, and the remainder 1/3 as the test set. Algorithm SKATER [3] was used to obtain the homogeneous regions in the regional model; this algorithm creates areas such that neighboring areas with similar characteristics belong to the same region (see Figure 2.b).

The prevalences were classified as low, medium or high [14], respectively defined by intervals $[0, 5]\%$, $(5, 15]\%$ and $(15, 100]\%$. Table 1 reproduces the results from [10] using R-Regr (regional basis and regression), G-Regr (global basis and regression), G-DT (global basis and decision trees) with the accuracy of the results for region R3 and details the results presented in [8], using similarity based weighting: R-Sim (regional basis and similarity approach) and G-Sim (regional basis and similarity approach), actually an instance of experiment type WR.

R-Regr	G-Regr	G-DT	G-Sim	R-Sim
28.57%	42.86%	35.71%	28.57%	35.71%

Table 1. Classification accuracy original methods test set region R3. [9] [10]

7. Classification of Schistosomiasis prevalence using weighted cluster based fuzzy approach

In the experiments, we used the same variables of the experiments described in Section 6, as well as the same the training and test case bases. We have also used the same proximity relations for description variables that resulted in Table 1, constructed by means of the parameterized family described in Section 2. The only exception is that we used the identity matrix for S_{out}, instead of the parameterization used in [8]. The arithmetic means was used to calculate the solution in the non-weighted framework, and the weighted means was used to both determine cluster strength and calculate the solution in the weighted frameworks.

A Fuzzy ART neural network was also used to calculate the clusters. Our best results have been obtained when we trained the network for 600 iterations with vigilance threshold equal to 0.45, with an initial number of 3 neurons on the output layer. Note that in our implementation each case in the training base belongs to a single cluster. We developed three types of training for Fuzzy ART in what regards the input layer. In T_1, each neuron in the input layer corresponds to an attribute in the case problem description. Then, in T_2, these neurons correspond to the (non-weighted) similarity of a given case in the training base with the other cases in this base, therefore the input layer has the same size as the training case base. Finally, in T_3 we used the same approach as in T_2 but with the weighted relation between the cases. The number of clusters for each of the experiments was 4, 5, and 3 respectively. Table 2 bring the results using the regional approach for the fuzzy approach in Section 3 and the approach using the Fuzzy ART neural network.

We can observe that the methodologies studied here (see Table 2) produced better results than previous works on the literature (see Table 1). The experiments show that for this data set, the division of the cases into clusters produce better results than taking the whole base as training set. For this data set, weighting of the cases led to the best results for the fuzzy formalism but proved to be insignificant in the neural network approach. Strategy W+R+ in yielded poorer results than WR+ for this data set, indicating that learning weights taking the whole base into account is a better choice than learning them on clusters and then taking as result the solution calculated using the strongest cluster. However, enlarging the clusters (W++R+) seems to somewhat compensate the gap, which is a very promising result. The resulting analysis for the other regions (not shown here) are similar to the one presented above for region R3 (see [20] for more details).

8. Conclusion and Future works

We have presented an approach to Case-Based Reasoning grounded on fuzzy proximity relations and residuated implications operators. We propose to create clusters of cases based on fuzzy gradual rules, modelling the principle "the more similar the problem descriptions are, the more similar the solution descriptions are". We have also proposed to create the clusters using the Fuzzy ART neural network. We have also proposed a set of strategies to obtain weights for cases in the training base, considering the existence of not of clusters, and how to calculated the solution to a new problem.

We have performed experiments to estimate the prevalence of Schistosomiasis, and obtained better results than those from the literature. The method was also applied successfully to classify deforesting patterns [20].

Learning vector weights considering clusters of cases did not always lead to better results than those produced by learning weights for the whole base. However, one of the strategies, that take negative information into account, did produce good results, indicating that it could be used to deal with large case bases where weighing the cases using the whole base for training would be unfeasible.

The clustering algorithm used in the proposed fuzzy approach is not very efficient and the use of neural networks such as Fuzzy ART is a promising direction to deal with large case bases. As future work, we intend to verify whether the introduction of the solutions of the cases from the training base in the neural network may lead to better results, to study the effect of allowing a case to belong to more than a single cluster in the neural network approach and also study the possibility of using parallelization in the learning process.

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REPORT

Projections of solutions for fuzzy differential equations

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In this paper we propose the concept of *fuzzy projections* on subspaces of $\mathcal{F}(\mathbb{R}^n)$, obtained from Zadeh's extension of canonical projections in $\mathbb{R}^n$, we study some of the main properties of such projections. Furthermore, we will review some properties of fuzzy projection solution of fuzzy differential equations. As we shall see, the concept of fuzzy projection can be interesting for the graphical representation of fuzzy solutions

1. Introduction

Consider the set $U \subset \mathbb{R}^n$. Denote by $\mathcal{F}(U)$ the set formed by the fuzzy subsets of U whose subsets has support compacts in U. Some properties metrics $\mathcal{F}(U)$ can be found in [9]. If A is a subset of U, we will use the notation χ_A to indicate a membership function for the fuzzy set called membership function or crisp of A.

Consider the autonomous equation defined by

$$\frac{dx}{dt} = f(x) \tag{28}$$

where $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth function. For each $x_o \in U$, denote by $\varphi_t(x_o)$ the solution for Equation (28) with initial condition x_o. Here we are assuming that the solution is defined for all $t \in \mathbb{R}_+$. The function $\varphi_t : U \to U$, The function $\varphi_t(x_o)$ will be called *deterministic flow*.

To consider initial conditions with inaccuracies modeled by fuzzy sets, [7] consider the proposed Zadeh's extension φ_t, the aplication $\hat{\varphi}_t : \mathcal{F}(U) \to \mathcal{F}(U)$, which takes the fuzzy set $\boldsymbol{x}_o \in \mathcal{F}(U)$ the fuzzy set $\hat{\varphi}_t(\boldsymbol{x}_o)$. In the context of this paper we call the application $\hat{\varphi}_t$ of *fuzzy flow*. Given $\boldsymbol{x}_o \in \mathcal{F}(U)$, we say $\hat{\varphi}_t(\boldsymbol{x}_o)$ is a *fuzzy solution* to Eq. (28) whose initial condition is the fuzzy set $\boldsymbol{x}_o$.

The conditions for existence of fuzzy equilibrium points and the nature of the stability of such spots were first presented in [7]. The concepts of stability and asymptotic stability for fuzzy equilibrium points are similar to those of equilibrium points of deterministic solutions and stability conditions for fuzzy equilibrium points can be found in [7]. Conditions for the existence of periodic fuzzy solutions and the stability of such solutions can be found in [4].

In this paper we propose the concept of *fuzzy projections* on subspaces of $\mathcal{F}(\mathbb{R}^n)$, obtained from the Zadeh's extension defined canonical projections in $\mathbb{R}^n$ and study some of the main properties such projections. Furthermore, we review some properties of fuzzy projection solution of fuzzy differential equations. As we shall see, the concept of fuzzy projection can be interesting for the graphical representation of fuzzy solutions.

2. Projections in fuzzy metric spaces

For application that will do following this work, we restrict our analysis to the set $\mathcal{F}(X)$ whose elements are subsets of a fuzzy set X whose α - levels are compact and non-empty subsets in X. The fuzzy subsets that are $\mathcal{F}(X)$ will be denoted by bold lower case letters to differentiate the elements X. So $\boldsymbol{x} \in \mathcal{F}(X)$ if and only if, $[\boldsymbol{x}]^\alpha$ is compact and non-empty subset for all $\alpha \in [0, 1]$.

We can define a structure of metric spaces in $\mathcal{F}(X)$ by the Hausdorff metric for compact subsets of X. Let $\mathcal{K}(X)$ the set formed by nonempty compact subsets of the metric space (X, d). Given two sets A, B in $\mathcal{K}(X)$ the distance between them can be defined by:

$$dist(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b)$$

The distance between sets defined above is a *pseudo-métric* to $\mathcal{K}(X)$, since $dist(A, B) = 0$ if and only if $A \subseteq B$, not necessarily equal value. However, *Hausdorff distance* between A, $B \in \mathcal{K}(X)$ defined by

$$\begin{aligned} d_H(A, B) &= \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\} \\ &= \max\{dist(A, B), dist(B, A)\} \end{aligned}$$

is a metric for all $\mathcal{K}(X)$, so that $(\mathcal{K}(X), d_H)$ is a metric space. It is also worth that (X, d) is a complete metric space, so $(\mathcal{K}(X), d_H)$ is also a complete metric space [1].

Through the Hausdorff metric d_H, we can define a metric for all $\mathcal{F}(X)$, Here we denote it by d_∞. Given two points $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{F}(X)$ the distance between $\boldsymbol{u}$ e $\boldsymbol{v}$ is defined by

$$d_\infty(\boldsymbol{u}, \boldsymbol{v}) = \sup_{\alpha\in[0,1]} d_H([\boldsymbol{u}]^\alpha, [\boldsymbol{v}]^\alpha).$$

It is not difficult to show that the distance defined above satisfies the properties of a metric and thus $(\mathcal{F}(X), d_\infty)$ is a metric space.

Nguyen's theorem provides an important link between α - levels image of a fuzzy subsets and the image of his α - levels by a function $f : X \times Y \to Z$. According [3], if $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ and $f : X \to Y$ is continuous, then the extension Zadeh $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is well defined and is worth

$$[\hat{f}(\boldsymbol{u})]^\alpha = f([\boldsymbol{u}]^\alpha) \tag{29}$$

for all $\alpha \in [0, 1]$ and $\boldsymbol{u} \in \mathcal{F}(X)$.

2.1 Projections fuzzy

Consider the application $P_n : \mathbb{R}^{n+m} \to \mathbb{R}^n$ that for each $(x, y) \in \mathbb{R}^{n+m}$ associating point $P_n(x, y) = x \in \mathbb{R}^n$.

Provided that $\mathbb{R}^n$ can be characterized as a subset of $\mathbb{R}^{n+m}$ by identifying the subset $\mathbb{R}^n \times \{0\}$, then the application P_n can be seen as the projection of $\mathbb{R}^{n+m}$ on the set $\mathbb{R}^n$. For this reason, we say that x is the projection in $\mathbb{R}^n$, the point $(x, y) \in \mathbb{R}^{n+m}$.

Notice that a point (u, v) is in the image of P_n if, and only if $v = 0$. Furthermore, $P_n(x, y) = x$ for all $y \in \mathbb{R}^m$. Thus, given a point $\boldsymbol{z} \in \mathcal{F}(\mathbb{R}^{n+m})$, with membership function $\mu_{\boldsymbol{z}} : \mathbb{R}^{n+m} \to [0, 1]$, the image $\hat{P}_n(\boldsymbol{z})$, obtained by extension of Zadeh's projection P_n, has the membership function

$$\mu_{\hat{P}_n(\boldsymbol{z})}(x) = \sup_{v\in\mathbb{R}^m} \mu_{\boldsymbol{z}}(x, v)$$

The aplication $\hat{P}_n : \mathcal{F}(\mathbb{R}^{n+m}) \to \mathcal{F}(\mathbb{R}^n)$, obtained from the extension of Zadeh P_n, that for all $\boldsymbol{z} \in \mathcal{F}(\mathbb{R}^{n+m})$ associates the point $\hat{P}_n(\boldsymbol{z}) \in \mathcal{F}(\mathbb{R}^n)$ can be seen as a *projection* of $\mathcal{F}(\mathbb{R}^{n+m})$ in $\mathcal{F}(\mathbb{R}^n)$, since this can be identified with the subset $\mathcal{F}(\mathbb{R}^n) \times \chi_{\{0\}}$. Also the projection P_n, is not difficult to see that the application $\hat{P}_n$ also satisfies:

$$\hat{P}_n(\hat{P}_n(\boldsymbol{z})) = \hat{P}_n(\boldsymbol{z}).$$

Based on this, we can define the *projection of fuzzy* $\boldsymbol{z} \in \mathcal{F}(\mathbb{R}^{n+m})$ in $\mathcal{F}(\mathbb{R}^n)$ as the point $\boldsymbol{x} \in \mathcal{F}(\mathbb{R}^n)$ with a membership function

$$\mu_{\boldsymbol{x}}(x) = \sup_{a\in\mathbb{R}^m} \mu_{\boldsymbol{z}}(x, a). \qquad (30)$$

We also consider the function $P_m : \mathbb{R}^{n+m} \to \mathbb{R}^m$ that for all $(x, y) \in \mathbb{R}^{n+m}$ associates the point $P_n(x, y) = y \in \mathbb{R}^m$. In this case, the image of a point $\boldsymbol{z} \in \mathcal{F}(\mathbb{R}^{n+m})$, with the membership function $\mu_{\boldsymbol{z}} : \mathbb{R}^{n+m} \to [0, 1]$, is a point $\boldsymbol{y} \in \mathcal{F}(\mathbb{R}^m)$ with the membership function

$$\mu_{\boldsymbol{y}}(y) = \sup_{a\in\mathbb{R}^n} \mu_{\boldsymbol{z}}(a, y) \qquad (31)$$

that which we call *fuzzy projection* $\boldsymbol{z}$ in $\mathcal{F}(\mathbb{R}^m)$. Thus the application $\hat{P}_m : \mathcal{F}(\mathbb{R}^{n+m}) \to \mathcal{F}(\mathbb{R}^m)$ can be viewed as a *fuzzy projection* $\mathcal{F}(\mathbb{R}^{n+m})$ in $\mathcal{F}(\mathbb{R}^m)$.

Here are some examples:

Example 6 *Let be* $\boldsymbol{a} \in \mathcal{F}(\mathbb{R}^n)$ *and* $\boldsymbol{b} \in \mathcal{F}(\mathbb{R}^m)$. *We can define* $\boldsymbol{z} = (\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{F}(\mathbb{R}^{n+m})$ *with membership function*

$$\mu_{\boldsymbol{z}}(x, y) = \min\{\mu_{\boldsymbol{a}}(x), \mu_{\boldsymbol{b}}(y)\}.$$

The image of $\boldsymbol{z}$ *by applying* $\hat{P}_n$, *in this case, has a membership function:*

$$\mu_{\hat{P}_n(\boldsymbol{z})}(x) = \sup_{v\in\mathbb{R}^m} \min\{\mu_{\boldsymbol{a}}(x), \mu_{\boldsymbol{b}}(v)\}$$

Since that $\min\{\mu_{\boldsymbol{a}}(x), \mu_{\boldsymbol{b}}(v)\} \leq \mu_{\boldsymbol{a}}(x)$ *so*

$$\sup_{v\in\mathbb{R}^m} \min\{\mu_{\boldsymbol{a}}(x), \mu_{\boldsymbol{b}}(v)\} \leq \mu_{\boldsymbol{a}}(x).$$

As $\boldsymbol{b} \in \mathcal{F}(\mathbb{R}^m)$, *so exist* $v \in \mathbb{R}^m$ *so that* $\mu_{\boldsymbol{b}}(v) = 1$. *So, the fuzzy projection* $\boldsymbol{x}$ *of* $\boldsymbol{z}$ *about* $\mathcal{F}(\mathbb{R}^n)$ *has a membership function:*

$$\mu_{\boldsymbol{x}}(x) = \sup_{v\in\mathbb{R}^m} \min\{\mu_{\boldsymbol{a}}(x), \mu_{\boldsymbol{b}}(v)\} = \mu_{\boldsymbol{a}}(x).$$

In figure 1 can see the membership functions of $\boldsymbol{z} \in \mathcal{F}(\mathbb{R}^2)$, *defined from* $\boldsymbol{a}$ *and* $\boldsymbol{b} \in \mathcal{F}(\mathbb{R})$, *and your fuzzy projection in* $\mathcal{F}(\mathbb{R})$, *respectively. In this figure,*

$$\mu_{\boldsymbol{a}}(x) = \mu_{\boldsymbol{b}}(x) = \max\{1 - x^2, 0\}.$$

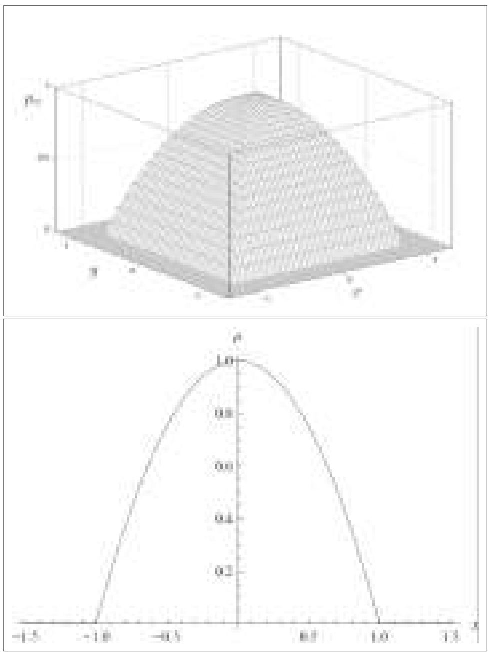

Fig 1. Membership function of z and a respectively

With similar argument we can show that $\boldsymbol{b} \in \mathcal{F}(\mathbb{R}^m)$ *is a fuzzy projection of de* $\boldsymbol{z}$ *in* $\mathcal{F}(\mathbb{R}^m)$.

We can also define $\boldsymbol{x} = (\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{F}(\mathbb{R}^{n+m})$ *through the t - norm product, i.e.,*

$$\mu_{\boldsymbol{z}}(x, y) = \mu_{\boldsymbol{a}}(x)\mu_{\boldsymbol{b}}(y).$$

The projection of $\boldsymbol{z}$ *in* $\mathcal{F}(\mathbb{R}^n)$ *has a membership function:*

$$\sup_{v\in\mathbb{R}^m} \mu_{\boldsymbol{x}}(x, v) = \sup_{v\in\mathbb{R}^m} \mu_{\boldsymbol{a}}(x)\mu_{\boldsymbol{b}}(v) = \mu_{\boldsymbol{a}}(x).$$

Moreover, the projection $\mathcal{F}(\mathbb{R}^m)$ *has a membership function:*

$$\sup_{u\in\mathbb{R}^n} \mu_{\boldsymbol{x}}(u, y) = \sup_{u\in\mathbb{R}^n} \mu_{\boldsymbol{a}}(u)\mu_{\boldsymbol{b}}(y) = \mu_{\boldsymbol{b}}(y).$$

Similarly, we can show that fuzzy projections $\boldsymbol{z} = (\boldsymbol{a}, \boldsymbol{b})$ *in* $\mathcal{F}(\mathbb{R}^n)$ *and* $\mathcal{F}(\mathbb{R}^m)$ *for all t - norm* Δ *are, respectively,* $\boldsymbol{a}$ *and* $\boldsymbol{b}$. *First, for any t - norm* Δ, *we have*

$$\Delta(\mu_{\boldsymbol{a}}(x), \mu_{\boldsymbol{b}}(y)) \leq \Delta(\mu_{\boldsymbol{a}}(x), 1) = \mu_{\boldsymbol{a}}(x).$$

So,

$$\sup_{v\in\mathbb{R}^m} \Delta(\mu_{\boldsymbol{a}}(x), \mu_{\boldsymbol{b}}(v)) \leq \mu_{\boldsymbol{a}}(x).$$

But the ultimate is reached if we take $v \in \mathbb{R}^n$ *so that* $\mu_{\boldsymbol{b}}(v) = 1$. *So, the projection of* $\boldsymbol{z} = (\boldsymbol{a}, \boldsymbol{b})$ *in* $\mathcal{F}(\mathbb{R}^n)$ *has membership function*

$$\mu_{\boldsymbol{x}}(x) = \sup_{v\in\mathbb{R}^m} \Delta(\mu_{\boldsymbol{a}}(x), \mu_{\boldsymbol{b}}(v)) = \mu_{\boldsymbol{a}}(x),$$

for all t - norm Δ.

Example 7 *Consider* $\boldsymbol{z} \in \mathcal{F}(\mathbb{R}^2)$ *determined by membership function*

$$\mu_{\boldsymbol{z}}(x, y) = \max\{1 - x^2 - 2y^2, 0\}.$$

For this case, we have the fuzzy projections $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *on* $\mathcal{F}(\mathbb{R})$ *respectively determined by:*

$$\mu_{\boldsymbol{x}}(x) = \sup_{v \in \mathbb{R}^m} \mu_{\boldsymbol{z}}(x, v) = \max\{1 - x^2, 0\};$$

$$\mu_{\boldsymbol{y}}(y) = \sup_{u \in \mathbb{R}^n} \mu_{\boldsymbol{z}}(u, y) = \max\{1 - 2y^2, 0\}.$$

In Figure 2 we cam see the membership function $\boldsymbol{z}$ *and* $\boldsymbol{x}$ *respectively.*

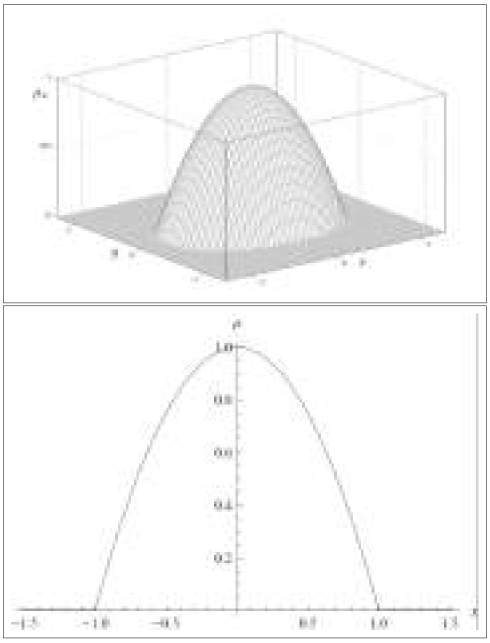

Fig 2. Membership function of $\boldsymbol{z}$ ***e*** $\boldsymbol{x}$ ***respectively***

Proposition 1 *Let be* $\bar{\boldsymbol{x}} = \hat{P}_n(\boldsymbol{x})$ *and* $\bar{\boldsymbol{y}} = \hat{P}_n(\boldsymbol{y})$, *with* $\boldsymbol{x}$ *and* $\boldsymbol{y} \in \mathcal{F}(\mathbb{R}^{n+m})$. *The distance between the fuzzy projections* $\bar{\boldsymbol{x}}$ *and* $\bar{\boldsymbol{y}}$ *is always limited by the distance between* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

Proof. In fact, for all $\alpha \in [0, 1]$ we have:

$$\begin{aligned}
dist([\boldsymbol{x}]^\alpha, [\boldsymbol{y}]^\alpha) &= \sup_{a \in [\boldsymbol{x}]^\alpha} \inf_{b \in [\boldsymbol{y}]^\alpha} \|a - b\| \\
&= \sup_{(a_1,a_2) \in [\boldsymbol{x}]^\alpha} \inf_{(b_1,b_2) \in [\boldsymbol{y}]^\alpha} \sqrt{\|a_1 - b_1\|^2 + \|a_2 - b_2\|^2} \\
&\geq \sup_{(a_1,a_2) \in [\boldsymbol{x}]^\alpha} \inf_{(b_1,b_1) \in [\boldsymbol{y}]^\alpha} \sqrt{\|a_1 - b_1\|^2} \\
&= \sup_{a_1 \in [\bar{\boldsymbol{x}}]^\alpha} \inf_{b_1 \in [\bar{\boldsymbol{y}}]^\alpha} \sqrt{\|a_1 - b_1\|^2} \\
&= dist([\bar{\boldsymbol{x}}]^\alpha, [\bar{\boldsymbol{y}}]^\alpha).
\end{aligned}$$

We can prove that $dist([\boldsymbol{y}]^\alpha, [\boldsymbol{x}]^\alpha) \geq dist([\bar{\boldsymbol{y}}]^\alpha, [\bar{\boldsymbol{x}}]^\alpha)$. Therefore,

$$d_\infty(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \leq d_\infty(\boldsymbol{x}, \boldsymbol{y}).\square$$

The fuzzy projection $\bar{\boldsymbol{p}} \in \mathcal{F}(\mathbb{R}^n)$ to a point $\boldsymbol{p} \in \mathcal{F}(\mathbb{R}^{n+m})$ still satisfies another important metric property when it comes to projections. Namely, the projection $\bar{\boldsymbol{p}}$ is the point that minimizes the distance between the point $\boldsymbol{p} \in \mathcal{F}(\mathbb{R}^{n+m})$ and the set $\mathcal{F}(\mathbb{R}^n)$, the latter, seen as a subset of $\mathcal{F}(\mathbb{R}^{n+m})$.

Proposition 2 *The fuzzy projection* $\bar{\boldsymbol{p}}$ *in* $\mathcal{F}(\mathbb{R}^n)$ *of* $\boldsymbol{p} \in \mathcal{F}(\mathbb{R}^{n+m})$ *is such that*

$$d_\infty(\boldsymbol{p}, \bar{\boldsymbol{p}}) = \inf_{\boldsymbol{z} \in \mathcal{F}(\mathbb{R}^n)} d_\infty(\boldsymbol{p}, \boldsymbol{z}).$$

Proof. First, let us note the abuse of notation in the statement. The term $d_\infty(\boldsymbol{p}, \boldsymbol{z})$ only makes sense because we can see $\mathcal{F}(\mathbb{R}^n)$ as a subset of $\mathcal{F}(\mathbb{R}^{n+m})$. Provided that, $[\boldsymbol{p}]^\alpha \subset \mathbb{R}^{n+m}$ and $[\bar{\boldsymbol{p}}]^\alpha \subset \mathbb{R}^n$, for $x \in \mathbb{R}^n$ and $y = (y_1, y_2) \in \mathbb{R}^{n+m}$ we have

$$\|x - y\| = \sqrt{\|x - y_1\|^2 + \|y_2\|^2},$$

since:

$$\begin{aligned}
dist([\boldsymbol{p}]^\alpha, [\bar{\boldsymbol{p}}]^\alpha) &= \sup_{y \in [\boldsymbol{p}]^\alpha} \inf_{x \in [\bar{\boldsymbol{p}}]^\alpha} \sqrt{\|y_1 - x\|^2 + \|y_2\|^2} \\
&= \sup_{y \in [\boldsymbol{p}]^\alpha} \|y_2\|.
\end{aligned}$$

Moreover, we have:

$$dist([\bar{\boldsymbol{p}}]^\alpha, [\boldsymbol{p}]^\alpha) = \sup_{x \in [\bar{\boldsymbol{p}}]^\alpha} \inf_{y \in [\boldsymbol{p}]^\alpha} \sqrt{\|y_1 - x\|^2 + \|y_2\|^2}.$$

Now, since $x \in [\bar{\boldsymbol{p}}]^\alpha$, so $(x, z) \in [\boldsymbol{p}]^\alpha$ for some $z \in \mathbb{R}^m$, where we have the inequality:

$$\begin{aligned}
dist([\bar{\boldsymbol{p}}]^\alpha, [\boldsymbol{p}]^\alpha) &= \sup_{x \in [\bar{\boldsymbol{p}}]^\alpha} \inf_{y \in [\boldsymbol{p}]^\alpha} \sqrt{\|y_1 - x\|^2 + \|y_2\|^2} \\
&\leq \|z\| \leq \sup_{y \in [\boldsymbol{p}]^\alpha} \|y_2\|.
\end{aligned}$$

Thus, the Hausdorff distance between $[\boldsymbol{p}]^\alpha$ and $[\bar{\boldsymbol{p}}]^\alpha$ in this case is:

$$\begin{aligned}
d_H([\boldsymbol{p}]^\alpha, [\bar{\boldsymbol{p}}]^\alpha) &= \max\{dist([\boldsymbol{p}]^\alpha, [\bar{\boldsymbol{p}}]^\alpha), dist([\bar{\boldsymbol{p}}]^\alpha, [\boldsymbol{p}]^\alpha)\} \\
&= dist([\boldsymbol{p}]^\alpha, [\bar{\boldsymbol{p}}]^\alpha).
\end{aligned}$$

Let be $\boldsymbol{q} \in \mathcal{F}(\mathbb{R}^n)$ such that $\boldsymbol{q} \neq \bar{\boldsymbol{p}}$. This implies that $[\boldsymbol{q}]^\alpha \neq [\bar{\boldsymbol{p}}]^\alpha$, for some $\alpha \in [0, 1]$. Consequently, there $y = (y_1, y_2) \in [\boldsymbol{p}]^\alpha$ such that $y_1 \notin [\boldsymbol{q}]^\alpha$ or exists $z_1 \in [\boldsymbol{q}]^\alpha$ such that $z = (z_1, z_2) \notin [\boldsymbol{p}]^\alpha$, for all $z_2 \in \mathbb{R}^m$. Namely, $z_1 \notin [\bar{\boldsymbol{p}}]^\alpha$. For the first case, we have

$$\sqrt{\|x - y_1\|^2 + \|y_2\|^2} > \|y_2\|$$

for all $x \in [\boldsymbol{q}]^\alpha$. For the second property follows directly from the projection inequality

$$\|z_1 - y\| = \sqrt{\|z_1 - y_1\|^2 + \|y_2\|^2} > \|y_2\|,$$

for all $y = (y_1, y_2) \in [\boldsymbol{p}]^\alpha$. Thus in both cases we have to

$$\begin{aligned}
dist([\boldsymbol{p}]^\alpha, [\boldsymbol{q}]^\alpha) &= \sup_{y \in [\boldsymbol{p}]^\alpha} \inf_{x \in [\bar{\boldsymbol{p}}]^\alpha} \sqrt{\|y_1 - x\|^2 + \|y_2\|^2} \\
&\geq \sup_{y \in [\boldsymbol{p}]^\alpha} \|y_2\| \\
&= dist([\boldsymbol{p}]^\alpha, [\bar{\boldsymbol{p}}]^\alpha).
\end{aligned}$$

Therefore, we have $d_H([\boldsymbol{p}]^\alpha, [\boldsymbol{q}]^\alpha) \geq d_H([\boldsymbol{p}]^\alpha, [\bar{\boldsymbol{p}}]^\alpha)$. Thus, we can conclude that for all $\boldsymbol{q} \in \mathcal{F}(\mathbb{R}^n)$, $d_\infty(\boldsymbol{p}, \boldsymbol{q}) \geq d_\infty(\boldsymbol{p}, \bar{\boldsymbol{p}})$, which proves the assertion.$\square$

We can also define fuzzy projections $\boldsymbol{z} \in \mathcal{F}(U \times P)$ in $\mathcal{F}(U)$ and $\mathcal{F}(P)$, where $U \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^m$. In this case, the supreme in membership functions (30) and (31) is taken on the sets P and U, respectively and properties shown above metrics remain valid.

We can also consider the projection $\pi_i : \mathbb{R}^n \to \mathbb{R}$ from a point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ in i - th coordinate axis, i.e. $\pi_i(x) = x_i$. As before, the projection of a Zadeh's extention π_i defines the application $\hat{\pi}_i : \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R})$ that we call for the i - *th fuzzy projection* of $\mathcal{F}(\mathbb{R}^n)$ on $\mathcal{F}(\mathbb{R})$. Thus, given a point $\boldsymbol{x} \in \mathcal{F}(\mathbb{R})$, the i - th fuzzy projection of $\boldsymbol{x}$ on $\mathcal{F}(\mathbb{R})$ is a point $\boldsymbol{x}_i$ with membership function given by:

$$\mu_{\boldsymbol{x}_i}(a) = \sup_{\substack{x \in \mathbb{R}^n \\ x_i = a}} \mu_{\boldsymbol{x}}(x). \tag{32}$$

Again, if $\boldsymbol{x} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n)$ is defined by fuzzy cartesian product so that the value i - th fuzzy projection of $\boldsymbol{x} \in \mathcal{F}(\mathbb{R}^n)$ in $\mathcal{F}(\mathbb{R})$ is a point $\boldsymbol{a}_i$. For simplicity, consider $\boldsymbol{x} \in \mathbb{R}^3$ defined by

$$\mu_{\boldsymbol{x}}(x, y, z) = \Delta(\Delta(\mu_{\boldsymbol{a}_1}(x), \mu_{\boldsymbol{a}_2}(y)), \mu_{\boldsymbol{a}_3}(z)).$$

By the properties of t - norm, it folllows that

$$\begin{aligned} \Delta(\Delta(\mu_{\boldsymbol{a}_1}(x), \mu_{\boldsymbol{a}_2}(y)), \mu_{\boldsymbol{a}_3}(z)) &\leq \Delta(\Delta(\mu_{\boldsymbol{a}_1}(x), \mu_{\boldsymbol{a}_2}(y)), 1) \\ &= \Delta(\mu_{\boldsymbol{a}_1}(x), \mu_{\boldsymbol{a}_2}(y)) \\ &\leq \mu_{\boldsymbol{a}_2}(y), \end{aligned}$$

for all $x, y, z \in \mathbb{R}$.

Thus, the second fuzzy projection $\boldsymbol{x}$ on $\mathcal{F}(\mathbb{R})$ is the point $\boldsymbol{x}_2$ where the membership function is:

$$\mu_{\boldsymbol{x}_2}(a) = \sup_{\substack{x \in \mathbb{R}^3 \\ x_2 = a}} \mu_{\boldsymbol{x}}(x).$$

For the previous inequality, we have

$$\mu_{\boldsymbol{x}_2}(a) = \sup_{\substack{x \in \mathbb{R}^3 \\ x_2 = a}} \mu_{\boldsymbol{x}}(x) \leq \mu_{\boldsymbol{a}_2}(a).$$

Taking $\bar{x}$ and $\bar{z}$ such that $\mu_{\boldsymbol{a}_1}(\bar{x}) = \mu_{\boldsymbol{a}_3}(\bar{z}) = 1$, equality is attained in the supreme and hence,

$$\mu_{\boldsymbol{x}_2}(a) = \sup_{\substack{x \in \mathbb{R}^3 \\ x_2 = a}} \mu_{\boldsymbol{x}}(x) = \mu_{\boldsymbol{a}_2}(a).$$

Induction proves the general case in which $\boldsymbol{x} \in \mathcal{F}(\mathbb{R}^n)$.

Through expression (30) can determine the α - levels of fuzzy projection $\boldsymbol{x} \in \mathcal{F}(\mathbb{R}^n)$ to a point $\boldsymbol{z} \in \mathcal{F}(\mathbb{R}^{n+m})$. Indeed, if $\mu_{\boldsymbol{x}}(x) \geq \alpha$ so exist $y \in \mathbb{R}^m$ such that $\mu_{\boldsymbol{z}}(x, y) \geq \alpha$ so that $(x, y) \in [\boldsymbol{z}]^\alpha$. The reciprocal is also true, because if $\mu_{\boldsymbol{z}}(x, y) \geq \alpha$ then by (30), $\mu_{\boldsymbol{x}}(x) \geq \alpha$. Thus, we conclude that:

$$x \in [\boldsymbol{x}]^\alpha \iff (x, y) \in [\boldsymbol{z}]^\alpha \quad \text{for some } y \in \mathbb{R}^m,$$

or

$$[\boldsymbol{x}]^\alpha = \{x \in \mathbb{R}^n : (x, y) \in [\boldsymbol{z}]^\alpha\}. \tag{33}$$

Since applying π_i is continuous, we can use the equality (29) to show that the i - th fuzzy projection $\boldsymbol{x}_i \in \mathcal{F}(\mathbb{R})$ of $\boldsymbol{x} \in \mathcal{F}(\mathbb{R}^n)$ has α - levels:

$$[\boldsymbol{x}_i]^\alpha = \{a \in \mathbb{R} : x \in [\boldsymbol{x}]^\alpha, \ x_i = a\}. \tag{34}$$

3. Projection of fuzzy solutions

3.1 Projection on the coordinate axes

Consider the flow $\varphi_t : U \subset \mathbb{R}^n \to U$ generated by the autonomous equation

$$\frac{dx}{dt} = f(x)$$

and is $\varphi_t^{(i)} : U \to \mathbb{R}$ the projection of the deterministic flow i - th coordinate axis, i.e. $\varphi_t^{(i)}(x_o)$ is the i - th solution component $\varphi_t(x_o)$ or even, $\varphi_t^{(i)}(x_o)$ is the solution of the equation

$$\frac{dx_i}{dt} = f_i(x), \qquad x(0) = x_o.$$

The extent of the application Zadeh $\varphi_t^{(i)}$ define the application $\hat{\varphi}_t^{(i)} : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ that for each $\boldsymbol{x}_o \in \mathcal{F}(U)$ associates the image $\hat{\varphi}_t^{(i)}(\boldsymbol{x}_o) \in \mathcal{F}(\mathbb{R})$. As in the deterministic case, we show that the application $\hat{\varphi}_t(i) : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ ia a i - th fuzzy projection to fuzzy flow $\hat{\varphi}_t : \mathcal{F}(U) \to \mathcal{F}(U)$ on $\mathcal{F}(\mathbb{R})$.

Proposition 3 *The application $\hat{\varphi}_t^{(i)} : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ is i-th fuzzy projection of fuzzy flow $\hat{\varphi}_t : \mathcal{F}(U) \to \mathcal{F}(U)$ on $\mathcal{F}(\mathbb{R})$.*

Proof. Let be $\boldsymbol{x}_o \in \mathcal{F}(U)$. By definition, i-th fuzzy projection $\hat{\varphi}_t(\boldsymbol{x}_o)$ on $\mathcal{F}(\mathbb{R})$ is the point $\boldsymbol{x}_i = \hat{\pi}_i(\hat{\varphi}_t(\boldsymbol{x}_o))$. Since the projection is a continuous map, then it is worth:

$$\begin{aligned} [\boldsymbol{x}_i]^\alpha &= \pi_i(\varphi_t([\boldsymbol{x}_o]^\alpha)) = \{\pi_i(\varphi_t(x_o)) : \ x_o \in [\boldsymbol{x}_o]^\alpha\} \\ &= \{\varphi_t^{(i)}(x_o) : \ x_o \in [\boldsymbol{x}_o]^\alpha\} \\ &= \varphi_t^{(i)}([\boldsymbol{x}_o]^\alpha). \end{aligned}$$

So, $[\hat{\varphi}_t^{(i)}(\boldsymbol{x}_o)]^\alpha = [\boldsymbol{x}_i]^\alpha$ for all $\alpha \in [0, 1]$ and the assertion is proved. □

We showed in [4] that the equilibrium point x_e deterministic flow $\varphi_t : U \to U$ depends on the initial condition $x_o \in U$, then the equilibrium point for flow fuzzy $\hat{\varphi}_t : \mathcal{F}(U) \to \mathcal{F}(U)$ is obtained by extension of Zadeh $x_e : U \to U$. Let be $x_e^{(i)}(x_o)$ a i-th coordinated of equilibrium point x_e. Similarly, we can prove that i-th projected of the equilibrium point fuzzy $\boldsymbol{x}_e = \hat{x}_e(\boldsymbol{x}_o) \in \mathcal{F}(U)$ is the point $\bar{\boldsymbol{x}}_i = \hat{x}_e^{(i)}(\boldsymbol{x}_o) \in \mathcal{F}(\mathbb{R})$ where $\hat{x}_e^{(i)} : \mathcal{F}(U) \to \mathcal{F}(\mathbb{R})$ is a extension of Zadeh of $x_e^{(i)} : U \to \mathbb{R}$. More briefly, for $\boldsymbol{x}_o \in \mathcal{F}(U)$, the bellow equality holds

$$\mu_{\bar{\boldsymbol{x}}_i}(x) = \mu_{\hat{x}_e^{(i)}(\boldsymbol{x}_o)}(x) \tag{35}$$

where $\bar{\boldsymbol{x}}_i$ is i-th fuzzy projection of the fuzzy equilibrium point $\boldsymbol{x}_e$.

Consider just a few examples of the results presented above.

Example 8 *The autonomous equation*

$$\begin{cases} \dfrac{dx_1}{dt} = x_2, & x_1(0) = x_{01}, \\ \\ \dfrac{dx_2}{dt} = -x_1, & x_2(0) = x_{02}, \end{cases}$$

determines the flow two-dimensional $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi_t = (\varphi_t^{(1)}, \varphi_t^{(2)})$, *given by:*

$$\varphi_t^{(1)}(x_{01}, x_{02}) = x_{01} \cos t + x_{02} \sin t;$$

$$\varphi_t^{(2)}(x_{01}, x_{02}) = x_{02} \cos t - x_{01} \sin t.$$

In [4] show that the fuzzy solution $\hat{\varphi}_t(\boldsymbol{x}_o)$ *this equation is periodic for any choice of initial condition* $\boldsymbol{x}_o \in \mathcal{F}(\mathbb{R}^2)$. *According to the previous proposition, projections of fuzzy* $\hat{\varphi}_t : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R}^2)$ *on* $\mathcal{F}(\mathbb{R})$ *are obtained by taking extensions of components Zadeh* $\varphi_t^{(1)}$ *and* $\varphi_t^{(2)}$.

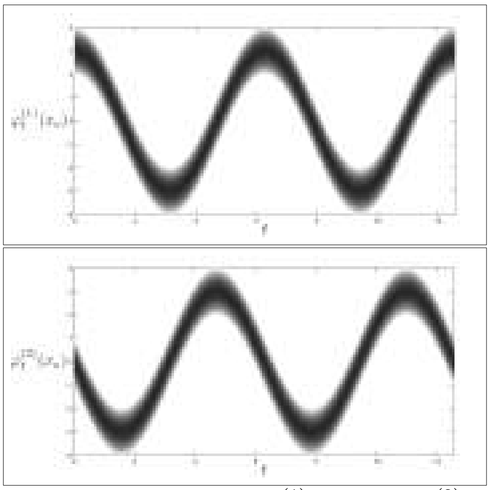

Fig 3. Time course of $\varphi_t^{(1)}(\boldsymbol{x}_o)$ ***and*** $\varphi_t^{(2)}(\boldsymbol{x}_o)$ ***respectively***

The Figure 3 shows the time evolution of the fuzzy projection of $\hat{\varphi}_t(\boldsymbol{x}_o)$ *on* x *and* y *respectively. Take the initial condition* x_0 *defined by the membership function:*

$$\mu_{\boldsymbol{x}_o}(x, y) = \max\{1 - (x-3)^2 - y^2, 0\}.$$

The graphical representation of fuzzy projections of this work is established as follows: given a $\alpha \in [0,1]$ the region in plan bounded by α-level $\hat{\varphi}^{(i)}_{[0,T]}(\boldsymbol{x}_o)$ is filled with a shade of gray. if $\alpha = 0$ then the region bounded by $\hat{\varphi}^{(i)}_{[0,T]}(\boldsymbol{x}_o)$ is filled with the white color, whereas if $\alpha = 1$ then the region bounded by $\hat{\varphi}^{(i)}_{[0,T]}(\boldsymbol{x}_o)$ is filled with black. Thus, the larger the degree of membership of a point x, the darker its color.

4. Parameters and initial condition fuzzy

In [7] include the problem of uncertainty in the parameters of a given autonomous equation is solved the strategy to consider such parameters as the initial condition of an equation with dimension higher than the original. More precisely, given an autonomous equation that depends on a parameter vector $p_o \in P \subset \mathbb{R}^m$

$$\frac{dx}{dt} = f(x, p_o), \quad x(0) = x_o \tag{36}$$

define the equation associated with increased

$$\begin{cases} \dfrac{dx}{dt} = f(x, p), & x(0) = x_o; \\ \\ \dfrac{dp}{dt} = 0, & p(0) = p_o, \end{cases} \tag{37}$$

and thus the parameter vector $p_o \in P \subset \mathbb{R}^m$ now is a part of the initial condition. Thus, the extension of Zadeh $\hat{\psi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U \times P)$ to the flow $\psi_t : U \times P \to U \times P$ generated by equation (37) incorporates the uncertainties of initial conditions and parameters of the equation (36).

Once the solution $\varphi_t : U \times P \to U$ generated by equation (36) is contínuos in the initial condition and parameters, so the extention of Zadeh $\hat{\varphi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U)$ of $\varphi_t(x_o, p)$ is well defined and according to the equation (29), for all $\boldsymbol{y}_o \in \mathcal{F}(U \times P)$ we have:

$$[\hat{\varphi}_t(\boldsymbol{y}_o)]^\alpha = \varphi_t([\boldsymbol{y}_o]^\alpha).$$

From the standpoint of applications, it is important to know the flow behavior of the deterministic phase space $U \subset \mathbb{R}^n$ of equation (36) instead space $U \times P \subset \mathbb{R}^{n+m}$ to the equation (37), since the flow components $\psi_t : U \times P \to U \times P$, that are $P \subset \mathbb{R}^m$, not have any additional information. It is worth noting that for all $y_o = (x_o, p_o)$, we have

$$P_n(\psi_t(y_o)) = P_n(\varphi_t(x_o, p_o), p_o) = \varphi_t(y_o).$$

Analogously to the deterministic case, we can also be interested only in the fuzzy flow behavior $\hat{\psi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U \times P)$ on the phase space $\mathcal{F}(U)$. The fuzzy projections defined at the outset of this work can then be used to obtain the fuzzy flow behavior $\hat{\psi}_t$ on the space $\mathcal{F}(U)$.

The following statement characterizes the relationship between the projection of fuzzy $\hat{\psi}_t$ on the space $\mathcal{F}(U)$ and the extension of Zadeh $\hat{\varphi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U)$ is a solution of equation (36).

Proposition 4 *The aplication* $\hat{\varphi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U)$, *given of the extension of zadeh* $\varphi_t : U \times P \to U$, *is the fuzzy projection of fuzzy flow* $\hat{\psi}_t : \mathcal{F}(U \times P) \to \mathcal{F}(U \times P)$ *on* $\mathcal{F}(U)$.

Proof. Let be $\boldsymbol{y}_o \in \mathcal{F}(U \times P)$ and fix $t \geq 0$. To prove the claim, it suffices to show that $\boldsymbol{y}$ is the fuzzy projection of $\hat{\psi}_t(\boldsymbol{y}_o)$ of $\mathcal{F}(U)$ so it is iqualdade $\boldsymbol{y} = \hat{\varphi}_t(\boldsymbol{y}_o)$.

To simplify, let be $Im(\varphi_t)$ the image set of $\varphi_t : U \times P \to U$. By definition, the membership function of $\hat{\varphi}_t(\boldsymbol{y}_o)$ is given by

$$\mu_{\hat{\varphi}_t(\boldsymbol{y}_o)}(x) = \begin{cases} \displaystyle\sup_{\substack{(x_o, p_o) \\ \varphi_t(x_o, p_o) = x}} \mu_{\boldsymbol{y}_o}(x_o, p_o) & \text{if } x \in Im(\varphi_t) \\ \\ 0 & \text{if } x \notin Im(\varphi_t). \end{cases} \tag{38}$$

Let be $\boldsymbol{y} \in \mathcal{F}(U)$ the projection of $\hat{\psi}_t(\boldsymbol{y}_o)$ on $\mathcal{F}(U)$. By definition of fuzzy projection, the membership function of $\boldsymbol{y} \in \mathcal{F}(U)$ is given by

$$\mu_{\boldsymbol{y}}(x) = \sup_{p \in P} \mu_{\hat{\psi}_t(\boldsymbol{y}_o)}(x,p)$$

Now, as $\psi_t(x_o,p_o) = (\varphi_t(x_o,p_o),p_o)$, so $x \in Im(\varphi_t)$ if, and only if, $(x,p) \in Im(\psi_t)$ for some $p \in P$. So, for all $x \in Im(\varphi_t)$, the membership function of $\hat{\psi}_t(\boldsymbol{y}_o)$ is:

$$\begin{aligned}\mu_{\hat{\psi}_t(\boldsymbol{y}_o)}(x,p) &= \sup_{\psi_t(x_o,y)=(x,p)} \mu_{\boldsymbol{y}_o}(x_o,y)\\ &= \sup_{\substack{\varphi_t(x_o,y)=x\\ y=p}} \mu_{\boldsymbol{y}_o}(x_o,y)\\ &= \sup_{\substack{x_o \in U\\ \varphi_t(x_o,p)=x}} \mu_{\boldsymbol{y}_o}(x_o,p).\end{aligned}$$

If $x \notin Im(\varphi_t)$ so $(x,p) \notin Im(\psi_t)$ for all $p \in P$ and so, $\mu_{\hat{\psi}_t(\boldsymbol{y}_o)}(x,p) = 0$.

But, the fuzzy projection of $\hat{\psi}_t(\boldsymbol{y}_o)$ on $\mathcal{F}(U)$ has the membership function

$$\begin{aligned}\sup_{p \in P} \mu_{\hat{\psi}_t(\boldsymbol{y}_o)}(x,p) &= \sup_{p \in P} \sup_{\substack{x_o \in U\\ \varphi_t(x_o,p)=x}} \mu_{\boldsymbol{y}_o}(x_o,p)\\ &= \sup_{\substack{(x_o,p)\\ \varphi_t(x_o,p)=x}} \mu_{\boldsymbol{y}_o}(x_o,p).\end{aligned}$$

Now, by definition, the point $\hat{\varphi}_t(\boldsymbol{y}_o) \in \mathcal{F}(U)$ has the membership function

$$\mu_{\hat{\varphi}_t(\boldsymbol{y}_o)}(x) = \sup_{\substack{(x_o,p)\\ \varphi_t(x_o,p)=x}} \mu_{\boldsymbol{y}_o}(x_o,p).$$

So, for all $x \in U$, value equality

$$\mu_{\boldsymbol{y}}(x) = \sup_{p \in P} \mu_{\hat{\psi}_t(\boldsymbol{y}_o)}(x,p) = \mu_{\hat{\varphi}_t(\boldsymbol{y}_o)}(x),$$

which proves the assertion. □

The proof of the proposition can also be made through the α-levels. In fact, we must show that

$$\hat{\varphi}_t(\boldsymbol{y}_o) = \hat{P}_n(\hat{\psi}_t(\boldsymbol{y}_o))$$

for all $\boldsymbol{y}_o \in \mathcal{F}(U \times P)$ and $t \in \mathbb{R}_+$. Using the continuity of applications P_n and ψ_t, we have

$$\begin{aligned}[\hat{P}_n(\hat{\psi}_t(\boldsymbol{y}_o))]^\alpha &= P_n(\hat{\psi}_t([\boldsymbol{y}_o]^\alpha))\\ &= \{P_n(\hat{\psi}_t(y_o)) :\ y_o \in [\boldsymbol{x}_o]^\alpha\}\\ &= \{P_n(\varphi_t(x_o,p_o),p_o) :\ (x_o,p_o) \in [\boldsymbol{y}_o]^\alpha\}\\ &= \{\varphi_t(x_o,p_o) :\ (x_o,p_o) \in [\boldsymbol{y}_o]^\alpha\}\\ &= \varphi_t([\boldsymbol{y}_o]^\alpha),\end{aligned}$$

for all $\alpha \in [0,1]$. The above equality concludes the proof proposition.

Unlike proposed by [6] and [8], when the equation depends on parameters such as the equation (36), the solution proposed by fuzzy Feuring and Buckley in [2] is obtained by extension of Zadeh flow deterministic $\varphi_t(x_o,p_o)$. This way, the proposition 4 ensures that the solution of fuzzy Buckley and Feuring is the fuzzy projection of the fuzzy solution proposed by [6] and [8].

Consider subjective parameters in equation (36) contributes to an increase in uncertainty, in the sense that we will see below, the flow fuzzy. Set a parameter $\bar{p} \in P$ and given a fuzzy initial condition $\boldsymbol{x}_o$, the α-levels to the fuzzy flow generated by equation (36) are the sets

$$[\hat{\varphi}_t(\boldsymbol{x}_o)]^\alpha = \{\varphi_t(x_o,\bar{p}) :\ x_o \in [\boldsymbol{x}_o]^\alpha\}.$$

On the other hand, if the α-levels of $\boldsymbol{p}_o \in \mathcal{F}(P)$ contains $\bar{p}$ so, by proposition 4, we have:

$$[\hat{\varphi}_t(\boldsymbol{x}_o,\boldsymbol{p}_o)]^\alpha = \{\varphi_t(x_o,p_o) :\ (x_o,p_o) \in [\boldsymbol{x}_o]^\alpha \times [\boldsymbol{p}_o]^\alpha\}.$$

So we have

$$[\hat{\varphi}_t(\boldsymbol{x}_o)]^\alpha \subseteq [\hat{\varphi}_t(\boldsymbol{x}_o,\boldsymbol{p}_o)]^\alpha. \tag{39}$$

Example 9 *Consider the case where the parameter k_o in the equation*

$$\frac{dx}{dt} = \beta(k_o - x)$$

is a fuzzy parameter. In the above equation, the solution $\varphi_t : \mathbb{R}^2 \to \mathbb{R}$, in terms of x_o e k_o, is given by

$$\varphi_t(x_o,k_o) = k_o + (x_o - k_o)e^{-\beta t}$$

and thus the flow 2-dimensional $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$, for the case in which the parameter is incorporated into the initial condition is given by

$$\psi_t(x_o,k_o) = (k_o + (x_o - k_o)e^{-\beta t}, k_o).$$

According to proposition 4, the extent of Zadeh $\hat{\varphi}_t : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R})$ of φ_t is the projection of $\mathcal{F}(\mathbb{R})$ fuzzy flow $\hat{\psi}_t : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R}^2)$. To illustrate, consider $\boldsymbol{y}_o \in \mathcal{F}(\mathbb{R}^2)$. By definition, we have:

$$\hat{\varphi}_t([\boldsymbol{y}_o]^\alpha) = \{\varphi_t(x_o,p_o) :\ (x_o,p_o) \in [\boldsymbol{y}_o]^\alpha\}.$$

Moreover, the projection $\hat{P}_1(\hat{\psi}_t(\boldsymbol{y}_o))$ has α-levels given by:

$$\begin{aligned}[\hat{P}_n(\hat{\psi}_t(\boldsymbol{y}_o))]^\alpha &= P_1(\psi_t(()[\boldsymbol{y}_o]^\alpha))\\ &= \{P_1(\psi_t(x_o,p_o)) :\ (x_o,p_o) \in [\boldsymbol{y}_o]^\alpha\}\\ &= \{P_1(\varphi_t(x_o,p_o),p_o) :\ (x_o,p_o) \in [\boldsymbol{y}_o]^\alpha\}\\ &= \{\varphi_t(x_o,p_o) :\ (x_o,p_o) \in [\boldsymbol{y}_o]^\alpha\},\end{aligned}$$

from which we conclude that $[\hat{P}_n(\hat{\psi}_t(\boldsymbol{y}_o))]^\alpha = \hat{\varphi}_t([\boldsymbol{y}_o]^\alpha)$ and consequently,

$$\hat{P}_1(\hat{\psi}_t(\boldsymbol{y}_o)) = \hat{\varphi}_t(\boldsymbol{y}_o).$$

For any initial condition $\boldsymbol{y}_o \in \mathcal{F}(\mathbb{R}^2)$, we show that $\hat{\psi}_t$ converges to the equilibrium points $\boldsymbol{y}_e$ which is the extension of Zadeh $y_e : \mathbb{R}^2 \to \mathbb{R}^2$ given by $y_e(x_o,k_o) = (k_o,k_o)$. That is, the equilibrium point $\boldsymbol{y}_e$ has membership function

$$\mu_{\boldsymbol{y}_e}(x,k) = \min\left\{\chi_{\{k\}}(x), \sup_{x_o} \mu_{\boldsymbol{y}_o}(x_o,k)\right\}.$$

In particular, if $\boldsymbol{y}_o = (\boldsymbol{x}_o,\boldsymbol{k}_o)$ is the fuzzy cartesian product of $\boldsymbol{x}_o$ and $\boldsymbol{k}_o \in \mathcal{F}(\mathbb{R})$, then the membership function in this case is given by

$$\mu_{\boldsymbol{y}_e}(x,k) = \sup_{x_o} \mu_{\boldsymbol{y}_o}(x_o,k) = \sup_{x_o} \Delta(\mu_{\boldsymbol{x}_o}(x_o),\mu_{\boldsymbol{k}_o}(x)) = \mu_{\boldsymbol{k}_o}(x)$$

when $x = k$ *and* $\mu_{\boldsymbol{y}_e}(x,k) = 0$ *when* $x \neq k$.

The projection $\bar{\boldsymbol{x}} \in \mathcal{F}(\mathbb{R})$ *for this equilibrium point has membership function*

$$\mu_{\bar{\boldsymbol{x}}}(x) = \sup_{k\in\mathbb{R}} \mu_{\boldsymbol{y}_e}(x,k) = \mu_{\boldsymbol{k}_o}(x)$$

and we have $d_\infty(\hat{\varphi}_t(\boldsymbol{y}_o), \bar{\boldsymbol{x}}) \to 0$ *as* $t \to \infty$.

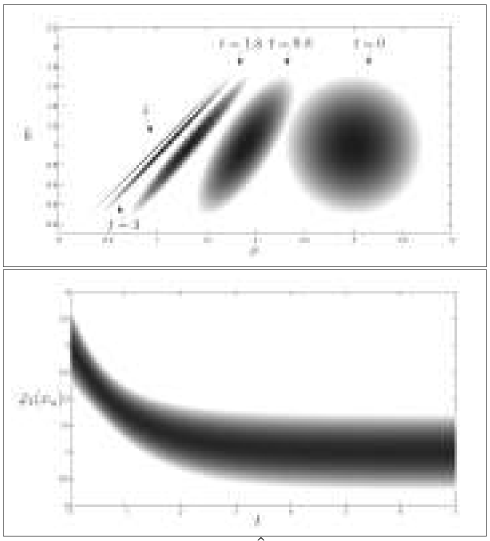

Fig 4. (a) - Evolution of $\hat{\psi}_t(\boldsymbol{y}_o)$. ***(b) - Evolution of fuzzy projection of*** $\hat{\varphi}_t(\boldsymbol{y}_o)$.

In figure 4, we have the graphical representation of the fuzzy solution $\hat{\psi}_t(\boldsymbol{y}_o)$ *and its fuzzy projection* $\hat{\varphi}_t(\boldsymbol{y}_o)$.

5. Conclusions

In this paper we define the concept of fuzzy projections and study some of its main properties, in addition, establish some results on projections of fuzzy differential equations. As we have seen, different concepts of fuzzy solutions of differential equations are related by fuzzy projections. Importantly, by means of fuzzy projections, we can analyze the evolution of fuzzy solutions over time.

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REPORT

A Fuzzy Decision Tree Algorithm Based on C4.5

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Decision trees have been successfully applied to many areas for tasks such as classification, regression, and feature subset selection. Decision trees are popular models in machine learning due to the fact that they produce graphical models, as well as text rules, that end users can easily understand. Moreover, their induction process is usually fast, requiring low computational resources. Fuzzy systems, on the other hand, provide mechanisms to handle imprecision and uncertainty in data, based on the fuzzy logic and fuzzy sets theory. The combination of fuzzy systems and decision trees has produced fuzzy decision tree models, which benefit from both techniques to provide simple, accurate, and highly interpretable models at low computational costs. In this paper, we expand previous experiments and present more details of the FuzzyDT algorithm, a fuzzy decision tree based on the classic C4.5 decision tree algorithm. Experiments were carried out using 16 datasets comparing FuzzyDT with C4.5. This paper also includes a comparison of some relevant issues regarding the classic and fuzzy models.

1. Introduction

Machine learning is concerned with the development of methods for the extraction of patterns from data in order to make intelligent decisions based on these patterns. A relevant concern related to machine learning methods is the issue of interpretability, which is highly desirable for end users. In this sense, Decision Trees (DT) [12] are powerful and popular models for machine learning since they are easily understandable, quite intuitive, and produce graphical models that can also be expressed as rules. The induction process of DTs is usually fast and the induced models are usually competitive with the ones generated by other interpretable machine learning methods. Also, DTs performs an embedded feature selection during their induction process.

Fuzzy systems, on the other hand, have also been successfully applied in many areas covered by machine learning. Fuzzy systems can handle uncertainty and imprecision by means of the fuzzy logic and fuzzy sets theories, producing interpretable models. A system can be considered 'fuzzy' if at least one of its attributes is defined by fuzzy sets, according to the fuzzy logic and fuzzy sets theory proposed by prof. Zadeh [9, 19]. A fuzzy system is usually composed of a Knowledge Base (KB) and an Inference Mechanism (IM). The KB contains a Fuzzy Rule Base (FRB) and a Fuzzy Data Base (FDB). The FRB has the rules that form the core of the system. These rules are constructed based on the fuzzy sets defining the attributes of the system, stored in the FDB. The FDB and FRB are used by the IM to classify new examples.

Regarding DTs, the ID3 [11], CART [1], and C4.5 [13] algorithms are among the most relevant ones. Fuzzy DTs have also been proposed in the literature [2, 7, 8, 10, 14]. Fuzzy DTs combine the powerful models of DTs with the interpretability and ability of processing uncertainty and imprecision of fuzzy systems. Moreover, fuzzy DTs inherit the desirable characteristics of DTs regarding their low computational induction cost, as well as the possibility of expressing the induced models graphically and as a set of rules.

In this work, we describe our fuzzy version of C4.5, named FuzzyDT [3], which has been applied for the induction of classifiers, presenting expanded experiments and further details on its algorithm. FuzzyDT has also been applied to a real-world problem, the prediction and control of the coffee rust disease in Brazilian crops [4]. This paper includes the experimental evaluation of FuzzyDT and C4.5 considering 16 datasets and a 10-fold cross-validation strategy.

The remainder of this paper is organized as follows. Section 2 introduces the fuzzy classification systems. Section 3 discusses DTs. The FuzzyDT algorithm is described in Section 4. Section 5 presents a comparison between classic and fuzzy DTs. Section 6 describes the experiments and comparisons, followed by the conclusions and future work in Section 7.

2. Fuzzy Classification Systems

Classification is a relevant task of machine learning that can be applied to pattern recognition, decision making, and data mining, among others. The classification task can be roughly described as: given a set of objects $E = \{e_1, e_2, ..., e_n\}$, also named *examples or cases*, which are described by m features, assign a class c_i from a set of classes $C = \{C_1, C_2, ..., C_j\}$ to an object e_p, $e_p = (a_{p_1}, a_{p_2}, ..., a_{p_m})$.

Rule-based fuzzy classification systems require the granulation of the features domain by means of fuzzy sets and partitions. The linguistic variables in the antecedent part of the rules represent features, and the consequent part represents a class. A typical fuzzy classification rule can be expressed by

$$R_k : \textbf{IF } X_1 \text{ is } A_{1l_1} \textbf{ AND } ... \textbf{ AND } X_m \text{ is } A_{ml_m}$$
$$\textbf{THEN } Class = C_i$$

where R_k is the rule identifier, $X_1, ..., X_m$ are the features of the example considered in the problem (represented by linguistic variables), $A_{1l_1}, ..., A_{ml_m}$ are the linguistic values used to represent the feature values, and $C_i \in C$ is the class. The inference mechanism compares the example to each rule in the fuzzy rule base aiming at determining the class it belongs to.

The classic and general fuzzy reasoning methods [5] are widely used in the literature. Given a set of fuzzy rules (fuzzy rule base) and an input example, the classic fuzzy reasoning method classifies this input example using the class of the rule with maximum compatibility to the input example, while the general fuzzy reasoning method calculates the sum of compatibility degrees for each class and uses the class with highest sum to classify the input example.

Next section introduces the DT algorithms.

3. Decision Trees

As previously mentioned, DTs provide popular and powerful models for machine learning. Some of the relevant characteristics of DTs include the following:

- they are easily understandable and intuitive;
- the induced model can be graphically expressed, as well as a set of rules;
- they are usually competitive with more costly approaches;
- their induction process performs an embedded feature subset selection, improving the interpretability of the induced models;
- decision trees are usually robust and scalable;
- they can handle discrete and continuous data;
- DTs can be applied to datasets including a large number of examples;
- their inference and induction process require low computational cost.

C4.5 [13] is one of the most relevant and well-known DT algorithm. A fuzzy version of the classic C4.5 algorithm was proposed in [8] (in Japanese). In this work, we present our fuzzy version of C4.5, named FuzzyDT [3].

The classic C4.5 algorithm uses the information gain and entropy measures to decide on the importance of the features, which can be numerical and/or categorical. C4.5 recursively creates branches corresponding to the values of the selected features, until a class is assigned as a terminal node. Each branch of the tree can be seen as a rule, whose conditions are formed by their attributes and respective tests. In order to avoid overfitting, C4.5, as well as most DT algorithms, includes a pruning process. Specifically, C4.5 adopts a post-pruning strategy, *i.e.*, the pruning takes place after the tree is completely induced. The pruning process basically assesses the error rates of the tree and its components directly on the set of training examples [12].

To understand the process of DT pruning, assume N training examples are covered by a leaf, E of them incorrectly. This way, the error rate for this leaf is defined by E/N. Considering this set of N training cases as a sample, it is possible to estimate the probability of error over the entire population of examples covered by this leaf. This probability cannot be precisely determined. However, it has a probability distribution that is usually summarized by a pair of confidence limits. For a given confidence level CF, the upper limit of this probability can be found from the confidence limits for the binomial distribution; this upper limit is here written as $U_{CF}(E, N)$. As the upper and lower binomial distribution limits are symmetrical, the probability that the real error rate exceeds $U_{CF}(E, N)$ is $CF/2$. As pointed out by Quinlan, although one might argue that this heuristic is questionable, it frequently yields acceptable results [12].

The default confidence limits used by C4.5 is 25%. However, it is important to notice that the smaller the confidence limit, the higher the chances of pruning, while the higher the confidence limit, the smaller the chances of pruning. Thus, if the confidence limit is set to 100%, the predicted error, obtained with the examples at hand, is defined as the real error, and no pruning is performed. This idea conflicts with the natural intuition that a 25% confidence limit will produce less pruning than an 80% confidence limit, for instance. This way, one should not associate the default 25% confidence limits of C4.5 with actually pruning 25% of the generated tree. Next we detail the FuzzyDT algorithm.

Algorithm 1: Fuzzyfication of the continuous values of the training set.

```
Input: A given dataset described by m attributes and n
       examples, and the predefined fuzzy data base;
for a = 1 to m do
    for b = 1 to n do
        if Attribute Att_a is continuous then
            for x = 1 to the total number of linguistic
            values defining Att_a do
                calculate A_{Att_a_x}, as the membership degree
                of the input value of attribute Att_a, example
                b, in the fuzzy set defining the x^th linguistic
                value of attribute Att_a;
            Replace the continuous value of attribute Att_a,
            example b, with the linguistic value with highest
            membership degree with it;
```

4. The FuzzyDT Algorithm

FuzzyDT[5], proposed by us in [3], uses the same measures of the classic C4.5 algorithm (entropy and information gain) to decide on the importance of the features. It also uses the same induction strategy to recursively partition the feature space creating branches until a class is assigned to each branch. However, for FuzzyDT, continuous features are defined in terms of fuzzy sets before the

[5] FuzzyDT is available at `http://dl.dropbox.com/u/16102646/FuzzyDT.zip`

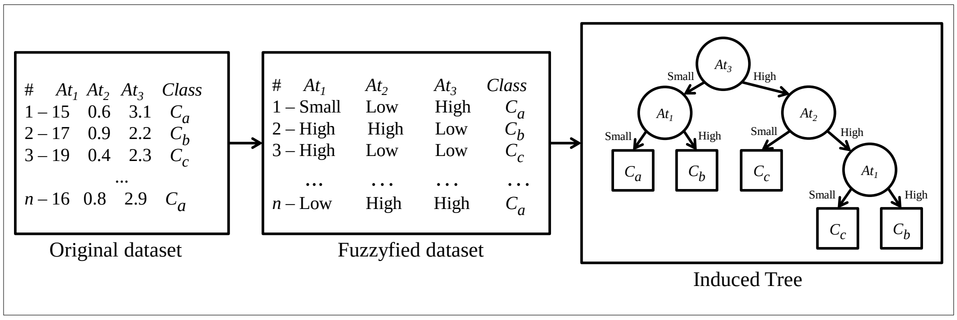

Fig 1. The FuzzyDT algorithm - a toy example

induction of the tree. This way, the process of inducing a tree using FuzzyDT takes a set of "discretized" features, *i.e.*, crispy features, since the continuous features are defined in terms of fuzzy sets and the training set is fuzzyfied before the DT induction takes place. Algorithm 1 details this fuzzyfication step.

Algorithm 2 describes FuzzyDT.

As the fuzzyfication of the training data is done before the induction of the tree, the third step of FuzzyDT corresponds to the same step of the classic DT algorithm. Figure 1 illustrates the process of data fuzzyfication and tree induction for a toy dataset with n examples, 3 attributes (At_1, At_2, and At_3), and 3 classes (C_a, C_b, and C_c).

Algorithm 2: The FuzzyDT algorithm.

1. *Define the fuzzy data base, i.e., the fuzzy granulation for the domains of the continuous features*;
2. *Replace the continuous attributes of the training set using the linguistic labels of the fuzzy sets with highest compatibility with the input values*;
3. *Calculate the entropy and information gain of each feature to split the training set and define the test nodes of the tree until all features are used or all training examples are classified*;
4. *Apply a post-pruning process, similarly to C4.5, using 25% confidence limits as default.*

The first block of Figure 1 illustrates a dataset with n examples, three attributes (At_1, At_2, and At_3) and a class attribute. The fuzzyfied version of this dataset is presented in the second block. This fuzzyfied set of examples is used to induce the final DT, illustrated in the last block of Figure 1.

It follows a detailed comparison between classic and fuzzy DTs.

5. Classic Versus Fuzzy Decision Trees

Classic and fuzzy DTs, although sharing the same basic idea of building a tree like structure by partitioning the feature spaces, also present some relevant differences. Next, we discuss some of these similarities and differences, using the classic C4.5 and FuzzyDT algorithms for the comparisons.

Evaluation of features— For the partitioning process, both versions use the same measures, entropy and information gain, in order to select the features to be used in the test nodes of the tree;

Induction process — Both versions use the same approach: repeated subdivision of the feature space using the most informative

features until a leaf node is reached or no features or examples remain;

Handling continuous features — The classic version splits the domain into crisp intervals according to the examples at hand by minimizing entropy and maximizing information gain. This process might cause unnatural divisions that reflect on a lower interpretability of the rules. As a practical illustration, let us consider the Vehicle dataset, from UCI [6], which has *Compactness* as its first test attribute of a DT induced by C4.5. *Compactness* is a continuous attribute with real values ranging from 73 to 119. For the tree induced by C4.5, it is possible to find the following tests using *Compactness*:

1. **IF** *Compactness* is ≤ 95 **AND** ... **AND** *Compactness* is ≤ 89
2. **IF** *Compactness* is ≤ 95 **AND** ... **AND** *Compactness* is > 89
3. **IF** *Compactness* is > 95
4. **IF** *Compactness* is ≤ 102
5. **IF** *Compactness* is > 102
6. **IF** *Compactness* is ≤ 109 **AND** ... **AND** *Compactness* is ≤ 106
7. **IF** *Compactness* is ≤ 109 **AND** ... **AND** *Compactness* is > 106
8. **IF** *Compactness* is > 109
9. **IF** *Compactness* is ≤ 82 **AND** ... **AND** *Compactness* is ≤ 81
10. **IF** *Compactness* is ≤ 82 **AND** ... **AND** *Compactness* is > 81
11. **IF** *Compactness* is > 82 **AND** ... **AND** *Compactness* is ≤ 84
12. **IF** *Compactness* is > 82 **AND** ... **AND** *Compactness* is > 84

These 12 tests make it difficult to understand the rules since, for a whole understanding of the model, the user has to keep in mind the subspaces defined by each condition of the rule that uses the same attribute. A particular problem happens when the subdivisions are relatively close, strongly restraining the domain of the features (rule 10: $81 < Compactness \leq 82$).

Another issue regarding the use of continuous features by C4.5 is the number of divisions used to split continuous attributes: it cannot be predefined, even if it is previously known. In fact, for the algorithm to use a previously defined number of divisions for any attribute, such attribute needs to be discretized before the induction of the DT, since the number of divisions splitting continuous attributes is dynamically determined during the tree induction process. This way, the number determined by the DT algorithm might be different from the number of divisions used by an expert, for example. Notice that in the example provided, the DT uses 8 different splitting points for the same attribute (81, 82, 84, 89, 95, 102, 106, and 109), some of them very close to each other.

FuzzyDT, on the other hand, is able to use the partitions (in terms of fuzzy sets) defined by an expert. Furthermore, even if this information is not available, automatic methods for the generation of fuzzy partitions can be used, most of them controlling and preventing the creation of unnatural splitting points.

Reuse of features — for C4.5, the same continuous feature can be included several times in one single rule (such as feature *Compactness* in the previous example). This repetition of the same feature and subdivision of the domain degrades the interpretation of the rule. On the other hand, the induction process of FuzzyDT can be seen as inducing a DT with fuzzyfied (discretized) attributes, thus a feature is never used more than once in the same rule. This fact favours the interpretability of the generated rules.

Inference — As previously stated, a special issue regarding classic DTs is the fact that they can be seen as a set of disjunct rules in which only one rule is fired to classify a new example. For fuzzy DTs, differently from the classic model, two branches are usually fired simultaneously, each one with a degree of compatibility with an input example. This characteristic is illustrated in Figure 2, which presents the partition of attribute At_n on the left, defined by fuzzy sets S1, S2, and S3, and the membership degrees y_1 and y_2 for input x_1, as well as the fuzzy DT on the right with two triggered branches in blue, S1 and S2.

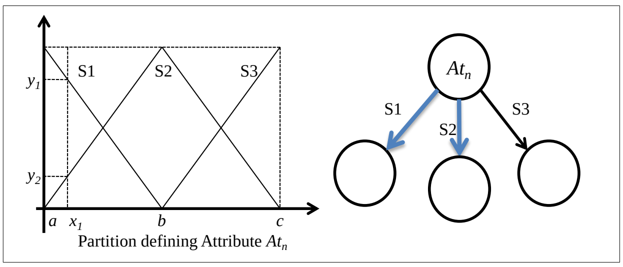

Fig 2. Partition defining attribute At_n (left) and triggered branches of a decision tree (right)

Notice that for an input value x_1, fuzzy sets S1 and S2 are intersected with membership degree values y_1 and y_2, respectively. This way, branches S1 and S2, indicated by blue (lighter) arrows, of the fuzzy DT are fired. For any input value ranging from a to b, the branches defined by fuzzy sets S1 and S2 are triggered, while for an input value ranging from b to c, the branches defined by fuzzy sets S2 and S3 are triggered.

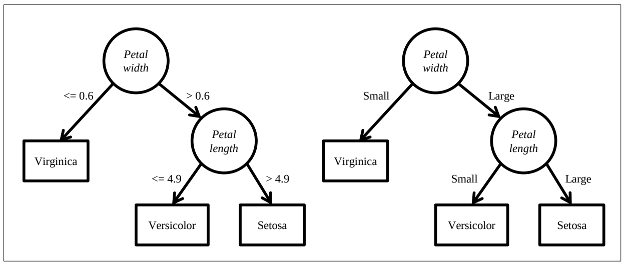

Fig 3. A classic(left) and a fuzzy (right) decision tree for the Iris dataset

Figure 4 presents the fuzzy sets (Small and Large) defining the attributes tested in the DTs of Figure 3, including the input values (*Petal Length*= 4.9, *Petal Width* = 0.6) and their corresponding membership degrees.

The inference process of the classic DT is straightforward: if *Petal Width* is ≤ 0.6, the example belongs to the *Virginica* class, otherwise, the *Petal Length* attribute is tested; if it is ≤ 4.9, the example is classified as *Versicolor*, otherwise it is classified as *Setosa*. This way, considering a new input example to be classified having *Petal Length* = 4.9 and *Petal Width* = 0.6, both values on the borderline of the crisp discretization of the classic DT, only the first rule is fired: **IF** *Petal Width* is ≤ 0.6 **THEN** *Class* is *Virginica*.

For the fuzzy DT, on the other hand, the membership degrees of the input example, shown in Figure 4 (*Petal Length Small* = 0.66; *Petal Length Large* = 0.34; *Petal Width Small* = 0.79; *Petal Width Large* = 0.21) are used to calculate the compatibility degree of the input example with each rule. For this particular example, using minimum as t-norm, the fuzzy rules and their compatibility degrees (in brackets) with the input example are:

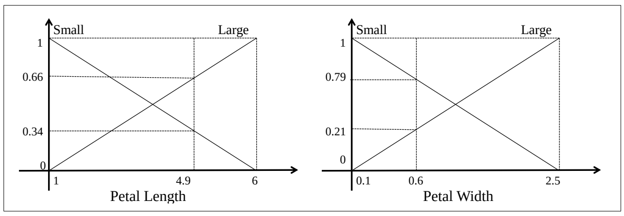

Fig 4. Fuzzy sets defining attributes *Petal Length* and *Petal Width*

1. **IF** *Petal Width* is *Small* **THEN** *Class* is *Virginica* (0.79)
2. **IF** *Petal Width* is *Large* **AND** *Petal Length* is *Small* **THEN** *Class* is *Versicolor* (0.21)
3. **IF** *Petal Width* is *Large* **AND** *Petal Length* is *Large* **THEN** *Class* is *Setosa* (0.21)

For this example, using the classic fuzzy reasoning method (best rule), the class of the first rule, which has highest compatibility degree with the input example, is used to classify the example as *Virginica*. Notice that this is the same class defined by the classic DT.

Now, let us assume that the *Petal Width* of the input example is 0.61, while the *Petal Length* remains the same. Notice that the difference in the *Petal Width* between this example and the last one is quite small (0.01). This way, we are likely to believe, intuitively, the class of such similar examples should be the same. Nevertheless, the classic

DT classifies this new example as belonging to the *Versicolor* class. The fuzzy DT, on the other hand, since it uses the compatibility degrees of the input values with the fuzzy sets defining the tree tests, still classifies this new example with the same class, *Virginica*. Notice that the same situation can occur for *Petal Length*, as well as with any continuous attribute with a crisp discretization. This robustness of fuzzy DTs is highly desirable.

Since fuzzy DTs use the compatibility degree of each rule to classify an input example, the classic and general fuzzy reasoning methods can be used in the inference process. Notice that, once multiple rules derived from the DT can be fired, an input instance can be classified with the class of the rule with highest compatibility with the input example (classic fuzzy reasoning method), or with the class with the highest combination from the set of rules with that given class (general fuzzy reasoning method).

In conclusion, the classic C4.5 algorithm and its fuzzy version, FuzzyDT, present relevant differences regarding the handling of continuous features, reuse of features, and inference procedures. C4.5 has a simpler and faster inference process, while FuzzyDT is able to avoid the reuse of features, the repeated splitting of continuous features. FuzzyDT also provides a more robust, although more costly, inference process. Next, we present the experiments and results.

6. Experiments

FuzzyDT was compared to C4.5 using 16 datasets from the UCI - Machine Learning Repository [6] and a 10-fold cross-validation strategy. C4.5 was selected for the comparisons since FuzzyDT presents many similarities with C4.5, while adding the advantages of fuzzy systems, regarding the processing of uncertainty and imprecision, as well as interpretability, to the induced models.

Table 1 summarizes the characteristics of the datasets, presenting the number of examples (E), features, including the number of continuous (c) and discrete (d) features in brackets, number of classes (C), as well as the majority error, *i.e.*, the error of the algorithm that always predicts the majority class.

The experiments were carried out using the classic fuzzy reasoning method [5] and post-pruning with a default confidence level of 25%.

Dataset	Examples	Features	(c	d)	C	ME	Dataset	E	Features	(c	d)	C	ME
Breast	682	9	(9	0)	2	65.10	Iono	351	34	(34	0)	2	35.90
Credit	653	15	(6	9)	2	45.33	Iris	150	4	(4	0)	3	66.67
Cylinder	277	32	(19	13)	2	35.74	Liver	345	7	(7	0)	2	57.97
Diabetes	769	8	(8	0)	2	34.90	Segment	210	19	(19	0)	7	85.71
Gamma	19020	10	(10	0)	2	64.84	Spam	4601	57	(57	0)	2	60.60
Glass	220	9	(9	0)	7	65.46	Steel	1941	29	(29	0)	7	32.82
Haberman	306	3	(3	0)	2	73.53	Vehicle	846	18	(18	0)	4	74.23
Heart	270	13	(13	0)	2	44.44	Wine	178	13	(13	0)	3	59.74

Table 1. Characteristics of the datasets

Table 2 presents the error rates of the experiments, including the average error rates. The error standard deviations are presented in brackets.

Table 3 presents the average number of rules (column Rules) and the average of the total number of conditions in each model (column Cond.). The standard deviations for the number of rules, as well as number of conditions, are presented in brackets (columns SD). The best (smallest) rates, are dark-gray shaded.

As can be observed, FuzzyDT obtained better results than C4.5, *i.e.* smaller error rates, for 10 of the 16 datasets. All the models obtained better error rates than the majority error of the datasets included in the experiments. In order to check for statistically significant differences, we executed the Mann-Whitney test [15], which, with a 95% confidence, found no statistically significant differences between FuzzyDT and C4.5.

Dataset	FuzzyDT		C4.5		Dataset	FuzzyDT		C4.5	
Breast	1.49	(0.00)	5.13	(3.03)	Ionosphere	3.99	(7.35)	11.40	(3.83)
Credit	7.81	(0.00)	13.00	(2.98)	Iris	8.00	(2.67)	5.33	(5.81)
Cylinder	6.16	(4.51)	30.65	(5.91)	Liver	36.76	(4.65)	32.74	(6.57)
Diabetes	21.84	(1.27)	25.52	(2.63)	Segmentation	12.38	(2.33)	2.86	(1.12)
Gamma	21.13	(0.70)	15.02	(0.58)	Spam	28.98	(0.00)	7.87	(1.35)
Glass	39.13	(0.00)	30.37	(6.87)	Steel	20.78	(0.93)	23.13	(2.48)
Haberman	26.67	(0.00)	29.09	(6.10)	Vehicle	25.37	(1.80)	27.07	(4.10)
Heart	14.44	(5.60)	23.11	(5.94)	Wine	5.00	(6.43)	7.25	(6.59)

Table 2. Error rates

For the evaluation of the interpretability of the models, we compared the average number of rules and the average of the total number of conditions in the models induced by FuzzyDT and C4.5. In this work, we adopt the average number of conditions in the induced models as the measure of the syntactic complexity of the models.

Regarding the number of rules of the induced models, C4.5 and FuzzyDT tie, both presenting better results (smallest number of rules) for 8 datasets. It is interesting to notice that for the *Gamma* dataset, although the error rate obtained by the model induced by C4.5 is smaller than the one of FuzzyDT, the number of rules for the C4.5 model is 8 times larger than for FuzzyDT: 43.00 rules for FuzzyDT against 328.70 for C4.5. No other discrepancies are present in the results regarding the average number of rules of the induced models.

	FuzzyDT		C4.5		FuzzyDT		C4.5	
Dataset	Rules	SD	Rules	SD	Cond.	SD	Cond.	SD
Breast	15.00	(0.00)	12.30	(3.32(	50.00	(0.00)	52.70	(19.83)
Credit	7.80	(1.83)	19.30	(6.48)	21.40	(5.50)	90.90	(33.76)
Cylinder	45.80	(4.87)	42.80	(9.45)	198.50	(26.16)	248.50	(102.69)
Diabetes	13.40	(5.50)	23.60	(7.55)	42.60	(22.00)	150.20	(64.15)
Gamma	43.00	(0.00)	328.70	(31.36)	228.00	(0.00)	3,634.80	(435.18)
Glass	24.00	(0.00)	24.10	(2.17)	99.00	(0.00)	137.80	(20.35)
Haber	4.60	(1.20)	3.10	(1.64)	8.30	(2.10)	6.90	(4.18)
Heart	22.40	(2.84)	23.60	(3.67)	78.90	(13.87)	95.70	(23.08)
Iono	21.00	(0.00)	13.90	(1.45)	89.00	(0.00)	72.40	(13.15)
Iris	5.00	(0.00)	4.60	(0.66)	9.00	(0.00)	12.10	(3.08)
Liver	2.40	(1.28)	24.50	(4.92)	3.70	(2.00)	139.90	(34.60)
Segment	28.80	(1.89)	41.80	(2.96)	127.30	(13.52)	314.70	(35.73)
Spam	55.00	(0.00)	100.40	(10.43)	660.00	(0.00)	1,045.50	(142.89)
Steel	225.60	(2.85)	159.90	(6.74)	1,761.70	(24.48)	1,909.00	(125.86)
Vehicle	82.00	(6.47)	66.30	(7.20)	530.50	(62.02)	503.00	(79.81)
Wine	15.40	(0.80)	5.10	(0.30)	48.20	(4.40)	12.50	(1.20)

Table 3. Number of rules and conditions

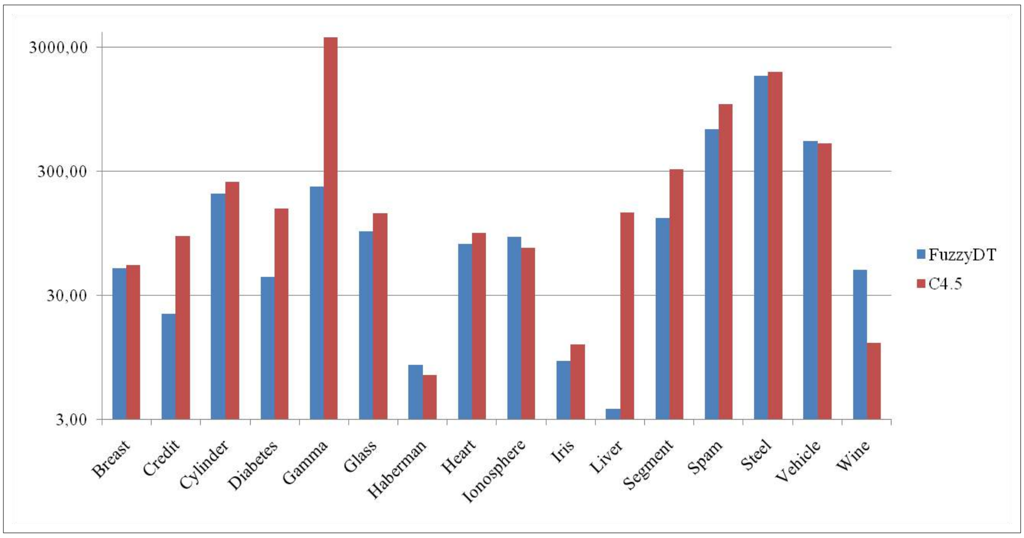

Fig 5. Average of the total number of conditions

Figure 5 present the average number of conditions in a graphical manner. Notice that a 10-base logarithm scale is used for the number of conditions. Thus, the lower portion of the graph varies from 3 to 30 conditions, while the upper area varies from 300 to 3,000 conditions.

Regarding the average number of conditions in the models, the ones induced by FuzzyDT were smaller than those of C4.5 for 12 of the 16 datasets. Moreover, notice in Figure 5 that for the *Credit*, *Gamma*, and *Liver* datasets the number of conditions of the C4.5 models is larger than for FuzzyDT. In fact, the number of conditions of the C4.5 models is 3 times larger than FuzzyDT for *Credit*, 16 times larger for *Gamma*, and 38 times larger for the *Liver* dataset.

In summary, FuzzyDT presented smaller error rates than C4.5 for most of the datasets, as well as smaller syntactic complexity for 12 datasets. The models induced by C4.5 for three datasets were considerably more complex than those induced by FuzzyDT. Next, we present the conclusions and future work.

7. Conclusions

DTs have been successfully applied to many areas for tasks such as classification, regression, and feature subset selection, among others. DTs are popular in machine learning due to the fact that they produce graphical models, as well as textual rules, that end users can easily understand. The induction process of DTs is usually fast, requiring low computational resources.

Fuzzy systems, on the other, provide mechanisms to handle imprecision and uncertainty in data based on the fuzzy logic and fuzzy sets theory. The combination of fuzzy systems and DTs has produced fuzzy DT models, which benefit from both techniques to provide simple, accurate, and highly interpretable models at low computational costs.

In this paper, we detailed the FuzzyDT algorithm, a fuzzy DT based on the classic C4.5 DT algorithm and expanded previous experiments. We also provided a thorough comparison of some relevant issues regarding the classic and the fuzzy models, and discussed the use of FuzzyDT for feature subset selection. FuzzyDT was experimentally evaluated and compared to C4.5 using 16 datasets and a 10-fold cross-validation strategy. FuzzyDT obtained smaller error for 10 datasets and was also able to induce models with less rules and less conditions in the rules when compared to C4.5.

As future work we intend to compare FuzzyDT with other fuzzy DTs, as well as with other classic DTs. We also intend to further evaluate FuzzyDT for the task of feature subset selection.

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REPORT

Performance Analysis of Evolving Fuzzy Neural Networks for Pattern Recognition

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The Evolving Fuzzy Neural Networks (EFuNNs), recently proposed by Kasabov are dynamic connectionist feed forward networks with five layers of neurons and they are adaptive rule-based systems. Several paper can be found in the literature comparing EFuNN with other methods. However, it is known that results of all pattern recognition methods depend on the training data and in particular from statistical distribution of training data. This work assesses the accuracy of EFuNNs) for pattern recognition tasks using seven different statistical distributions data. Results of assessment are provided and show different accuracy according to the statistical distribution of data.

1. Introduction

In 2001 Kasabov [7] proposed a new class of Fuzzy Neural Networks named Evolving Fuzzy Neural Networks (EFuNNs). EFuNNs are structures that evolve according determined principles: quick learning, open structure for new features and new knowledge, representing space and time and analyse itself of errors. EFuNNs have low complexity and high accuracy and it has been successfully applied to the solution of several pattern recognition problems in the last years [16]. In several papers EFuNN showed best results when compared to Multilayer Perceptron Neural Networks (MLP) [6], [11], [18] and Support Vector Machine [17].

However, it is known that results of all those methods depend on the training data for a specific application. In particular, results are dependent on statistical distribution of training data. In the literature, articles about analysis of results on EFuNN with respect the statistical distributions of data were not found. Even detailed studies as [13], which analysed the optimal number of features to provide the most effective classification using EFuNN did not explored this aspect. Thus, the performance of EFuNN when using data from different statistical distributions is not known, for pattern recognition applications.

In this paper we made an assessment of accuracy of EFuNNs in pattern recognition tasks using seven different statistical distributions for data, statistical with 1, 2, 3 and 4 dimensions for each. Results of those comparisons are provided with an analysis about the better kind of statistical distribution of data to be used for better EFuNN performance for each dimension of data.

The paper is organized as following: in the Section 2 is presented theoretical aspects of Evolving Fuzzy Neural Networks. Section 3 brings details about assessment criteria, statistical distributions used and Kappa Coefficient. Section 4 presents an analysis of results and Section 5 shows the conclusions of the comparisons. This paper is a revised and extended version of a conference paper [12].

2. Evolving Fuzzy Neural Networks

As mentioned before, Evolving Fuzzy Neural Networks (EFuNNs) are structures that evolve according ECOS (Evolving COnnectionist System) principles [5], which are: quick learning, open structure for new features and new knowledge, representing space and time and analyse itself of errors. The EFuNN is a connectionist feed forward network with five layers of neurons [14], but nodes and connections are created or connected when data examples are presented [7]. The Figure 1 presents a schematic diagram of an EFuNN. The input layer represents input variable of the network as crisp value x. The second layer represents fuzzy quantization of inputs variables. Here, each neuron implements a fuzzy set [19] and its membership function (MF) as triangular membership, gaussian membership or other [9]. The function of this layer is convert input values for membership degrees using some MF (neuron) available. If given an input value, it is not possible assign to it a membership degree greater than a threshold, new neurons can be created (evolving).

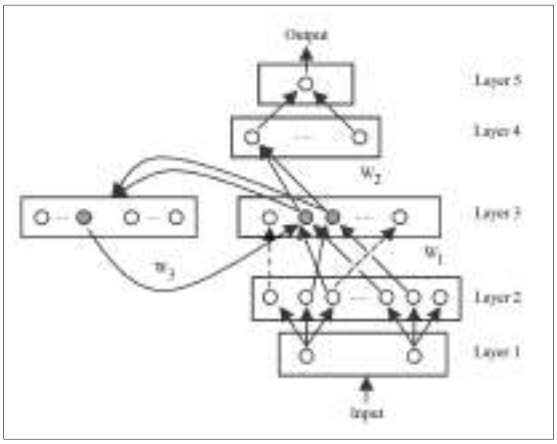

Fig 1. Diagram of an EFuNN. Adapted from [7]

The third layer of a EFuNN is the kernel of this network. This layer contains rule nodes (r_j) that evolve through training. The rule nodes represent groups of input-output data associations. Each node is defined by two connections vectors: $W_1(r_j)$ from fuzzy input layer to rule nodes and $W_2(r_j)$ from rule nodes to fuzzy output layer. These nodes are created during network learning and they represent prototypes of data mapping from fuzzy input to fuzzy output space. So, $W_1(r_j)$ is estimated from data and using output error and $W_2(r_j)$ is estimated based on similarity with respect to a group. The activation function used in this layer can be a linear activation function or a Gaussian function. The existence of temporal dependence between consecutive data can be estimated by a weighted connection W_3.

In the third layer, each $W_1(r_j)$ represents the coordinates of the center of a hypersphere in the fuzzy input space and each $W_2(r_j)$ represents the coordinates of the center of a hypersphere in the fuzzy output space. The radius of the hypersphere of a rule node r_j is defined as $R_j = 1 - S_j$, where S_j is the sensitive threshold parameter for activation of r_j from a new example (x, y). The pair of fuzzy data (x_f, y_f) will be allocated to r_j if x_f is into the r_j input hypersphere and if y_f is into the r_j output hypersphere. For this, two conditions must be satisfied:

a) The local normalized fuzzy distance between x_f and $W_1(r_j)$ must be smaller than R_j. The local normalized fuzzy distance between these two fuzzy membership vectors is done by:

$$D(x_f, W_1(r_j)) = \|x_f - W_1(r_j)\| \,/\, \|x_f + W_1(r_j)\| \quad (40)$$

where $\|a - b\|$ and $\|a + b\|$ are the sum of all the absolute values of a vector that is obtained after vector subtraction $a - b$ or summation $a + b$. In the equation (40), the bar $''/''$ denotes the division between two scalars.

b) The normalized output error $Err = \|y - y'\| / N_{out}$ must be smaller than an error threshold E, where y is as defined before, y' is the output produced by EFuNN, N_{out} is the number of outputs in EFuNN and E is the error tolerance of the system for fuzzy output.

If the conditions (a) or (b) are not satisfied, a new rule node can be created. The weights of rule r_j are updated according to an interactive process, where superscript (t) denotes a step of interaction:

$$\begin{aligned} W_1(r_j^{(t+1)}) &= W_1(r_j^{(t)}) + l_{j,1}\left(W_1(r_j^{(t)}) - x_f\right) \\ W_2(r_j^{(t+1)}) &= W_2(r_j^{(t)}) + l_{j,2}(A_2 - y_f)A_1(r_j^{(t)}) \end{aligned} \quad (41)$$

where $W_1(r_j^{(t)})$ and $W_2(r_j^{(t)})$ are the values of weights W_1 and W_2, respectively at the interaction (t); $W_1(r_j^{(t+1)})$ and $W_2(r_j^{(t+1)})$ are the updated values of the same weights at the interaction $(t+1)$; $l_{j,1}$ is the learning rate for the first layer and $l_{j,2}$ is the learning rate for the second layer. In general, it can be assumed they have the same value done by: $l_j = 1/N_{ex}(r_j)$, where $N_{ex}(r_j)$ is the number of examples associated with rule node r_j.

The function:

$$A_1(r_j^{(t)}) = f_1(D(W_1(r_j^{(t)}), x_f) \quad (42)$$

is the activation function of the rule $r_j^{(t)}$ and

$$A_2 = f_2(W_2(r_j^{(t)}, A_1(r_j^{(t)}) \quad (43)$$

is the activation of the fuzzy output neurons, when x is presented [7]. For the functions f_1 and f_2 a simple linear function can be used.

When a new example is associated with a rule r_j, the parameters R_j and S_j are changed:

$$\begin{aligned} R_j^{(t+1)} &= R_j^{(t)} + D\left(W_1(r_j^{(t+1)}), W_1(r_j^{(t)})\right) \\ S_j^{(t+1)} &= S_j^{(t)} - D\left(W_1(r_j^{(t+1)}), W_1(r_j^{(t)})\right) \end{aligned} \quad (44)$$

If exists temporal dependencies between consecutive data, the connection weight W_3 can capture that. The connection W_3 works as a *Short-Term Memory* and as a feedback connection from rule nodes layer. If the winning rule node at time $(t-1)$ was $r_{max}^{(t-1)}$ and at time (t) was $r_{max}^{(t)}$, then a link between the two nodes is established by [8]:

$$\begin{aligned} &W_3\left(r_{\max}^{(t-1)}, r_{\max}^{(t)}\right) = W_3\left(r_{\max}^{(t-1)}, r_{\max}^{(t)}\right) + \\ &l_3 A_1\left(r_{\max}^{(t-1)}\right) A_1\left(r_{\max}^{(t)}\right) \end{aligned} \quad (45)$$

where $A_1\left(r_{\max}^{(t)}\right)$ denotes the activation of a rule node r at a time (t) and l_3 defines a learning rate. If $l_3 = 0$, no temporal associations are learned in an EFuNN. More details about connection weight W_3 can be obtained in [8].

The fourth layer represents fuzzy quantization of the output variables from a weighted sum function of inputs and from an activation function (similar to second layer), the degree to which an output vector belongs to some output MF is computed. The last layer uses an activation function to calculate defuzzified values for output variables y.

In a simplified way, the EFuNN learning algorithm starts with initial values for parameters [7]. According to mentioned above, the EFuNN is trained by examples until convergence. When a new data example $d = (x, y)$ is presented, the EFuNN either creates a new rule r_n to memorize the new data (input vector $W_1(r_n) = x$ and output vector $W_2(r_n) = y$) or adjusts the winning rule node r_j [7]. More details about the EFuNN algorithm can be found in [17].

3. Assessment

In pattern recognition tasks we need a methodology that can provide better performance over any kind of data. Unfortunately, several methods proposed in the literature can obtain their better performance when they use specific statistical distribution of data. Some methods can obtain good results with different kind of distributions, as neural networks. As a neural network, an EFuNN is a structure that evolves, but in some papers better results were showed when its use is compared with neural networks [6], [11], [18] and Support Vector Machine [17]. However, it remains unclear the EFuNN performance with respect to different statistical distributions of data. Thus, in this paper, we investigated the behavior of EFuNN for pattern recognition tasks, according on kinds of statistical distributions of data utilized. We made simulations with seven different statistical distributions: Binomial, Continuous Uniform, Discrete Uniform, Exponential, Gaussian, Poisson and Weibull [3]. Those statistical distributions are used in different applications, as for example:

a) Binomial distribution: Quality Control and Epidemiology;

b) Uniform distributions (discrete or continuous): random studies where all classes are possibles;

c) Exponential distribution: used in Quality Control and also to estimate time of failure in products or components;

d) Gaussian distribution: widely used in image and signal classification;

e) Poisson distribution: used in Quality Control and also to model the number of occurrences of an event for a certain period in time or space or both;

f) Weibull distribution: used in industries to stablish the time of warranty for a product.

For each statistical distribution was analysed data with up to four dimensions. Results of this analysis are provided and our goal is to know for what statistical distribution we can use EFuNN and what performance is expected for each statistical distribution. Besides that, we can know for what statistical distribution of data the better performance of EFuNN can be expected.

For performance assessment, we use the Mean Square Error (MSE) computed between the values expected (y) and that one estimated by EFuNN (y') and it is defined as:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(y_i - {y'}_i)^2, \tag{46}$$

where N is total number of elements to be classified. A critical analysis of this measure can be found in the [15]. We use also the matrix of classification for performance assessment. From this one, we obtain the percentile of correct classification (relation between sum of values in the diagonal of classification matrix and the total number of elements to be classified - N - [4]), the number of classifications mistakes (sum of the values outside of the diagonal) and Kappa Coefficient [1]. The percentile of correct classification is a measure of easy interpretation, but is criticized for taking into account only the successes in classification [10]. The Kappa Coefficient was proposed by Cohen in 1960 and is used to evaluate classification performance of a classifier, using a reference classification and a classification matrix. Kappa Coefficient is the coefficient widely used in the literature of pattern recognition [2] to take into account not only successes, but also classification mistakes [10]. It is defined by (47):

$$K = \frac{P_0 - P_c}{1 - P_c}, \tag{47}$$

where:

$$P_0 = \frac{\sum_{i=1}^{M} n_{ii}}{N}; \quad P_c = \frac{\sum_{i=1}^{M} n_{i+}n_{+i}}{N^2}; \tag{48}$$

where n_{i+} is the sum of i line of classification matrix and n_{+i} is the sum of i colum of the same matrix. The Kappa variance σ_K^2 is given by [10]:

$$\sigma_K^2 = \frac{P_0(1-P_0)}{N(1-P_c)^2} + \frac{2(1-P_0)+2P_0P_c-\theta_1}{N(1-P_c)^3} + \frac{(1-P_0)^2\theta_2 - 4P_c^2}{N(1-P_c)^4}. \tag{49}$$

where:

$$\theta_1 = \frac{\sum_{i=1}^{M} n_{ii}(n_{i+}+n_{+i})}{N^2}; \quad \theta_2 = \frac{\sum_{i=1}^{M} n_{ii}(n_{i+}+n_{+i})^2}{N^3} \tag{50}$$

For calculations, it can be used an approximation to the first component of equation (49), as pointed out by [10]. However, in this paper we use the complete formula for variance σ_K^2.

4. Results

In this section we presented the main results of tests in pattern recognition tasks using EFuNN. Only better results are presented in this paper due to space restrictions. We generated two sets of samples: the first with 500 samples for training and the second with 4000 samples for classification test. Each sample had four different classes, with four dimensions. Two sets of each were generated for each one of seven statistical distributions. All the results presented bellow are about samples for classification test.

In all Figures (2, 3, 4, 5, 6, 7, 8) in the following, we are presenting sample data for training an EFuNN in the first line. The next line shows the sample data for testing with the same parameters. For sample data, we generated 1500 samples, but the first 1000 were discarded. For sample data for testing, we generated 5000 samples, but the first 1000 were discarded. The first two lines in the Figures correspond to the first dimension (sample of training and sample for testing). The next two for the second dimension and following in the same sequence. Thus, the figures have 8 lines to show four sets of sample of training and sample for testing for each dimension and 4 columns to show the four classes for each one. It is possible to observe for some classes the great intersection between the distributions. This has been done intentionally to make simulations were as similar as possible to real conditions.

4.1 Binomial Distribution

Let be an essay where only two results are possible and each result denotes a type of occurrence. We wish to know the number of occurrences of only one of these results in a finite sequence of such essays. The Binomial distribution models a discrete random variable which made a counting of number of only one of types of occurrence.

In a simulation with one dimension data and Binomial distribution, the parameters used were: number of membership functions (MF) 6, no-prunning, sensitivity threshold value 0.999 and error threshold value 0.001. With that configuration, Mean Square Error was 0.2116, the percentile of correct classification was 78.8%, with 848 classification mistakes and Kappa Coefficient was 71.7333%, with variance 6.6945×10^{-5}. When dimension of data was changed for 2, the parameters used were the same, except the MF, which was changed for 5. The

Mean Square Error was 0.1347, the percentile of correct classification was 86.875%, with 525 classification mistakes and Kappa Coefficient was 82.5000%, with variance 4.9923×10^{-5}.

For data with three dimensions, all parameters remained the same and all points were classified correctly, i. e. Kappa Coefficient was 100%, with variance zero. The same was observed for data with four dimensions. From that results, we can observe that EFuNN is able to classify data from Binomial distribution with good accuracy. For dimensions over three it can obtain 100% of correct classification.

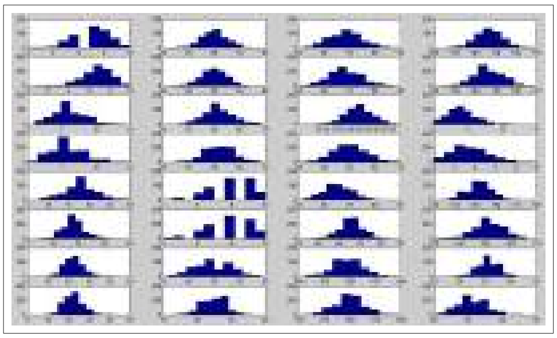

Fig 2. Random numbers generated for Binomial distribution - four sets of training and test samples (lines) for four classes (columns)

4.2 Continuous Uniform Distribution

The Continuous Uniform Distribution is used when all possible results have the same probability of occurrence in continuous space. The parameters used for one dimension were: MF 6, no-prunning, sensitivity threshold value 0.999 and error threshold value 0.001. From that configuration, Mean Square Error obtained was 0, 0645, the percentile of correct classification was 93.575%, with 257 classification mistakes and Kappa Coefficient was 91.4333%, with variance 2.6721×10^{-5}.

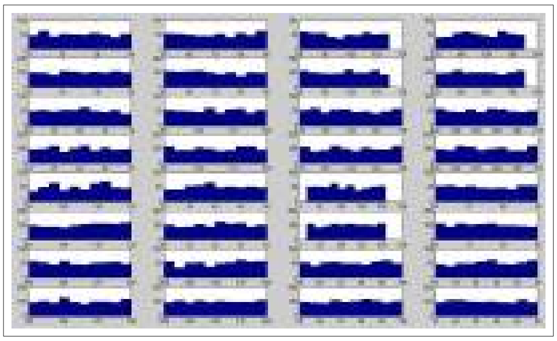

Fig 3. Random numbers generated for Continuous Uniform distribution - four sets of training and test samples (lines) for four classes (columns)

In other simulation, data with two dimensions were generated, the parameters used were the same, except the MF, wich was changed for 2. In this case, all points were classified correctly (Kappa Coefficient was 100% and variance was zero). The same was observed for data with 3 and 4 dimensions. The EFuNN seems to be adapted to classify data with this distribution. It is capable to classify data with relative accuracy even in low dimensions and when classes intersection happens.

4.3 Discrete Uniform Distribution

The Discrete Uniform Distribution is used when all possible results have the same probability of occurrence in discrete space. From data randomly generated with this distribution and with one dimension, the parameters used were: MF 4, no-prunning, sensitivity threshold value 0.999 and error threshold value 0.001. Using that configuration, the Mean Square Error obtained was 0, 0164, the percentile of correct classification was 98.35%, with 66 classification mistakes and Kappa Coefficient was 97.8000%, with variance 7.2072×10^{-6}.

Similarly to what occurred in Continuous Uniform Distribution, the same was observed for data with two, three and four dimensions: all points were classified correctly and the parameters were the same as those ones used for a dimension. So, the EFuNN also seems to be adapted to classify data with this distribution. It is capable to classify data with relative accuracy, even in low dimensions and when classes intersection happens. The same verification done for the continuous uniform distribution can be done for the discrete uniform distribution.

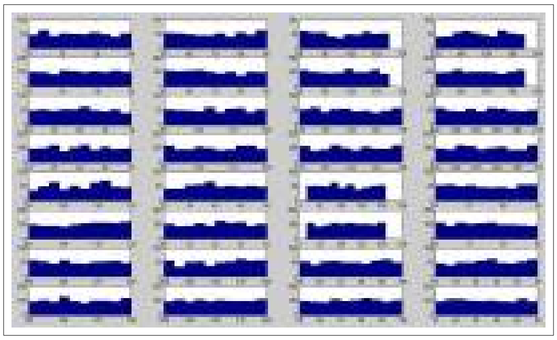

Fig 4. Random numbers generated for Discrete Uniform distribution - four sets of training and test samples (lines) for four classes (columns)

4.4 Exponential Distribution

The Exponential Distribution calculates the time elapsed between two events of interest, as for instance, time interval between two patterns of failures in a textile manufacturing. Using data with this distribution and with one dimension, the parameters utilized were: number of membership functions (MF) 9, no-prunning, sensitivity threshold value 0.999 and error threshold value 0.001. With that configuration, Mean Square Error was 1, 3110, the percentile of correct classification was 40.025%, with 2399 classification mistakes and Kappa Coefficient was 20.0333%, with variance 1.1082×10^{-4}.

In other simulation, we used data with two dimensions and the parameters used were the same, except the MF, which was changed for 5. The Mean Square Error was 1, 1751, the percentile of correct classification was 46.025%, with 2159 classification mistakes and Kappa Coefficient was 28.0333%, with variance 1.1191×10^{-4}. For data with three dimensions, all parameters remained the same and the Mean Square Error was 1, 0161, the percentile of correct classification was 47.15%, with 2114 classification mistakes and Kappa Coefficient was 29.5333%, with variance 1.1105×10^{-4}. For data with four dimensions, the parameters used were the same, except the MF, which was changed for 3. The Mean Square Error was 0, 9584, the percentile of correct classification was 52.0250%, with 1919 classification mistakes and Kappa Coefficient was 52.0250%, with variance 1.1056×10^{-4}. In opposition to the uniform distribution, the EFuNN is not adequated to classify data from exponential distribution and it did not obtain good results for none of the researched dimensions.

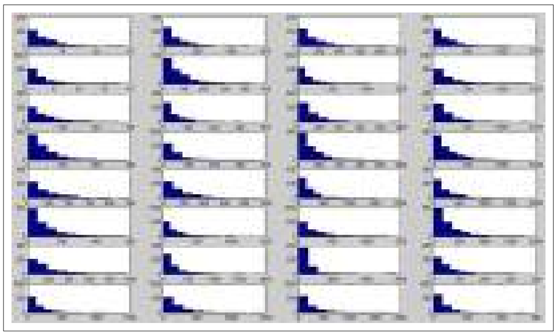

Fig 5. Random numbers generated for Exponential distribution - four sets of training and test samples (lines) for four classes (columns)

4.5 Gaussian Distribution

The Gaussian or Normal Distribution is the most typical distribution and it is widely used in image processing and to model several independent variables as the socio-economic variables. For data randomly generated with this distribution and with one dimension, the parameters used were: MF 7, no-prunning, sensitivity threshold value 0.999 and error threshold value 0.001. From that configuration, Mean Square Error obtained was 0.2642, the percentile of correct classification was 72.70005%, with 1092 classification mistakes and Kappa Coefficient was 63.6000%, with variance 8.6015×10^{-5}.

When the dimension of data was increased for 2, the parameters used were the same and the Mean Square Error was 0.2007, and the percentile of correct classification was 80.1750%, with 793 classification mistakes. The Kappa Coefficient was 73.5667%, with variance 7.0074×10^{-5}. After, we generated data with three dimensions using this distribution and the Mean Square Error obtained was 0, 1296, the percentile of correct classification was 88.9000%, with 444 classification mistakes and Kappa Coefficient was 85.2000%, with variance 4.3852×10^{-5}.

For data with four dimensions and with Gaussian distribution, the parameters used were the same, except the MF, which was changed for 6. The Mean Square Error was 0, 0484, the percentile of correct classification was 97.2250%, with 111 classification mistakes and Kappa Coefficient was 96.3000%, with variance 1.1987×10^{-5}. We can observe that EFuNN obtained good results, however it did not obtain total correct classification for none of the researched dimensions.

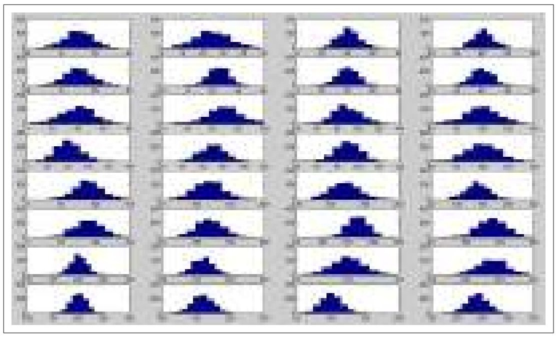

Fig 6. Random numbers generated for Gaussian distribution - four sets of training and test samples (lines) for four classes (columns)

4.6 Poisson Distribution

The Poisson Distribution is used to model discrete particular events in continuous space or time or both. For example, number of failures patterns in a textile fabric. We generated data with Poisson distribution and with one dimension. For this case, the parameters used were: MF 3, no-prunning, sensitivity threshold value 0.999 and error threshold value 0.001. For this configuration, the Mean Square Error obtained was 0, 040, the percentile of correct classification was 94.6750%, with 213 classification mistakes and Kappa Coefficient was 92.9000%, with variance 2.2283×10^{-5}.

After that we generate data with this statistical distribution and with two dimensions. The parameters used were the same, except the MF, which was changed for 2, and all points were correctly classified (Kappa Coefficient was 100% and variance was zero). The same was observed for data with 3 and 4 dimensions. The EFuNN seems to be adapted to classify data from Poisson distribution better than any other researched. Even with data with one dimension, the EFuNN is able to obtain a high rate of correct classification.

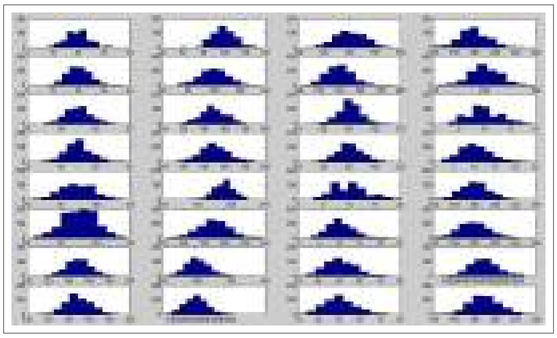

Fig 7. Random numbers generated for Poisson distribution - four sets of training and test samples (lines) for four classes (columns)

4.7 Weibull Distribution

The Weibull Distribution is used to model rates of displacement in equipments and their durability. Initially, we generated data with this statistical distribution and with one dimension. The parameters used for this data were: MF 5, no-prunning, sensitivity threshold value 0.999 and error threshold value 0.001. From that configuration, Mean Square Error obtained was 1, 5351, the percentile of correct classification was 37.3000%, with 2508 classification mistakes and Kappa Coefficient was 16.4000%, with variance 1.0455×10^{-4}.

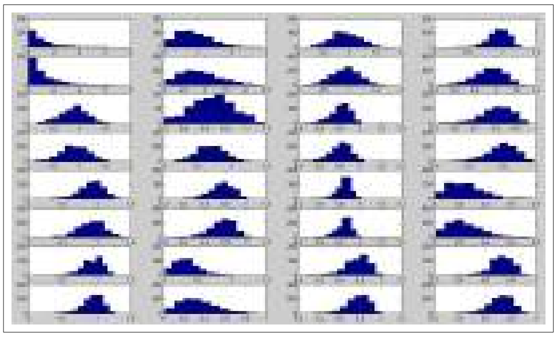

Fig 8. Random numbers generated for Weibull distribution - four sets of training and test samples (lines) for four classes (columns)

After, we increase the data dimension for two, but the parameters used were the same, except the MF, changed for 2. In this case, the Mean Square Error obtained was 1.0363, the percentile of correct classification was 47.4000%, with 2104 classification mistakes and Kappa Coefficient was 29.8667%, with variance 1.1141×10^{-4}.

For dimension 3, the parameters used were the same, except the MF, which was changed again for 6. In this case, the Mean Square Error obtained was 0.4516, the percentile of correct classification was 73.4250%, with 1063 classification mistakes and Kappa Coefficient was 64.5667%, with variance 8.6754×10^{-5}. Again the MF parameter was altered for the dimension 4, taking on the value 3. For that dimension, the Mean Square Error obtained was 0.2362, the percentile of correct classification was 90.7750%, with 369 classification mistakes and Kappa Coefficient was 87.7000%, with variance 3.7221×10^{-5}. We can observe that EFuNN obtained good results, however it did not obtain total correct classification for any of the researched dimensions.

The Table 1 presents a summary of best results obtained by each statistical distribution in those simulations. For distributions Binomial, Discrete and Continuous Uniform and Poisson distributions, using EFuNN is possible to achieve 100% in pattern recognition tasks, according to the Kappa Coefficient. However, for Exponential distribution, its performance is modest. For data which follows Gaussian or Weibull distributions, EFuNN is a competitive approach and the results are better according to higher dimensions of data.

Statistical Distribution	N° of Dimensions	Kappa Coefficient
Binomial	3 or more	100.000%
Continuous Uniform	2 or more	100.000%
Discrete Uniform	2 or more	100.000%
Exponential	4	52.025%
Gaussian	4	96.300%
Poisson	2 or more	100.000%
Weibull	4	87.700%

Table 1. Summary of bests results by statistical distributions, according to the Kappa Coefficient

5. Conclusions and Further Works

In this paper we presented an assessment of Evolving Fuzzy Neural Networks (EFuNNs) accuracy for pattern recognition using data with different statistical distributions. We made simulations with seven different statistical distributions: Binomial, Continuous Uniform, Discrete Uniform, Exponential, Gaussian, Poisson and Weibull. For each statistical distribution was analysed four different dimensions according to Mean Square Error, percentile of correct classification, number of classification mistakes, Kappa Coefficient and its variance.

According to the results obtained, it seems to be advisable to use EFuNN to classify data from the Binomial, Discrete and Continuous Uniform and Poisson distributions. In opposition, it seems to be not advisable to use EFuNN to classify data from the Exponential. For the Gaussian and Weibull distributions, EFuNN is able to perform a good classification, but the increasing of accuracy is related to higher dimensions of data.

As future work, we intend to expand the studies to data dimensions higher than those studied in this work. Another possibility to be explored is combining different distributions for a same simulation.

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REPORT

Fuzzy Differential Equations with Arithmetic and Derivative via Zadeh's Extension

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We adopt the derivative for fuzzy functions obtained via Zadeh's extension of the classical derivative operator and implement it in dynamical systems. Particularly, we find solutions to fuzzy initial value problems (FIVPs) that preserve the main properties and characteristics of functions of the base space, as periodicity and stability. This is a known feature of fuzzy differential inclusions (FDIs). However, unlike solving inclusions, we study a theory for fuzzy differential equations (FDEs). Some examples are provided to illustrate the theory and the solutions to FIVP are compared with those from other approaches.

1. Introduction

Zadeh's extension is considered a very powerful technique in fuzzy sets theory. It is a case of united extension, that is, it extends functions whose arguments are points to functions whose arguments are sets. Denoting by $\mathcal{F}(E)$ the space of fuzzy subsets of a topological space E, Zadeh's extension, in particular, returns functions over the space $\mathcal{F}(E)$, given that these functions were originally defined over the space E. What is interesting is that such extended functions inherit the main properties and characteristics of the original function.

In fuzzy dynamical systems, many approaches make use of Zadeh's extension. One example is the approach first proposed by Oberguggenberger in [1] and later studied by Mizukoshi *et al.* in [2], which consists of solving an initial value problem (IVP) and extending the solution according to fuzzy parameters. It has been proved in [2] that under certain conditions, these solutions have the same attainable sets as those obtained via FDIs.

It is also a common practice to define the field of a FIVP as Zadeh's extension of some classical function. Chalco-Cano *et al.* claim in [3] that this is an optimal interpretation and argue for the resulting arithmetic.

The usual fuzzy interval arithmetic (Moore's interval arithmetic) is very simple to compute, since it operates only with interval endpoints. However, it leads to overestimations in general. Zadeh's extension, on the other hand, defines a fuzzy arithmetic that is equivalent to using constraint interval arithmetic and extending it to fuzzy intervals, state Chalco-Cano *et al.* ([3]).

By studying FIVPs via the strongly generalized Hukuhara derivative (G-derivative) with the field given by Zadeh's extension, Chalco-Cano *et al.* ([3]) intended to avoid the undesired property resulting from blending Moore's interval arithmetic with the Hukuhara derivative (H-derivative). In the latter, it is well known that the diameter of the solution of a fuzzy initial value problem is always non-decreasing, as Kaleva asserts in [4].

In this article we study the FIVPs obtained by applying Zadeh's extension to classical IVPs. Different from Chalco-Cano *et al.*'s paper, [3], not only the field is extended, but also the the derivative operator is obtained via fuzzification of the classical derivative.

To start with, it is important to define the type of fuzzy function that we operate with. For our purposes, we define fuzzy function as the fuzzy subset of a classical function space. In other words, a fuzzy function is an element X that belongs to $\mathcal{F}(E)$, where E is a function space. We are specially concerned with the case when E is the space of absolutely continuous functions, which we denote $\mathcal{AC}([0,T];R^n)$. The H-derivative and the G-derivative, in turn, operate with functions of type $X:[0,T]\to\mathcal{F}(R^n)$.

In order to compare these two kind of fuzzy functions, $X\in\mathcal{F}(\mathcal{AC}([0,T];R^n))$ and $X:[0,T]\to\mathcal{F}(R^n)$, for each $X\in\mathcal{F}(\mathcal{AC}([0,T];R^n))$ we define the attainable sets at time t, $X(t)$, as the fuzzy subset of R^n whose α-levels are

$$[X(t)]^\alpha=[X]^\alpha(t)=\{x(t):x\in[X]^\alpha\}\subset R^n\ .$$

We investigate a concept of derivative for fuzzy function which is similar to that suggested by Chang and Zadeh ([5]) and later briefly studied by Dubois and Prade ([6]). We employ the interpretation of Barros *et al.* in [7]: the idea is to extend the classical derivative operator via Zadeh's extension (see Section 2 for more details).

Let $\mathcal{F}_\mathcal{K}(E)$ stand for the space of fuzzy subsets in a given topological space E with nonempty compact α-levels.

Briefly, given an IVP

$$\begin{cases} x'(t) &= f(t,x(t)) \\ x(0) &= x_0 \end{cases}, \tag{51}$$

where $x_0\in R^n$ and $f:R\times R^n\to R^n$, our purpose is to explore the FIVP

$$\begin{cases} \widehat{D}X(t) &= \widehat{f}(t,X(t)) \\ X(0) &= X_0 \end{cases} \tag{52}$$

where $X_0\in\mathcal{F}_\mathcal{K}(R^n)$, $\widehat{f}:R\times\mathcal{F}_\mathcal{K}(R^n)\to\mathcal{F}_\mathcal{K}(R^n)$ with $\widehat{f}$ obtained by applying Zadeh's extension to the second argument of a continuous function $f:R\times R^n\to R^n$ (or $f:R\times R^n\to\mathcal{F}_\mathcal{K}(R^n)$) and $\widehat{D}$ stands for the derivative (see Section 2).

In [7], Barros *et al.* propose this kind of FDEs in a more general context and prove an existence theorem. As we shall see, this approach, in fact, leads to a theory of FDE, as does the H-derivative ([8]) and the G-derivative ([9]). The solutions, however, can be related to those from

FDIs (more details about FDIs can be found in [10], [11] and [12]).

This paper is organized as follows. In Section 2, we display basic definitions and essential theorems. The notion of fuzzy derivative explored in this study, $\widehat{D}$-derivative, is also presented. Results in FIVPs employing the $\widehat{D}$-derivative are illustrated with some examples and we compare it to other existing approaches, namely FDIs and H-derivative. We conclude the study adding relevant final comments in Section 3.

2. Development

In abstract spaces U and V, given a function $f : U \to V$, we adopt the definition

$$\widehat{f}(u)(y) = \begin{cases} \sup_{s \in f^{-1}(y)} u(s), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{if } f^{-1}(y) = \emptyset \end{cases} \tag{53}$$

for Zadeh's extension. Thus, $\widehat{f}$ is a fuzzy function such that $\widehat{f} : \mathcal{F}(U) \to \mathcal{F}(V)$.

This means that $f(x)$ belongs to $\widehat{f}(u)$ to the same degree as x belongs to u, provided that f is an injective function.

It is well established that if f is a continuous function then $[\widehat{f}(A)]^\alpha = f([A]^\alpha)$ for each $\alpha \in [0,1]$ (for more details, see [13]). According to Cecconello in [14], this result is valid if the base space is a Hausdorff space.

When $f : U \to \mathcal{F}(V)$, Chang and Zadeh ([5]) and Huang and Wu ([15]) define $\widetilde{f} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that

$$\widetilde{f}(u)(y) = \sup_{x \in U} \{f(x)(y) \wedge u(x)\} \ . \tag{54}$$

As we shall see shortly, this definition of extension will play an important role in this paper.

Provided the metric

$$d_\infty(u,v) = \sup\{d_H([u]^\alpha,[v]^\alpha) : 0 \leq \alpha \leq 1\} \ ,$$

in $\mathcal{F}_\mathcal{K}(R^n)$, [15] also states the following theorem.

Theorem 4 *[15]*
Let $f : R \to \mathcal{F}(R)$ be a d_∞-continuous function. Then $\widetilde{f}$ is d_∞-continuous and $[\widetilde{f}(u)]^\alpha = \bigcup_{x \in [u]^\alpha} [f(x)]^\alpha$.

Derivative of fuzzy function

We will use D to represent the derivative operator, i.e.,

$$\begin{array}{rccc} D: & \mathcal{AC}([0,T];R^n) & \to & L^\infty([0,T];R^n) \\ & w & \to & Dw = w' \end{array}$$

where w' is the derivative in the sense of distributions (see [16]). Thus, there exists $Dw(t)$ *a.e.*, in $[0,T]$.

Definition 1 *Derivative of fuzzy function*
Let $W \in \mathcal{F}(\mathcal{AC}([0,T];R^n))$ be a fuzzy function. We define $\widehat{D}W$ as the derivative of W, where $\widehat{D}$ is given by Zadeh's extension of operator D, according to formula (53).

Hence, w' belongs to $\widehat{D}(W)$ with the same membership degree that $w+k$ belongs to W, for some $k \in R^n$. Since the operator D is not continuous for uniform convergence (sup norm), the identity $[\widehat{D}W]^\alpha = D([W]^\alpha)$ is not immediate. Nevertheless, since D is a closed operator, this property can be proved (see [7]).

Theorem 5 *[7]*
Let $W \in \mathcal{F}_\mathcal{K}(\mathcal{AC}([0,T];R^n))$. Then

$$[\widehat{D}(W)]^\alpha = D([W]^\alpha) \ .$$

Fuzzy differential equations

Consider the FIVP

$$\begin{cases} \widehat{D}X(t) &=& F(t,X(t)) \\ X(0) &=& X_0 \end{cases} \tag{55}$$

where $F : R \times \mathcal{F}_\mathcal{K}(R^n) \to \mathcal{F}_\mathcal{K}(R^n)$ and $X_0 \in \mathcal{F}_\mathcal{K}(R^n)$.

A solution to (55) is a fuzzy function $X(\cdot) \in \mathcal{F}_\mathcal{K}(\mathcal{AC}([0,T];R^n))$ that satisfies (55) *a.e.* in $[0,T]$.

If F meets certain conditions, including

$$[F(t,X)]^\alpha = \bigcup_{x \in [X]^\alpha} [F(t,x)]^\alpha \tag{56}$$

the existence of a solution to (55) is assured [7]. In fact, it was proved that the solution of the FDI associated (that is, solving the problem using the theory of FDIs) is also a solution to (55). In particular, if F is Zadeh's extension of some classical function, (56) is satisfied. Actually (see Theorem 4), if a fuzzy-set-valued function is d_∞-continuous, then its extension in the sense of (54) satisfies (56).

In this paper we consider the field F as Zadeh's extension, i.e., we deal with FIVPs of type

$$\begin{cases} \widehat{D}X(t) &=& \widehat{f}(t,X(t)) \\ X(0) &=& X_0 \end{cases} \tag{57}$$

where $X_0 \in \mathcal{F}_\mathcal{K}(R^n)$ and $\widehat{f} : R \times \mathcal{F}_\mathcal{K}(R^n) \to \mathcal{F}_\mathcal{K}(R^n)$ in which $\widehat{f}$ is Zadeh's extension of $f : R \times R^n \to R^n$ (or $f : R \times R^n \to \mathcal{F}_\mathcal{K}(R^n)$ like in formula (54)).

From Theorem 5, in α-levels, (57) is equivalent to the family of IVPs

$$\begin{cases} D[X]^\alpha(t) &=& [\widehat{f}(t,X(t))]^\alpha \\ [X(0)]^\alpha &=& [X_0]^\alpha \end{cases} \tag{58}$$

for all $\alpha \in [0,1]$.

We present some examples in which the field is given by Zadeh's extension of some function with real-valued argument. It is worth mentioning that, when we write equations such as $X + X^2$, we do not mean that we have applied Moore's interval arithmetic to the α-levels. This is simply a notation for extending the function $x + x^2$. Details of the resulting arithmetics can be found in [3] and [17].

Example 10 *Kaleva [4] considers the FIVP*

$$\begin{cases} X'(t) &=& X(t)^2 \\ X(0) &=& X_0 \end{cases} \tag{59}$$

where X_0 is the triangular fuzzy number $(1,2,3)$.

- ***FDI***

We will first find a solution to the FDI associated to (59):

$$\begin{cases} x'(t) &=& x(t)^2 \\ x_0 &\in& [1+\alpha, 3-\alpha] \end{cases} \tag{60}$$

where f is the continuous function $f(x) = x^2$, for each $\alpha \in [0,1]$.

The solution to (60) has the α-levels

$$\begin{aligned} &[X(\cdot)]^\alpha = \\ &\{x(.) : x'(t) = (x(t))^2, x_0 \in [1+\alpha, 3-\alpha]\} = \\ &\left\{x(.) : x(t) = \frac{x_0}{1-x_0 t}, x_0 \in [1+\alpha, 3-\alpha]\right\}. \end{aligned} \tag{61}$$

- ***H-derivative***

According to [4], the solution via the H-derivative calculated in time t is equal to the attainable set in t of the solution to the FDI associated (60).

The attainable fuzzy sets, $X(t)$, are given by the α-levels

$$\begin{aligned} [X(t)]^\alpha &= \left\{\frac{x_0}{1-x_0 t}, x_0 \in [1+\alpha, 3-\alpha]\right\} \\ &= \left[\frac{1+\alpha}{1-(1+\alpha)t}, \frac{3-\alpha}{1-(3-\alpha)t}\right], \end{aligned} \tag{62}$$

because the function $\frac{x_0}{1-x_0 t}$ is continuous with regard to x_0, if $0 \le t < 1/3$.

- ***$\widehat{D}$-derivative***

*Since the multiplication in $X \times X$ is interpreted as extension of $f(x) = x^2$, it is possible to verify that (**??**) is also a solution to (59) with the $\widehat{D}$-derivative. That is, the attainable sets $X(t)$ of both solutions, via FDI and $\widehat{D}$-derivative, are the same. And these attainable sets have the same α-levels as the solution via the H-derivative in time t. These solutions are illustrated in Figure 1.*

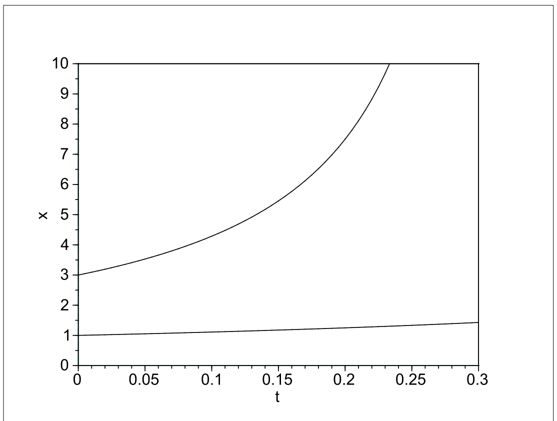

Fig 1. The upper and lower bounds of the support of a solution to $\widehat{D}X(t) = X(t)^2$ of Example 10 with the triangular fuzzy number $(1,2,3)$ as initial condition. The attainable sets of the solution obtained employing the $\widehat{D}$-derivative are the same attainable sets as the solution via FDI. They also coincide with the solution using the H-derivative

Example 11 *Consider a decay model, in which the parameter is uncertain and modeled in a fuzzy context:*

$$\begin{cases} X'(t) &=& -\Lambda X(t) \\ X(0) &=& x_0 \in R \end{cases} \tag{63}$$

with Λ a fuzzy number whose α-levels are $[\Lambda]^\alpha = [\lambda_1^\alpha, \lambda_2^\alpha]$ and $\lambda_1^\alpha > 0$, for all $\alpha \in [0,1]$.

Note that $F(X) = -\Lambda X = \widetilde{f}(X)$, where $\widetilde{f}$ is extension of $f(x) = -\Lambda x$ (according to the definition of extension in [15]).

We will first find a solution to (63) via the H-derivative and then offer another one by applying the $\widehat{D}$-derivative.

- ***H-derivative***

For each α-level, applying the H-derivative in (63) leads to the classical system

$$\begin{cases} (x_1^\alpha)'(t) &=& -\lambda_2^\alpha x_2^\alpha(t) \\ (x_2^\alpha)'(t) &=& -\lambda_1^\alpha x_1^\alpha(t) \\ x_1^\alpha(0) &=& x_0 \\ x_2^\alpha(0) &=& x_0 \end{cases} \tag{64}$$

whose solution is

$$\begin{cases} x_1^\alpha(t) &=& c_1^\alpha e^{\sqrt{\lambda_1^\alpha \lambda_2^\alpha}t} + c_2^\alpha e^{-\sqrt{\lambda_1^\alpha \lambda_2^\alpha}t} \\ x_2^\alpha(t) &=& -\sqrt{\lambda_1^\alpha/\lambda_2^\alpha}\, c_1^\alpha e^{\sqrt{\lambda_1^\alpha \lambda_2^\alpha}t} + \sqrt{\lambda_1^\alpha/\lambda_2^\alpha}\, c_2^\alpha e^{-\sqrt{\lambda_1^\alpha \lambda_2^\alpha}t} \end{cases} \tag{65}$$

with

$$c_1^\alpha = \frac{\left(-\sqrt{\lambda_1^\alpha/\lambda_2^\alpha}+1\right)x_0}{2} \quad \text{and} \quad c_2^\alpha = \frac{\left(\sqrt{\lambda_1^\alpha/\lambda_2^\alpha}+1\right)x_0}{2}.$$

Thus, the solution $X(t)$ of (63) via the H-derivative has the α-levels $[X(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)]$.

- ***$\widehat{D}$-derivative***

To find the solution $X(\cdot)$ via the $\widehat{D}$-derivative, we calculated affine combinations of the endpoints λ_1^α and λ_2^α:

$$\beta_1^{(\alpha,\gamma)} = \gamma\lambda_2^\alpha + (1-\gamma)\lambda_1^\alpha \quad \text{and} \quad \beta_2^{(\alpha,\gamma)} = \gamma\lambda_1^\alpha + (1-\gamma)\lambda_2^\alpha$$

with $\gamma \in [0,1]$ and $\alpha \in [0,1]$.

The idea here is to find curves in the format (65) such that they are between $x_1^\alpha(t)$ and $x_2^\alpha(t)$ of the solution via the H-derivative. It is also necessary that they fill the space between $x_1^\alpha(t)$ and $x_2^\alpha(t)$, that is, we want a compact subset of functions. For this, it is sufficient to use the affine combinations of λ_1^α and λ_2^α as coefficients of system (64). This technique is explored by [17] to represent fuzzy intervals.

Using this as new parameters for problem (64), we obtain the family of curves

$$\begin{cases} x_1^{(\alpha,\gamma)}(t) &=& c_1^{(\alpha,\gamma)} e^{\sqrt{\beta_1^{(\alpha,\gamma)}\beta_2^{(\alpha,\gamma)}}t} + \\ && c_2^{(\alpha,\gamma)} e^{-\sqrt{\beta_1^{(\alpha,\gamma)}\beta_2^{(\alpha,\gamma)}}t} \\ x_2^{(\alpha,\gamma)}(t) &=& k^{(\alpha,\gamma)}\left(-c_1^{(\alpha,\gamma)} e^{\sqrt{\beta_1^{(\alpha,\gamma)}\beta_2^{(\alpha,\gamma)}}t} + \right. \\ && \left. c_2^{(\alpha,\gamma)} e^{-\sqrt{\beta_1^{(\alpha,\gamma)}\beta_2^{(\alpha,\gamma)}}t}\right) \end{cases} \tag{66}$$

with

$$k^{(\alpha,\gamma)} = \sqrt{\beta_1^{(\alpha,\gamma)}/\beta_2^{(\alpha,\gamma)}},$$

$$c_1^{(\alpha,\gamma)} = \frac{\left(-\sqrt{\beta_1^{(\alpha,\gamma)}/\beta_2^{(\alpha,\gamma)}}+1\right)x_0}{2} \quad \text{and}$$
$$c_2^{(\alpha,\gamma)} = \frac{\left(\sqrt{\beta_1^{(\alpha,\gamma)}/\beta_2^{(\alpha,\gamma)}}+1\right)x_0}{2}.$$

The union of all these curves produces the α-levels of a fuzzy function $X(\cdot)$ which is the solution of (63) employing the $\widehat{D}$-derivative. In order to see this, note that, since each pair $(x_1^{(\alpha,\gamma)}(t), x_2^{(\alpha,\gamma)}(t))$ is solution to problem (64) with affine combinations of λ_1^α and λ_2^α as coefficients, the union of the derivative of all these curves is equal to $\{-\beta_i^{(\alpha,\gamma)} x_i^{(\alpha,\gamma)} : 0 \le \alpha,\gamma \le 1; i = 1,2\}$. Calculating the attainable sets in time t gives us $\{-\beta_i^{(\alpha,\gamma)} x_i^{(\alpha,\gamma)}(t) : 0 \le \alpha,\gamma \le 1; i = 1,2\} = [-\Lambda]^\alpha[-X(t)]^\alpha$, and so $[\widehat{D}X(t)]^\alpha = [-\Lambda X(t)]^\alpha$. Thus, in fact, $X(\cdot)$ is a solution to

$$\begin{cases} \widehat{D}X(t) &= -\Lambda X(t) \\ x(0) &= x_0 \end{cases}. \tag{67}$$

The attainable sets in time t are the solution to (63) via the H-derivative calculated in time t. This is due to the fact that the attainable sets of the α-levels of the solution are the intervals $[x_1^{(\alpha,0)}(t), x_2^{(\alpha,0)}(t)]$ given by (66), which are the same as $[x_1^\alpha(t), x_2^\alpha(t)]$ given by (65).

Hence, in this case, the attainable sets of a solution to (67) are the same as those obtained via the H-derivative (illustrated in Figure 2).

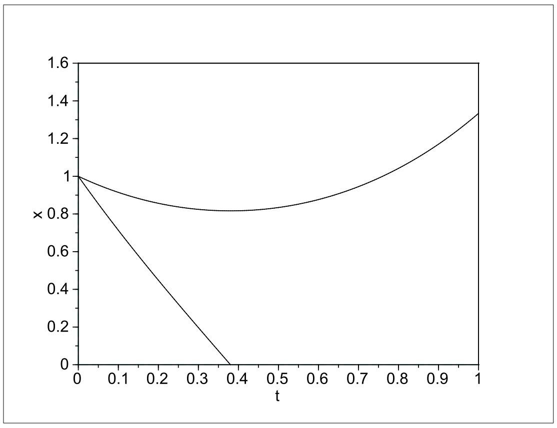

Fig 2. The upper and lower bounds of the support of a solution to $\widehat{D}X(t) = -\Lambda X(t), X(0) = x_0$, from Example 11. Here Λ is the fuzzy triangular number $(1,2,3)$ and initial condition is $x_0 = 1$. The attainable set in time t of this solution is the solution via the H-derivative calculated in time t

At this point, it is worth mentioning that the solution found for (67) using the $\widehat{D}$-derivative is not necessarily unique. This will be illustrated by solving another FIVP in the Example 12 below.

Example 12 *Consider the decay model in which the initial condition is fuzzy:*

$$\begin{cases} X'(t) &= -\lambda X(t) \\ X(0) &= X_0 \end{cases} \tag{68}$$

where X_0 is a fuzzy number and $\lambda > 0$.

- ***FDI***

  Solving the associated FDI,

$$\begin{cases} x'(t) &= -\lambda x(t) \\ x_0 &\in [X_0]^\alpha \end{cases} \tag{69}$$

for each $\alpha \in [0,1]$ we obtain $[X(\cdot)]^\alpha = \{x_0 e^{-\lambda\cdot}, x_0 \in [x_{01}^\alpha, x_{02}^\alpha]\}$, whose attainable sets are $[X(t)]^\alpha = [x_{01}^\alpha, x_{02}^\alpha]e^{-\lambda t}$.

- ***$\widehat{D}$-derivative: first solution***

  From a direct calculation, one can confirm that $X(\cdot)$ is also a solution to (68) employing the $\widehat{D}$-derivative. Actually, the existence theorem (see Theorem 5.2 in [7]) guarantees this result.

- ***H-derivative***

  Using the H-derivative, we find a solution to (68) solving

$$\begin{cases} (x_1^\alpha)'(t) &= -\lambda^\alpha x_2^\alpha(t) \\ (x_2^\alpha)'(t) &= -\lambda^\alpha x_1^\alpha(t) \\ x_1^\alpha(0) &= x_{01}^\alpha \\ x_2^\alpha(0) &= x_{02}^\alpha \end{cases}. \tag{70}$$

So, the solution $X(t)$ is such that $[X(t)]^\alpha = [x_1^\alpha(t), x_2^\alpha(t)]$ with

$$\begin{cases} x_1^\alpha(t) &= c_1^\alpha e^{\lambda t} + c_2^\alpha e^{-\lambda t} \\ x_2^\alpha(t) &= -c_1^\alpha e^{\lambda t} + c_2^\alpha e^{-\lambda t} \end{cases} \tag{71}$$

and

$$c_1^\alpha = \frac{x_{01}^\alpha - x_{02}^\alpha}{2} \quad \text{and} \quad c_2^\alpha = \frac{x_{01}^\alpha + x_{02}^\alpha}{2}.$$

- ***$\widehat{D}$-derivative: second solution***

  There is another solution whose attainable sets coincide with the solution using the H-derivative

  To obtain a solution $X(\cdot)$ to the approach using the $\widehat{D}$-derivative, we construct $[X(t)]^\alpha$ by calculating affine combinations of the initial conditions in (70):

$$z_{01}^\alpha = \gamma x_{02}^\alpha + (1-\gamma)x_{01}^\alpha \quad \text{and} \quad z_{02}^\alpha = \gamma x_{01}^\alpha + (1-\gamma)x_{02}^\alpha$$

with $\gamma \in [0,1]$.

We employ the same procedure as in Example 11. We use the system obtained using the H-derivative to find curves to construct the solution for the $\widehat{D}$-derivative.

Using z_{01}^α and z_{02}^α as new initial conditions for problem (70), we obtain a family of curves:

$$\begin{cases} x_1^{(\alpha,\gamma)}(t) &= c_1^{(\alpha,\gamma)} e^{\lambda t} + c_2^{(\alpha,\gamma)} e^{-\lambda t} \\ x_2^{(\alpha,\gamma)}(t) &= -c_1^{(\alpha,\gamma)} e^{\lambda t} + c_2^{(\alpha,\gamma)} e^{-\lambda t} \end{cases}. \tag{72}$$

with

$$c_1^{(\alpha,\gamma)} = \frac{(1-2\gamma)x_{01}^\alpha - (1-2\gamma)x_{02}^\alpha}{2} \quad \text{and}$$
$$c_2^{(\alpha,\gamma)} = \frac{x_{01}^\alpha + x_{02}^\alpha}{2}$$

This family compounds a solution that is also a solution to (68) with the $\widehat{D}$-derivative. So, as mentioned previously, it has been shown that the $\widehat{D}$-derivative does not lead to unique solutions to FDEs (see Figures 3 and 4).

This is not specific to the $\widehat{D}$-derivative. The H-derivative is a particular case of the G-derivative. So, the solution obtained via the H-derivative is a solution to problem (68) using the G-derivative. It can be shown, however, that $[X(t)]^\alpha = [x_{01}^\alpha, x_{02}^\alpha]e^{-\lambda t}$ also satisfies (68) with the G-derivative [9].

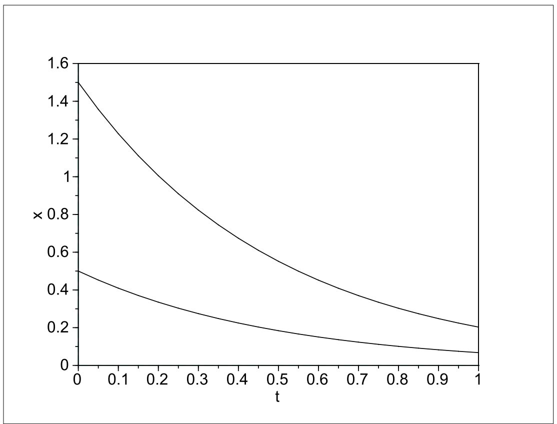

Fig 3. The upper and lower bounds of the support of a solution to FIVP** $\widehat{D}X(t) = -\lambda X(t), X(0) = X_0$**, from Example 12.** $\lambda = \{2\}$ **and** X_0 **is the triangular fuzzy number** $(0.5, 1, 1.5)$**. The solution has the same attainable sets as the one obtained by FDIs

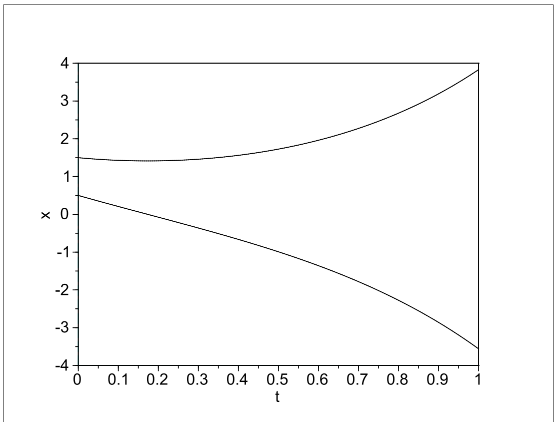

Fig 4. The upper and lower bounds of the support of another solution to the same FIVP** $\widehat{D}X(t) = -\lambda X(t), X(0) = X_0$**, from Example 12.** $\lambda = \{2\}$ **and** X_0 **is the triangular fuzzy number** $(0.5, 1, 1.5)$**. The solution has the same attainable sets as solution via H-derivative

1. Conclusion

We explored FIVPs employing fuzzification of the classical derivative operator and called it the $\widehat{D}$-derivative. Solutions to some FIVPs using this derivative have been proposed and compared with those from other approaches. It has also been shown that the property of uniqueness of the solution is not valid, even for a linear field (Example 12). Different solutions to the same problem were explicitly exhibited.

Unlike the approach based on differential inclusions (DIs), the $\widehat{D}$-derivative enables us to develop a theory of FDEs. Such theory is not limited to this case. Barros *et al.* ([7]) have presented other cases and have also provided theoretical basis required in this study.

The solutions X under the proposed approach belong to the space of fuzzy subsets of functions, that is, $X \in \mathcal{F}(E)$, where E is a classical space of functions. Thus, we focussed in fuzzy functions that preserve the main properties and characteristics of functions of the base space, a feature of the theory of FDIs. An example of a desired property that does not occur with the H-derivative is a solution for a decay model going to zero when $t \to \infty$, for all α-levels (see Figure 3). This behavior characterizes convergence, which leads us to conlude that it is possible to analyse stability in FIVPs using the $\widehat{D}$-derivative.

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REPORT

A Total Order for Symmetric Triangular (Interval) Fuzzy Numbers

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In type-2 fuzzy sets it is possible to model not only vague information (lack of sharp class boundaries, i.e., gradations in the notion of membership), but also uncertainty (lack of information about the membership function). In particular, interval-valued fuzzy numbers are specified by interval-valued membership functions. In the modeling of some applications (e.g, decision making, game theory), it is enough to consider symmetric triangular interval fuzzy numbers. One particular problem that may arise in such applications is the ordering or ranking interval fuzzy numbers. The aim of this paper is to introduce a total order for symmetric triangular (interval) fuzzy numbers, based on the also proposed total order for fuzzy numbers.

1. Introduction

In the modeling of different types of fuzzy applications, one may face the problem of the decision making based on the comparison among several fuzzy values. However, there is no universally accepted method for the ordering or ranking fuzzy numbers, since there are many forms for their representation. Different methods may provide different results. Thus, the choice of method to be used is an important factor in decision-making. [1, 2, 3, 4, 5, 6, 7]

When, in addition to the uncertainty modeled by fuzzy numbers, there is also uncertainty in how these numbers are constructed, interval fuzzy numbers may be used to model this other level of uncertainty [8]. The ordering of interval fuzzy numbers is even more complex than ordering fuzzy numbers and there are few studies in the literature specifically addressing this problem. [6, 9, 10, 11, 12]

There are many applications that demand that the order relation must be total to ensure a solution. For example, in game theory, the search for solutions requires a comparison of the rewards given to the choices of strategies by the players. If the rewards are vague, uncertain, in a way that their representations are given by interval fuzzy numbers, then a suitable ordering method is needed for the solution analysis [13, 14, 15, 16, 17, 18, 19, 20]. The problem of sorting interval fuzzy numbers becomes particularly important in certain applications such as the determination of equilibrium solutions in interval fuzzy Bayesian games, in which the estimated probabilities for the types of players are given by symmetric triangular interval fuzzy numbers. [21, 22, 23, 24, 25]

Also, for decision making, preference fuzzy relations are often used, which provide certain degrees represented by intervals, fuzzy numbers or interval fuzzy numbers. The comparison between these degrees is required to choose the best alternative to be adopted. [9, 10, 11, 12]

We have analyzed various methods available in the literature, which are discussed in Section 3. We observed that there is no practical method for ordering symmetrical triangular fuzzy numbers, providing intuitive and meaningful results, considering not only their position relation on the real line, but also taking into consideration both the accuracy of the information contained in the fuzzy number and the acceptable error for the considered application. Then, the so-called AD-order is introduced in this paper for addressing this specific type of problem.[6] This paper also introduces a method for ordering symmetrical triangular interval fuzzy numbers, based on the total AD-order relation for symmetrical triangular fuzzy numbers.[7]

The particular interest in symmetric triangular fuzzy numbers lies in its applications we are developing in (interval) fuzzy Bayesian games, where its sufficient to consider symmetric interval fuzzy numbers to represent interval fuzzy probabilities. [21, 23, 28, 29]

This paper is organized as follows. Section 2 summarizes the main concepts of fuzzy sets and numbers needed for the development of this work. Section 3 presents a discussion around the ordering of fuzzy numbers, introducing the AD-order and the proofs of its properties. In Section 4, the main concepts about interval fuzzy sets and numbers are presented. Section 5 extends the definition of the AD-order for interval fuzzy numbers, with the analysis of the properties under this definition.

2. Preliminary Concepts on Fuzzy Sets

In this section, we present the basic concepts of Fuzzy Sets Theory[8], which was introduced by Zadeh [34, 35].

The main concept of this theory is the membership function, which is an extension of the characteristic function of the classical theory of sets. Given a universe U, when one admits that an element $x \in U$ can belong gradually to a set F, then F is said to be fuzzy set of universe U (or just a fuzzy set F, whenever the universe U is not relevant), and one define a function $\varphi_F : U \to [0, 1]$, called the *membership function* of the fuzzy set F, indicating the degree to which x belongs to the set F. $\varphi_F(x) = 0$ and $\varphi_F(x) = 1$ represent, respectively, the complete pertinence and non-pertinence of x to the fuzzy set F.

[6] A preliminary version of the definition of the AD-order was presented as a poster in [26].
[7] A preliminary version of this work was presented at CBSF [27].
[8] For details on the theory of fuzzy sets, see [30, 31, 32, 33].

A *fuzzy subset* F of U can be represented by a set of ordered pairs:

$$F = \{(x, \varphi_F(x)) \mid x \in U\}, \tag{73}$$

where $\varphi_F(x)$ gives the membership degree in F for each element x of the domain U.

The *support* of F is defined as the set

$$supp_F = \{x \in U | \varphi_F(x) > 0\}, \tag{74}$$

meaning the set of all elements of U belonging, even partially, to the fuzzy set F. For $0 \leq \alpha \leq 1$, the *α-cuts* of F are defined as the classic subsets of U defined by

$$F[\alpha] = \begin{cases} \{x \in U | \varphi_F(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1 \\ \widehat{supp_F} & \text{if } \alpha = 0 \end{cases} \tag{75}$$

where $supp_F$ is the support of F defined in Eq. (74) and $\widehat{supp_F}$ is the closure of $supp_F$.[9] A fuzzy set is completely defined by its α-cuts. [8]

A fuzzy set $\bar{F}$ is called a *fuzzy number* when $\varphi_{\bar{F}}$ is defined in the set of real numbers $\mathbb{R}$, and the following conditions hold [8, 30, 31, 32, 33, 34, 35]:

a) all α-cuts of $\bar{F}$ are closed intervals in $\mathbb{R}$;

b) the support of $\bar{F}$ is bounded.

A fuzzy number is called *triangular* if its membership function is defined as[10]:

$$\varphi_{\bar{F}}(x) = \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{x-a}{u-a}, & \text{if } a < x \leq u \\ \dfrac{x-b}{u-b}, & \text{if } u < x \leq b \\ 0, & \text{if } x \geq b, \end{cases} \tag{76}$$

for $a, u, b \in \mathbb{R}$ and $a \leq u \leq b$. Thus, the triangular fuzzy number $\bar{F}$ is determined by the real numbers a, u and b, and is denoted by the ordered tuple $(a/u/b)$, where u is called the *core* of the fuzzy number, because it is the only value that belongs completely to the fuzzy set $\bar{F}$, that is, $\varphi_{\bar{F}}(u) = 1$.

The graphic representation of the membership function of a triangular fuzzy number can be seen in Figure 1. The support of $\bar{F}$ is an open interval $supp_{\bar{F}} = (a; b)$, and its representation through its α-cuts is given by

$$\bar{F}[\alpha] = [(u-a)\alpha + a, (u-b)\alpha + b]. \tag{77}$$

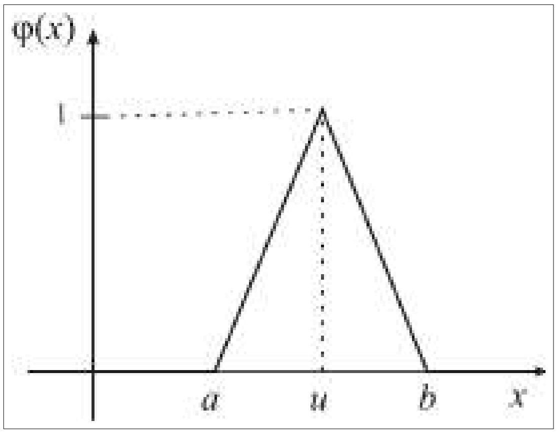

Fig 1. Membership function of a triangular fuzzy number

Note that the α-cuts of a fuzzy number are closed intervals. So to operate with fuzzy numbers, we can represent them through their α-cuts and apply interval arithmetic [36].

Of particular interest for this work are the *symmetric triangular fuzzy numbers*, when $b - u = u - a$, because they are used to model the expression "around of", commonly used for the modeling of fuzzy probabilities [28, 29], which are been used in our ongoing work concerning (interval) fuzzy Bayesian games [21, 25, 37].

3. Ordering Fuzzy Numbers

Given two fuzzy numbers $\bar{F_1}$ and $\bar{F_2}$, there are several ordering methods for analyzing if $\bar{F_1} \leq \bar{F_2}$, some of them given by partial orders and others not (see, e.g., [2, 3, 4, 5, 29, 38, 39]). The choice of which method to use depends on the membership functions of $\bar{F_1}$ and $\bar{F_2}$ and the problem being modeled. A comparison between various existing methods may be found in the work by Bortolan and Degani [1].

A common approach for ordering fuzzy numbers is through an ordering function that maps each fuzzy number on the line of real numbers, and then compare these real values. For example, in the work of Abbasbandy and Hajjari [40], a magnitude value, given by a real number, is associated to each fuzzy number and the ordering of fuzzy numbers are done by ordering those values. However, when two fuzzy number are symmetric, this approach fails in ranking them appropriately, since they present the same magnitude value. Other approaches based on similar principles also fails by analogous reason (see, e.g., the methods by Chu and Tsao [41], Wang and Lee [42] and Yao and Wu [43]). Ezzati et al. [44] have improved Abbasbandy-Hajjari's method, so that different symmetric fuzzy numbers having the same magnitude value can be distinguished. However, this method becomes trivial for symmetric triangular fuzzy numbers since the magnitude value coincides with the fuzzy number's core. Although the method succeeds in ordering symmetric triangular fuzzy numbers with the same core, the resulting ordering is not intuitive, since the method does not consider the quality of the information carried by the fuzzy number, as we discuss in the following.

[9] The closure of a set A is defined as the union of A and its boundary.
[10] See [30, 31, 32, 33, 34, 35] for other types of membership functions

There exist in the literature other kinds of ordering methods, e.g., based on obtaining a fuzzy set of optimal alternatives considering a degree that measures how much each alternative can be the best, determining, in other words, the "greatest" fuzzy number [1].

Thus, analyzing several methods available in the literature, we notice that none of them is efficient and practical to work with symmetrical triangular fuzzy numbers and, at the same time, takes into consideration the accuracy of the information contained in the fuzzy number [1], providing meaningful results. Then, the so-caalled AD-order was developed to address this specific type of problem. Before defining it, first we present two partial order relations that served as inspiration for the AD-order.

Order relation (1) [29]:

Given two fuzzy numbers $\bar{F_1}$ and $\bar{F_2}$ and their respective α-cuts, $\bar{F_1}[\alpha] = [a_1(\alpha), a_2(\alpha)]$ and $\bar{F_2}[\alpha] = [b_1(\alpha), b_2(\alpha)]$, one has that:

$$\bar{F_1} \leq_K \bar{F_2} \Leftrightarrow \forall\alpha \in [0;1] : \bar{F_1}[\alpha] \leq_K \bar{F_2}[\alpha], \qquad (78)$$

where

$$\forall\alpha \in [0;1] : \bar{F_1}[\alpha] \leq_K \bar{F_2}[\alpha] \Leftrightarrow a_1(\alpha) \leq_K b_1(\alpha) \ \wedge \ a_2(\alpha) \leq_K b_2(\alpha). \qquad (79)$$

Note that the relation $\leq_K$ defined in Eq. (79) consitute a parcial relation order of Kulisch-Miranker from Interval Mathematics [29]. This relation has the following properties, considering arbitrary fuzzy numbers $\bar{F_1}$, $\bar{F_2}$ and$\bar{F_3}$: [36, 45]

(a) Reflexivity: $\bar{F_1} \leq_K \bar{F_1}$;

(b) Anti-symmetry: $\bar{F_1} \leq_K \bar{F_2} \ \wedge \ \bar{F_2} \leq_K \bar{F_1} \ \Rightarrow \ \bar{F_1} = \bar{F_2}$;

(c) Transitivity: $\bar{F_1} \leq_K \bar{F_2} \wedge \bar{F_2} \leq_K \bar{F_3} \ \Rightarrow \ \bar{F_1} \leq_K \bar{F_3}$.

Thus, $\leq_K$ defined in Eq. (78) is a partial order relation, which is not total [36, 45].

Order relation (2) [28]:

$$\bar{F_1} \leq_I \bar{F_2} \Leftrightarrow \forall\alpha \in [0;1] : \bar{F_1}[\alpha] \subseteq \bar{F_2}[\alpha]. \qquad (80)$$

The relation $\leq_I$ in Eq. (80) is defined based on the interval arithmetic inclusion property, from Interval Mathematics [36, 45, 46]. It's also a partial order relation, but not total.

AD order:

The AD-order relation for symmetric triangular fuzzy numbers, denoted by $\bar{\leq}$, extends the order relation (1) and reinterprets the order relation (2), in the sense of an information order [47]. In the following, consider the symmetric triangular fuzzy numbers

$$\bar{F_1} = (a_1/u_1/b_1) \text{ e } \bar{F_2} = (a_2/u_2/b_2),$$

with their respective α-cuts characterized as

$$\bar{F_1}[\alpha] = [(u_1 - a_1)\alpha + a_1, (u_1 - b_1)\alpha + b_1] \qquad (81)$$
$$\bar{F_2}[\alpha] = [(u_2 - a_2)\alpha + a_2, (u_2 - b_2)\alpha + b_2]. \qquad (82)$$

To define the relation $\bar{\leq}$, we consider a degree of imprecision $\rho \in (0;1]$, which determines in which extension the cores' values of the compared fuzzy numbers are relevant for ordering fuzzy numbers that are too close. When assuming values approaching zero for ρ, we take into account the precision of the information contained in the fuzzy number more than the numeric quantity that this fuzzy number represents. Graphically, "narrow" fuzzy numbers tend to be greater than "wide" fuzzy numbers, because the former carry better information than the latter. [47, 48, 49]

Thus, the relation $\bar{<}$ is defined by four mutually exclusive conditions, which cover all possible relative positionings of two symmetric triangular fuzzy numbers, taking the imprecision degree ρ into account according to the modeled problem. So, we define:

Definition 2 *$\bar{F_1}\bar{<}\bar{F_2}$ if and only if one of the following conditions hold:*

(1)$(a_1 < a_2) \ \wedge \ (b_1 \leq b_2)$;
(2)$(a_1 < a_2) \ \wedge \ (b_2 < b_1) \ \wedge \ (u_1 \leq u_2)$
(3)$(a_1 < a_2) \ \wedge \ (b_2 < b_1) \ \wedge \ (u_2 < u_1) \ \wedge$
$[\forall\alpha : (0 \leq \alpha \leq \rho) \ \Rightarrow \ (u_1 - a_1)\alpha + a_1 \leq (u_2 - a_2)\alpha + a_2]$
(4)$(a_2 \leq a_1) \ \wedge \ (b_1 < b_2) \ \wedge \ (u_1 < u_2) \ \wedge$
$[\exists\alpha : (0 \leq \alpha \leq \rho) \ \wedge \ (u_1 - a_1)\alpha + a_1 < (u_2 - a_2)\alpha + a_2]$.

So, for any two symmetric triangular fuzzy numbers $\bar{F_1}$ and $\bar{F_2}$, one has that:

$$\bar{F_1}\bar{\leq}\bar{F_2} \Leftrightarrow \bar{F_1} = \bar{F_2} \ \vee \ \bar{F_1}\bar{<}\bar{F_2}. \qquad (83)$$

The $\bar{\leq}$ relation satisfies the properties of reflexivity, antisymmetry and transitivity, then constituting an order relation. Furthermore, it holds that that

$$\forall\bar{F_1}, \bar{F_2} : \quad \bar{F_1} = \bar{F_2} \ \underline{\vee} \ \bar{F_1}\bar{<}\bar{F_2} \ \underline{\vee} \ \bar{F_2}\bar{<}\bar{F_1},$$

where $\underline{\vee}$ denotes the exclusive disjunction, and, therefore, $\bar{\leq}$ is a total order relation.

In the following, we give a sketch of the proofs of these properties. Let $\bar{F_1} = (a_1/u_1/b_1)$ and $\bar{F_2} = (a_2/u_2/b_2)$ be two symmetric triangular fuzzy numbers. Then:

Proposition 5 $\forall\bar{F_1}, \bar{F_2} : \quad \bar{F_1} = \bar{F_2} \ \underline{\vee} \ \bar{F_1}\bar{<}\bar{F_2} \ \underline{\vee} \ \bar{F_2}\bar{<}\bar{F_1}$.

Proof. Since $\bar{F_1}$ and $\bar{F_2}$ are symmetric triangular fuzzy numbers, if $a_1 = a_2$, $u_1 = u_2$ and $b_1 = b_2$, then $\bar{F_1} = \bar{F_2}$. Then, we analyze the other alternatives below, some of them following directly from Definition 2:

Case 1: $(a_1 = a_2) \wedge (u_1 < u_2) \wedge (b_1 < b_2) \Rightarrow \bar{F_1}\bar{<}\bar{F_2}$. This follows from fact that, since $\rho > 0$, then for any $0 < \alpha < \rho$, the condition (4) of Definition 2 holds.

Case 2: $(a_1 < a_2) \wedge (u_1 < u_2) \wedge (b_1 \leq b_2) \Rightarrow \bar{F_1}\bar{<}\bar{F_2}$. This follows from the condition (1) of Definition 2. See the example in Figure 2.

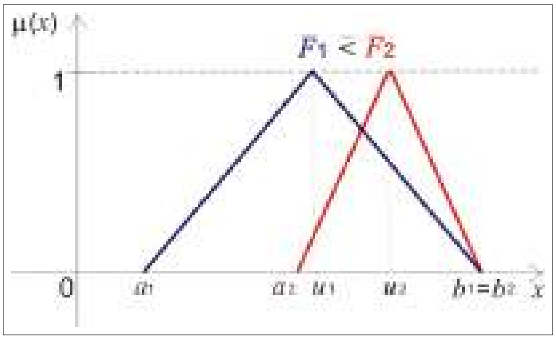

Fig 2. Example of the Case 2

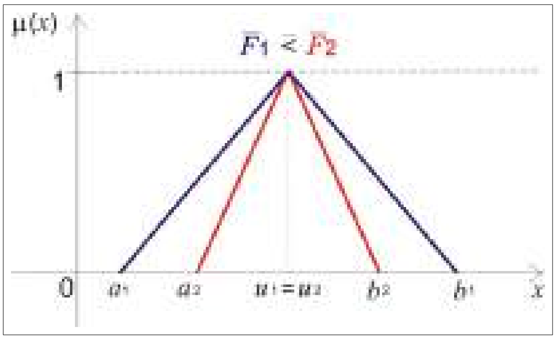

Fig 3. Example of the Case 3

Case 3: $(a_1 < a_2) \wedge (u_1 = u_2) \wedge (b_2 < b_1) \Rightarrow \bar{F}_1 \bar{<} \bar{F}_2$. This follows from the condition (2) of Definition 2. See the example in Figure 3.

Case 4: $(a_1 < a_2) \wedge (u_1 < u_2) \wedge (b_2 < b_1) \Rightarrow \bar{F}_1 \bar{<} \bar{F}_2$. This is similar of Case 5.

Case 5: If $(a_1 < a_2) \wedge (u_2 < u_1) \wedge (b_2 < b_1)$ then there are two possible alternatives for this case, depending on the assigned value for ρ. Then, if

$$\forall \alpha : (0 \leq \alpha \leq \rho) \;\; \Rightarrow \;\; (u_1 - a_1)\alpha + a_1 \leq (u_2 - a_2)\alpha + a_2,$$

then $\bar{F}_1 \bar{<} \bar{F}_2$. Otherwise, if

$$\exists \alpha : (0 \leq \alpha \leq \rho) \wedge (u_2 - a_2)\alpha + a_2 < (u_1 - a_1)\alpha + a_1$$

then $\bar{F}_2 \bar{<} \bar{F}_1$. Both conclusions follow directly form Definition2. Consider δ as the intersection between the lines $(u_1 - a_1)\alpha + a_1$ and $(u_2 - a_2)\alpha + a_2$. Thus, if $\delta \leq \rho$ then $\bar{F}_2 \bar{<} \bar{F}_1$, and if $\rho < \delta$, then $\bar{F}_1 \bar{<} \bar{F}_2$. It is immediate that both conclusions cannot be true simultaneously, thus, we have $\bar{F}_1 \leq \bar{F}_2 \;\underline{\vee}\; \bar{F}_2 \leq \bar{F}_1$. Both cases are exemplified in Figure 4.

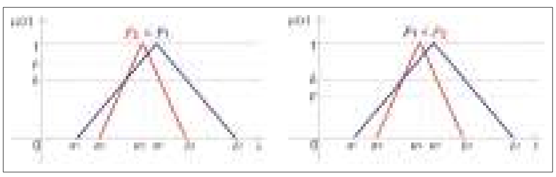

Fig 4. Examples of the Case 5, with different ρ

All remaining cases are analogous to the ones presented abouve, just switching the positioning between $\bar{F}_1$ and $\bar{F}_2$. Thus, for all possible cases one has that

$$\bar{F}_1 = \bar{F}_2 \;\underline{\vee}\; \bar{F}_1 \bar{<} \bar{F}_2 \;\underline{\vee}\; \bar{F}_2 \bar{<} \bar{F}_1 \square$$

Proposition 6 *The relation $\bar{\leq}$is reflexive.*

Proof. From Definition 2, it is immediate that for any fuzzy number $\bar{F}_1 = (a_1/u_1/b_1)$ it holds that $\bar{F}_1 \bar{\leq} \bar{F}_1$.□

Proposition 7 *The relation $\bar{\leq}$ is antisymmetric.*

Proof. We show that for any fuzzy numbers $\bar{F}_1$ and $\bar{F}_2$, it holds that

$$\bar{F}_1 \bar{\leq} \bar{F}_2 \wedge \bar{F}_2 \bar{\leq} \bar{F}_1 \Rightarrow \bar{F}_1 = \bar{F}_2.$$

For that suppose that $\bar{F}_1 \bar{\leq} \bar{F}_2$ and $\bar{F}_2 \bar{\leq} \bar{F}_1$, but $\bar{F}_1 \neq \bar{F}_2$. Then, it follows that $\bar{F}_1 \bar{<} \bar{F}_2$ and $\bar{F}_2 \bar{<} \bar{F}_1$, which is a contradiction by Proposition 5. Then, one concludes that $\bar{F}_1 = \bar{F}_2$.□

Proposition 8 *The relation $\bar{\leq}$ is transitive.*

Proof. For any $\bar{F}_1 = (a_1/u_1/b_1)$, $\bar{F}_2 = (a_2/u_2/b_2)$ and $\bar{F}_3 = (a_3/u_3/b_3)$, we show that $\bar{F}_1 \bar{\leq} \bar{F}_2 \wedge \bar{F}_2 \bar{\leq} \bar{F}_3 \Rightarrow \bar{F}_1 \bar{\leq} \bar{F}_3$. Assuming $\bar{F}_1 \bar{<} \bar{F}_2 \wedge \bar{F}_2 \bar{<} \bar{F}_3$ and an arbitrary $0 < \rho \leq 1$, then, from Definition 2, one has that:

$$\begin{array}{l}
\{(a_1 < a_2 \wedge b_1 \leq b_2) \vee (a_1 < a_2 \wedge b_2 < b_1 \wedge u_1 \leq u_2) \vee \\
(a_1 < a_2 \wedge b_2 < b_1 \wedge u_2 < u_1 \wedge \\
[\forall \alpha : 0 \leq \alpha \leq \rho \rightarrow (u_1 - a_1)\alpha + a_1 \leq (u_2 - a_2)\alpha + a_2]) \vee \\
(a_2 \leq a_1 \wedge b_1 \leq b_2 \wedge u_1 < u_2 \wedge \\
[\exists \alpha : 0 \leq \alpha \leq \rho \wedge (u_1 - a_1)\alpha + a_1 < (u_2 - a_2)\alpha + a_2])\} \\
\wedge \\
\{(a_2 < a_3 \wedge b_2 \leq b_3) \vee (a_2 < a_3 \wedge b_3 < b_2 \wedge u_2 \leq u_3) \vee \\
(a_2 < a_3 \wedge b_3 < b_2 \wedge u_3 < u_2 \wedge \\
[\forall \alpha : 0 \leq \alpha \leq \rho \rightarrow (u_2 - a_2)\alpha + a_2 \leq (u_3 - a_3)\alpha + a_3]) \vee \\
(a_3 \leq a_2 \wedge b_2 \leq b_3 \wedge u_2 < u_3 \wedge \\
[\exists \alpha : 0 \leq \alpha \leq \rho \wedge (u_2 - a_2)\alpha + a_2 < (u_3 - a_3)\alpha + a_3])\}.
\end{array}$$

Then, taking into account Definition 2, one has to analyze the following possibilities:

Case 1: $(a_1 < a_2 \wedge b_1 \leq b_2) \wedge (a_2 < a_3 \wedge b_2 \leq b_3)$.
As the relation $<$ for real numbers is transitive, then we have:

$$(a_1 < a_2 \wedge b_1 \leq b_2) \wedge (a_2 < a_3 \wedge b_2 \leq b_3) \Rightarrow$$
$$a_1 < a_3 \wedge b_1 \leq b_3 \Rightarrow \bar{F}_1 \bar{<} \bar{F}_3.$$

Case 2: $(a_1 < a_2 \wedge b_1 \leq b_2) \wedge (a_2 < a_3 \wedge b_3 < b_2 \wedge u_2 \leq u_3)$.
Similarly to the **Case 1**, we have:

$$(a_1 < a_2 \wedge b_1 \leq b_2) \wedge (a_2 < a_3 \wedge b_3 < b_2 \wedge u_2 \leq u_3) \Rightarrow$$
$$a_1 < a_3 \wedge u_1 < u_3.$$

Since there is no direct relation between b_1 and b_3, we analise all positioning possibilities between these real numbers:

(i) $a_1 < a_3 \wedge u_1 < u_3 \wedge b_1 \leq b_3 \Rightarrow \bar{F}_1 \bar{<} \bar{F}_3$;
(ii) $a_1 < a_3 \wedge u_1 < u_3 \wedge b_3 < b_1 \Rightarrow \bar{F}_1 \bar{<} \bar{F}_3$.

Thus, we observe that:

$$(a_1 < a_2 \wedge b_1 \leq b_2) \wedge (a_2 < a_3 \wedge b_3 < b_2 \wedge u_2 \leq u_3) \Rightarrow$$
$$\bar{F}_1 \bar{<} \bar{F}_3.$$

Case 3: If it is the case that:

$$(a_1 < a_2 \wedge b_1 \le b_2)$$
$$\wedge\ (a_2 < a_3 \wedge b_3 < b_2 \wedge u_3 < u_2 \wedge$$
$$[\forall\alpha : 0 \le \alpha \le \rho \wedge (u_2 - a_2)\alpha + a_2 \le (u_3 - a_3)\alpha + a_3]),$$

then, the only direct conclusion is that $a_1 < a_3$. Assuming $\bar{F_3}\bar{\le}\bar{F_1}$, we have two possibilities:

(i) $\bar{F_3}\bar{=}\bar{F_1}$, which is false, as $a_1 < a_3$;

(ii) $\bar{F_3}\bar{<}\bar{F_1}$. Considering Definition 2, the only condition that does not contradict $a_1 < a_3$ is:

$$(a_1 \le a_3) \wedge (b_3 \le b_1) \wedge (u_3 < u_1) \wedge$$
$$[\exists\alpha : 0 \le \alpha \le \rho \wedge$$
$$(u_3 - a_3)\alpha + a_3 < (u_1 - a_1)\alpha + a_1].$$

From Equation (84) and Equation (84), we obtain:

$$(\forall\alpha : 0 \le \alpha \le \rho \Rightarrow (u_2 - a_2)\alpha +$$
$$a_2 \le (u_3 - a_3)\alpha + a_3) \wedge$$
$$(\exists\alpha : 0 \le \alpha \le \rho \wedge (u_3 - a_3)\alpha +$$
$$a_3 < (u_1 - a_1)\alpha + a_1) \Rightarrow$$
$$\exists\alpha : 0 \le \alpha \le \rho \wedge (u_2 - a_2)\alpha +$$
$$a_2 < (u_1 - a_1)\alpha + a_1.$$

To analise the implication of Equation (84), we let α assume its extreme values, as $(u_2 - a_2)\alpha + a_2$ and $(u_1 - a_1)\alpha + a_1$ are both linear and crescent functions of α. For $\alpha = 0$, we have $a_2 < a_1$ and for $\alpha = 1$ we obtain $u_2 < u_1$, and both are false sentences, as they contradict Equation (84). Thus, Equation (84) is false, because there is no value of α that satisfies $(u_2 - a_2)\alpha + a_2 < (u_1 - a_1)\alpha + a_1$, so $\bar{F_3}\bar{<}\bar{F_1}$ is also false.

As $\bar{F_3}\bar{<}\bar{F_1}$ and $\bar{F_3} = \bar{F_1}$ are both false, according to the Proposition 5, we conclude that $\bar{F_1}\bar{<}\bar{F_3}$.

Case 4:

$$(a_1 < a_2 \wedge b_2 < b_1 \wedge u_1 \le u_2) \wedge$$
$$(a_2 < a_3 \wedge b_3 < b_2 \wedge u_2 \le u_3)$$

.

The consequence is immediate, since

$$(a_1 < a_2 \wedge b_2 < b_1 \wedge u_1 \le u_2) \wedge$$
$$(a_2 < a_3 \wedge b_3 < b_2 \wedge u_2 \le u_3) \Rightarrow$$
$$(a_1 < a_3 \wedge b_3 < b_1 \wedge u_1 \le u_3) \Rightarrow$$
$$\bar{F_1}\bar{<}\bar{F_3}.$$

Case 5: This is the case that:

$$(a_1 < a_2 \wedge b_2 < b_1 \wedge u_2 < u_1 \wedge$$
$$[\forall\alpha : 0 \le \alpha \le \rho \wedge (u_1 - a_1)\alpha +$$
$$a_1 \le (u_2 - a_2)\alpha + a_2]) \wedge$$
$$(a_3 \le a_2 \wedge b_2 \le b_3 \wedge u_2 < u_3 \wedge$$
$$[\exists\alpha : 0 \le \alpha \le \rho \wedge$$
$$(u_2 - a_2)\alpha + a_2 < (u_3 - a_3)\alpha + a_3]).$$

From Equation (84), we observe that

$$\exists\alpha : 0 \le \alpha \le \rho \wedge (u_1 - a_1)\alpha + a_1 < (u_3 - a_3)\alpha + a_3],$$

making the sentence $\bar{F_1}\bar{=}\bar{F_3}$ false. For $\alpha = 0$, we have $a_1 < a_3$, and for $\alpha = 1$, we obtain $u_1 < u_3$. Thus, all sentences from the Equation (2) for $\bar{F_3}\bar{<}\bar{F_1}$ contradict at least one of the consequences of the Equation (84). As $\bar{F_3}\bar{<}\bar{F_1}$ and $\bar{F_3} = \bar{F_1}$ are false, according to Proposition 5, we conclude that $\bar{F_1}\bar{<}\bar{F_3}$.

Case 6: This is the case that:

$$\{a_2 \le a_1 \wedge b_1 \le b_2 \wedge u_1 < u_2 \wedge$$
$$[\exists\alpha : 0 \le \alpha \le \rho \wedge (u_1 - a_1)\alpha$$
$$+ a_1 < (u_2 - a_2)\alpha + a_2]\} \wedge (a_2 < a_3 \wedge b_2 \le b_3).$$

It is immediate that $b_1 < b_3$ and $u_1 < u_3$, so $\bar{F_1} = \bar{F_3}$ is false. Assuming $\bar{F_3}\bar{<}\bar{F_1}$, the only sentence from Equation (2) that does not contradict neither $b_1 < b_3$ nor $c_1 < c_3$ is:

$$(a_3 < a_1) \wedge (b_1 < b_3) \wedge (u_1 < u_3) \wedge$$
$$[\forall\alpha : (0 \le \alpha \le \rho) \wedge (u_3 - a_3)\alpha + a_3 \le (u_1 - a_1)\alpha + a_1].$$

From Equation (84) and Equation (84), onde has that:

$$[\exists\alpha : 0 \le \alpha \le \rho \wedge (u_1 - a_1)\alpha + a_1 < (u_2 - a_2)\alpha + a_2]\} \wedge$$

$$[\forall\alpha : (0 \le \alpha \le \rho) \Rightarrow (u_3 - a_3)\alpha + a_3 \le (u_1 - a_1)\alpha + a_1] \Rightarrow$$

$$\exists\alpha : 0 \le \alpha \le \rho \wedge (u_3 - a_3)\alpha + a_3 < (u_2 - a_2)\alpha + a_2. \quad (84)$$

Assuming different values for α in Equation (84), we observe that $a_3 < a_2$ for $\alpha = 0$, and $u_3 < u_2$ for $\alpha = 1$, and both cases contradict Equation (84). Thus, $\bar{F_3}\bar{<}\bar{F_1}$ is false. According to Proposition 5, we conclude that$\bar{F_1}\bar{<}\bar{F_3}$.

The remaining cases are similar to the ones presented above, using the transitivity of the relation $<$ and the linearity of the compared functions of α (the complete proof is shown in [37]). Thus, through the analysis of all possibilities we conclude that:

$$\bar{F_1}\bar{<}\bar{F_2} \wedge \bar{F_2}\bar{<}\bar{F_3} \Rightarrow \bar{F_1}\bar{<}\bar{F_3} \quad (85)$$

proving the transitivity for the relation $\bar{<}$. The relations $\bar{<}$ and $=$ are both transitive, so the remaining possible cases are analogous from the ones shown here. □

Finally, we observed that the relation $\bar{\le}$ is reflexive, antisymmetric and transitive, configuring an order relation. Then, from Proposition (5), it follows that:

Theorem 6 *The relation $\bar{\le}$ is a total order relation.*

4. Interval Fuzzy Sets

The fuzzy set theory is a useful tool for modeling uncertainty. However, sometimes it is difficult to determine the membership degree to be used for certain problems. To work around this situation, several authors (e.g., as discussed in [48, 49, 50, 51]) represent the membership degrees through real intervals, thus extending fuzzy sets to interval fuzzy sets.

Considering $\mathbb{IR}$ as the set of all real intervals, let $UI = [0,1] \in \mathbb{IR}$ be the real unit interval, and define $\mathbb{U} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$ as the set of all sub-intervals of UI.

Thus, an interval fuzzy subset A of a universe $\mathbb{X}$ is defined as the set of ordered pairs $A = \{(x, \mu_A(x)) \mid x \in \mathbb{X}\}$, where $\mu_A : \mathbb{R} \to \mathbb{U}$ is the interval membership function of A.

If the interval membership function μ_A is continuous[11], then there are continuous functions $\mu_{A_l}, \mu_{A_u} : \mathbb{X} \to UI$ called respectively as lower membership function (LMF) and upper membership function (UMF), such that, for every $x \in \mathbb{X}$:

$$\mu_A(x) = [\mu_{A_l}(x), \mu_{A_u}(x)], \tag{86}$$

where $\mu_{A_l}(x) \leq \mu_{A_u}(x)$. The inner and outer supports of an interval fuzzy set A of $\mathbb{X}$ are defined, respectively, by:

$$lsupp_A = \{x \in \mathbb{X} \mid \mu_{A_l}(x) > 0\}; \tag{87}$$

$$usupp_A = \{x \in \mathbb{X} \mid \mu_{A_u}(x) > 0\}. \tag{88}$$

For the same A, the core of this interval fuzzy subset is defined by:

$$core_A = \{x \in \mathbb{X} \mid \mu_A(x) = [1; 1]\}. \tag{89}$$

In other words:

$$core_A = \{x \in \mathbb{X} \mid \mu_{A_l}(x) = \mu_{A_u}(x) = 1\}. \tag{90}$$

For $[\alpha_1, \alpha_2] \in \mathbb{U}$, we define the $[\alpha_1, \alpha_2]$-cuts of A as:

$$A[\alpha_1, \alpha_2] = \begin{cases} \{x \in \mathbb{X} \mid \mu_A(x) \geq_K [\alpha_1, \alpha_2]\}, & \text{if } \alpha_1 \neq 0 \\ A_l[0] \cap \{x \in \mathbb{X} \mid \mu_A(x) \geq_K [\alpha_1, \alpha_2]\}, & \text{if } \alpha_1 = 0 \wedge \alpha_2 \neq 0 \\ A_l[0] \cap A_u[0], & \text{if } \alpha_1 = \alpha_2 = 0. \end{cases} \tag{91}$$

where $A_l[0]$ is the closure of the support of A_l and $A_u[0]$ is the closure of the support of A_u. As in the classical fuzzy theory, an interval fuzzy set is completely determined by its $[\alpha_1, \alpha_2]$-cuts.

An interval fuzzy number is defined as an interval extension of the definition of fuzzy number given in Section 2, considering the approach of interval fuzzy sets presented here. Therefore, an interval fuzzy number $\hat{N}$ is defined as a interval fuzzy set of $\mathbb{R}$ with the following characteristics: [8]

(a) the $[\alpha_1, \alpha_2]$-cuts and the core of $\hat{N}$ are real intervals, that is, $\hat{N}[\alpha_1, \alpha_2], core_{\hat{N}} \in \mathbb{IR}$;

(b) $lsupp_{\hat{N}}$ and $usupp_{\hat{N}}$ are bounded.

If the functions LMF and UMF are both linear, then $\hat{N}$ is called a linear interval fuzzy number, which can be defined by an interval membership function $\mu_{\hat{N}}$, with supports $lsupp_{\hat{N}} = (a_l, b_l)$, $usupp_{\hat{N}} = (a_u, b_u)$ and core $core_{\hat{N}} = [u_1, u_2]$, and is denoted by the tuple $([a_u, a_l]/[u_1, u_2]/[b_l, b_u])$. If $u_1 = u_2 = u$, then $\hat{N}$ is a triangular (linear) interval fuzzy number. The set of all interval fuzzy numbers is denoted by $\hat{\mathbb{F}}(\mathbb{R})$.

One can observe that the functions LMF and UMF describe, respectively, the fuzzy numbers $\bar{N}_l$ e $\bar{N}_u$, which can represent the interval fuzzy number $\hat{N}$. Assuming that $\bar{N}_l$ and $\bar{N}_u$ are both symmetric triangular fuzzy numbers described respectively by the functions $\mu_{\hat{N}_i}$ (LMF) and $\mu_{\hat{N}_s}$ (UMF), and represented respectively by the tuples $(a_l/u/b_l)$ and $(a_u/u/b_u)$, then $\hat{N}$ is a symmetric triangular interval fuzzy number. The graphic representation of $\bar{N}_l, \bar{N}_u$ and $\hat{N}$ is shown in Figure 5.

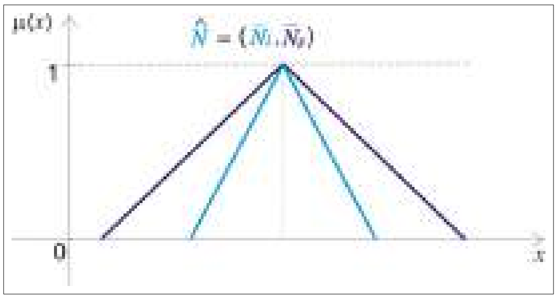

Fig 5. Graphical representation of an interval fuzzy number

Thus, an interval fuzzy number $\hat{N}$ may be represented as an ordered pair of fuzzy numbers $\hat{N} = (\bar{N}_l, \bar{N}_u)$, where $\bar{N}_l$ and $\bar{N}_u$ are called the $\hat{N}$'s lower generator fuzzy number and the $\hat{N}$'s upper generator fuzzy number, respectively.

5. Ordering Symmetric Interval Fuzzy Numbers

Based on the concept that two symmetric triangular fuzzy generator numbers with the same core produce a symmetric triangular interval fuzzy number, an order relation was defined, based on the AD-order for comparing the fuzzy generator numbers, and analyzing those comparisons to determine the ordering of symmetric triangular interval fuzzy numbers.

Let $\bar{F}_{1l}$ and $\bar{F}_{1u}$ be two symmetric triangular interval fuzzy numbers obtained through the functions LMF and UMF, respectively, of an interval fuzzy number $\hat{F}_1$. Analogously, let $\bar{F}_{2l}$ e $\bar{F}_{2u}$ be, respectively, the lower and upper fuzzy generator numbers of another symmetric triangular interval fuzzy number, $\hat{F}_2$. Then, the relation $\hat{<}$ is defined by:

$$\hat{F}_1 \hat{<} \hat{F}_2 \Leftrightarrow (\bar{F}_{1u} \bar{<} \bar{F}_{2u}) \vee (\bar{F}_{1u} = \bar{F}_{2u} \wedge \bar{F}_{1l} \bar{<} \bar{F}_{2l}) \tag{92}$$

where $\bar{<}$ is given in Definition 2.

[11] The continuity of interval functions was defined by Moore as an extension of the continuity of real functions. More information on this subject can be seen in [45, 52]

Then, one has that:

$$\hat{F}_1 \hat{\lesssim} \hat{F}_2 \Leftrightarrow \hat{F}_1 = \hat{F}_2 \vee \hat{F}_1 \hat{<} \hat{F}_2. \quad (93)$$

This relation, as the AD-order, take the information precision contained in each number into consideration as a factor to be analyzed when interval fuzzy numbers are ordered. It also satisfies the properties of reflexivity, antisymmetry and transitivity, so it is an order relation. Finally, one has that:

$$\forall \hat{F}_1, \hat{F}_2 : \quad \hat{F}_1 = \hat{F}_2 \ \underline{\vee} \ \hat{F}_1 \hat{<} \hat{F}_2 \ \underline{\vee} \ \hat{F}_2 \hat{<} \hat{F}_1. \quad (94)$$

making the relation defined by the Equation (92) to be total.

In the following, consider the interval fuzzy numbers $\hat{F}_1 = (\bar{F}_{1l}, \bar{F}_{1u})$, $\hat{F}_2 = (\bar{F}_{2l}, \bar{F}_{2u})$ and $\hat{F}_3 = (\bar{F}_{3l}, \bar{F}_{3u})$.

Proposition 9 $\forall \hat{F}_1, \hat{F}_2 : \quad \hat{F}_1 = \hat{F}_2 \ \underline{\vee} \ \hat{F}_1 \hat{<} \hat{F}_2 \ \underline{\vee} \ \hat{F}_2 \hat{<} \hat{F}_1.$

Proof. Suppose that $\hat{F}_1 \neq \hat{F}_2$ and $\hat{F}_1 \hat{\nless} \hat{F}_2$. Then, considering the fuzzy generator numbers of $\hat{F}_1$ and $\hat{F}_2$, by Equation (92), one has that:

$$\bar{F}_{2u} \bar{\lesssim} \bar{F}_{1u} \wedge (\bar{F}_{2u} \neq \bar{F}_{1u} \vee \bar{F}_{2l} \bar{\lesssim} \bar{F}_{1l}).$$

In other words, this means that:

$$\bar{F}_{2u} \bar{<} \bar{F}_{1u} \vee (\bar{F}_{2u} \bar{\lesssim} \bar{F}_{1u} \wedge \bar{F}_{2l} \bar{\lesssim} \bar{F}_{1l}).$$

Since $\hat{F}_1 \neq \hat{F}_2$, it follows that:

$$\bar{F}_{2u} \bar{<} \bar{F}_{1u} \vee (\bar{F}_{2u} = \bar{F}_{1u} \wedge \bar{F}_{2l} \bar{<} \bar{F}_{1l}),$$

which means that $\hat{F}_2 \hat{<} \hat{F}_1$. Analogously one proves that if $\hat{F}_1 \neq \hat{F}_2$ and $\hat{F}_2 \hat{\nless} \hat{F}_1$ then $\hat{F}_1 \hat{<} \hat{F}_2$. Finally, it is immediate that if $\hat{F}_1 \hat{\nless} \hat{F}_2$ and $\hat{F}_2 \hat{\nless} \hat{F}_1$ then $\hat{F}_1 = \hat{F}_2$. □

Proposition 10 *The relation $\hat{\lesssim}$ is reflexive.*

Proof. It is immediate, following from Equation (93). □

Proposition 11 *The relation $\hat{\lesssim}$ is antisymmetric.*

Proof. We show that

$$\hat{F}_1 \hat{\lesssim} \hat{F}_2 \wedge \hat{F}_2 \hat{\lesssim} \hat{F}_1 \Rightarrow \hat{F}_1 = \hat{F}_2.$$

Considering the fuzzy generator numbers of $\hat{F}_1$ and $\hat{F}_2$, suppose that $\hat{F}_1 \hat{\lesssim} \hat{F}_2$ and $\hat{F}_2 \hat{\lesssim} \hat{F}_1$ and $\hat{F}_1 \neq \hat{F}_2$. Then, by Equation (92), it holds that:

$$(\bar{F}_{1u} \bar{<} \bar{F}_{2u}) \vee (\bar{F}_{1u} = \bar{F}_{2u} \wedge \bar{F}_{1l} \bar{<} \bar{F}_{2l})$$

and

$$(\bar{F}_{2u} \bar{<} \bar{F}_{1u}) \vee (\bar{F}_{2u} = \bar{F}_{1u} \wedge \bar{F}_{2l} \bar{<} \bar{F}_{1l}).$$

From the antisymmetry of $\bar{\lesssim}$ (Proposition 7), it follows that $\bar{F}_{1u} = \bar{F}_{2u}$ and $\bar{F}_{1l} = \bar{F}_{2l}$, which is a contradiction. Therefore, one concludes that $\hat{F}_1 = \hat{F}_2$. □

Proposition 12 *The relation $\hat{\lesssim}$ is transitive.*

Proof. We show that

$$\hat{F}_1 \hat{\lesssim} \hat{F}_2 \wedge \hat{F}_2 \hat{\lesssim} \hat{F}_3 \Rightarrow \hat{F}_1 \hat{\lesssim} \hat{F}_3.$$

Considering the fuzzy generator numbers of $\hat{F}_1$, $\hat{F}_2$ and $\hat{F}_3$, if $\hat{F}_1 \hat{<} \hat{F}_2$ and $\hat{F}_2 \hat{<} \hat{F}_3$, from Equation (92), it follows that:

$$(\bar{F}_{1u} \bar{<} \bar{F}_{2u}) \vee (\bar{F}_{1u} = \bar{F}_{2u} \wedge \bar{F}_{1l} \bar{<} \bar{F}_{2l})$$

and

$$(\bar{F}_{2u} \bar{<} \bar{F}_{3u}) \vee (\bar{F}_{2u} = \bar{F}_{3u} \wedge \bar{F}_{2l} \bar{<} \bar{F}_{3l}).$$

By the transitivity of $\bar{<}$ (Proposition 8), it follows that

$$(\bar{F}_{1u} \bar{<} \bar{F}_{3u}) \vee (\bar{F}_{1u} = \bar{F}_{3u} \wedge \bar{F}_{1l} \bar{<} \bar{F}_{3l}).$$

Then, one concludes that $\hat{F}_1 \hat{<} \hat{F}_3$, and the result follows immediately. □

Finally, since the relation $\hat{\lesssim}$ is reflexive, antisymmetric and transitive, configuring an order relation, then, from Proposition 9, it follows that:

Theorem 7 *The relation $\hat{\lesssim}$ is a total order relation.*

Example 13 *Figures 6, 7, 8 and 9 show some examples of comparisons of interval fuzzy number using the interval AD-order.*

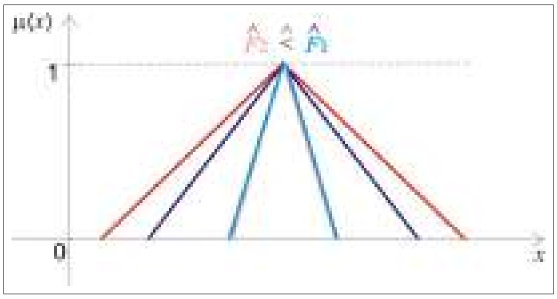

Fig 6. Example 1

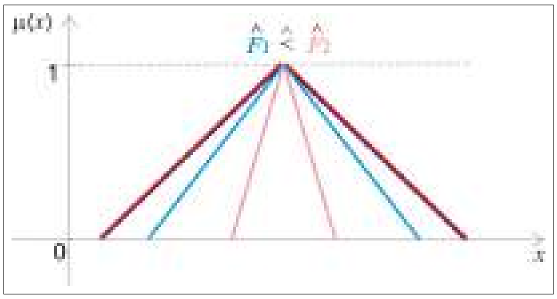

Fig 7. Example 2

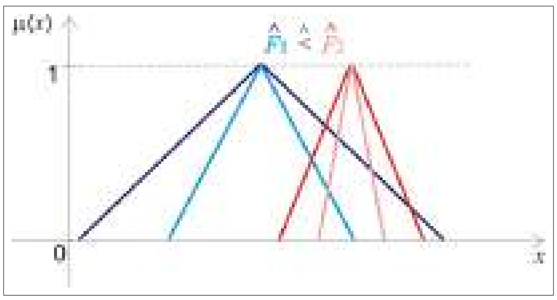

Fig 8. Example 3

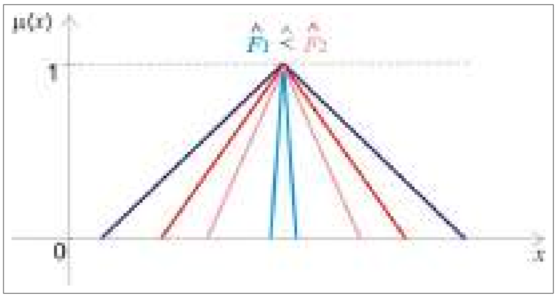

Fig 9. Example 4

6. Conclusion

In several applications applying fuzzy numbers, a total order is required, as in game theory, where the search for solutions requires a comparison of the rewards given to the choices of strategies by the players, or in decision making, where preference fuzzy relations are often used.

Analyzing various methods available in the literature, we observed that there is no practical method for ordering symmetrical triangular fuzzy numbers that considers both their position on the real line and the accuracy of the information contained in the fuzzy number and the acceptable error for the considered application.

Then, this paper introduced the total order for symmetrical triangular interval fuzzy numbers, based on the also introduced total AD-order relation for symmetrical triangular fuzzy numbers. The ordering method of interval fuzzy numbers was developed as an extension of the AD-order for fuzzy numbers, and as such, takes into account the accuracy of the information contained in the analyzed numbers, and not only their magnitudes. The ordering of interval fuzzy numbers by comparing their fuzzy generator numbers was shown as a practical option that is independent of auxiliary functions, which facilitates its application in real problems.

The next step in this work will be focused on a comparative analysis between the methods studied here and others found in the literature, especially when applied to situations of strategic interactions, such as interval fuzzy Bayesian games.

Furthermore, there are other possibilities of analogous order relation developed for fuzzy numbers and interval fuzzy numbers that are not necessarily symmetrical triangular.

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REPORT

Fuzzy Computing from Quantum Computing - Case Study in Reichenbach Implication Class

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This paper shows that quantum computing extends the class of fuzzy sets, taking advantage of properties such as quantum parallelism. The central idea associates the states of a quantum register with membership functions of fuzzy subsets, and the rules for the processes of fuzzyfication are performed by unitary quantum transformations. Besides studying the construction of logical operators, such as negation, using quantum, this paper also introduces the definition of t-norms and t-conorms based on unitary and controlled quantum gates. Such constructors allow modelling and interpreting union, intersection and difference between fuzzy sets. As the main contribution, an interpretation for the Reichenbach implication from quantum computing is obtained.An evaluation of the corresponding computation is implemented and simulated in the VPE-qGM visual programming environment.

1. Introduction

Fuzzy Logic (FL) and Quantum Computing (QC) are research areas that collaborate to the description of uncertainty: the former refers to uncertainty of human being's reasoning, while the latter studies the uncertainty of the subatomic world according to the principles of Quantum Mechanics. Many similarities between these two areas were highlighted in several scientific papers [1, 2, 3, 4] and [5]. Despite their similarities, they are not mathematically identical. While quantum logic has a special square root of NOT operation which is very useful in QC, FL lacks such an operation [6]. So, this paper investigates how QC can be used to model FL. But it does not address the relevant discussion about how fuzzy logic relates to probability theory and how quantum probability (amplitudes) extends the classical probability, as presented in [7, 8, 9] and [10]. Some studies of the relationship between probabilistic (polling) and Likert-scale approaches to elicit membership degrees clarified by QC can be found in [11].

The membership functions describing the uncertainty in the membership degree in terms of the fuzzy set theory (FST) can be modeled by quantum transformations and quantum states [12]. Following such approach, this paper has its two main goals: (i) to extend the modeling of fuzzy connectives by quantum transformations mainly considering the fuzzy implications and difference operator; and (ii) to simulate them on a visual programming environment. In the development of such quantum algorithms representing operations on fuzzy sets (FS) as union, intersection and difference, quantum gates are applied to model their corresponding logical connectives, t-conorms, t-norms and implications.

Quantum algorithms are, in many scenarios, exponentially faster than their classical versions, see e.g. [13]. However, such algorithms can only be efficiently executed by quantum computers, which are being developed and are restricted by the number of qubits. The simulation using classical computers allows the development and validation of basic quantum algorithms, anticipating the knowledge related to their behaviors when executed in a quantum hardware. The *VPE-qGM* (Visual Programming Environment for the Quantum Geometric Machine Model), is a quantum simulator that adds both characterizations, visual modeling and distributed simulation of quantum algorithms, showing the evolution of quantum states through integrated graphical interfaces (see, e.g. [14] and [15]).

As main contributions we provide analysis and implementation of the quantum algorithms for the basic fuzzy operations of union and intersection together with new quantum algorithms for the fuzzy operations of difference and implication. Furthermore, all operations are also studied and simulated in theVPE-qGM environment.

This paper is organized as follows: Section 2 presents the fundamental concepts of fuzzy logic. Section 3 brings the main concepts of QC. In Section 4, the approach for describing FS using the QC is depicted. Section 5 presents the operations on FS modeled from quantum transformations, considering the fuzzy operations of intersection, union, difference and implication. Finally, conclusions and further work are discussed in Section 6.

2. Preliminary on Fuzzy Logic

The sets with non well-defined borders, called fuzzy sets (FS), aim to overcome the limitations when the transitions from one class to another are smooth. The definition, properties and operations of FSs are obtained from the generalization of classical set theory (CST), which is a particular case of FST. A **membership function** $f_A(x) : \mathcal{X} \to [0,1]$ determines the membership degree (MD) of the element $x \in \mathcal{X}$ to the set A, such that $0 \leq f_A(x) \leq 1$. Thus, a **fuzzy set A** related to a set $\mathcal{X} \neq \emptyset$ is given by the expression: $A = \{(x, f_A(x)) : x \in \mathcal{X}\}$.

2.1 Fuzzy Connectives A function $N : [0,1] \to [0,1]$ is a **fuzzy negation** when the conditions hold:

N1 $N(0) = 1$ and $N(1) = 0$;

N2 If $x \leq y$ then $N(x) \geq N(y)$, for all $x, y \in [0,1]$.

Fuzzy negations verifying the involutive property

N3 $N(N(x)) = x$, for all $x \in [0,1]$.

are called strong fuzzy negations. See, e.g. the standard negation: $N_S(x) = 1 - x$.

Definitions of **intersection** and **union** between fuzzy subsets can be obtained through aggregation functions. Herein, we consider triangular norms (t-norms) and triangular conorms (t-conorms) [16].

Definition 3 *A **t-(co)norm** is a binary operation $(S)T : [0,1] \times [0,1] \to [0,1]$ such that, $\forall x, y, z \in [0,1]$, the following properties hold (commutativity, associativity, monotonicity and boundary conditions):*

***T1**: $T(x,y) = T(y,x)$;*
***T2**: $T(T(x,y),z) = T(x,T(y,z))$;*
***T3**: if $x \leq z$ then $T(x,y) \leq T(z,y)$;*
***T4**: $T(x,0) = 0$ and $T(x,1) = x$;*
***S1**: $S(x,y) = S(y,x)$;*
***S2**: $S(S(x,y),z) = S(x,S(y,z))$;*
***S3**: if $x \leq z$ then $S(x,y) \leq S(z,y)$*
***S4**: $S(x,1) = 1$ and $S(x,0) = x$*

Among different definitions of t-norms and t-conorms [17], in this work we consider the *Algebraic Produtc* and *Algebraic Sum*, respectively given as:

$$T_P(x,y) = x \cdot y; \quad S_P(x,y) = x + y - x \cdot y. \tag{95}$$

A binary function $I : [0,1]^2 \to [0,1]$ is an implication operator (implicator) if the following boundary conditions hold:

I 0: $I(1,1) = I(0,1) = I(0,0) = 1$ and $I(1,0) = 0$.

In [18, 19], additional properties are considered for defining a fuzzy implication. Thus, a **fuzzy implication** $I : [0,1]^2 \to [0,1]$ is an implicator verifying, for all $x, y, z \in [0,1]$, the following conditions:

I 1: Antitonicity in the first argument: if $x \leq z$ then $I(x,y) \geq I(z,y)$;

I 2: Isotonicity in the second argument: if $y \leq z$ then $I(x,y) \leq I(x,z)$;

I 3: Falsity dominance in the antecedent: $I(0,y) = 1$;

I 4: Truth dominance in the consequent: $I(x,1) = 1$.

Among the implication classes with explicit representation by fuzzy connectives (negations and aggregation functions) this work considers the class of (S,N)-implication, extending the classical equivalence $p \to q \Leftrightarrow \neg p \vee q$.

Let S be a t-conorm and N be a fuzzy negation. A (S,N)**-implication** is a fuzzy implication $I_{(S,N)} : [0,1]^2 \to [0,1]$ defined by:

$$I_{(S,N)}(x,y) = S(N(x),y), \forall x, y \in [0,1]. \tag{96}$$

If N is a strong negation, Eq. (96) defines an S**-implication**[20].

The Reichenbach implication (I_{RB}) expressed as:

$$I_{RB}(x,y) = 1 - x + x \cdot y, \forall x, y \in [0,1], \tag{97}$$

is an S-implication, obained by a fuzzy negation $N_S(x) = 1 - x$ and a t-conorm $S_P(x,y) = x + y - x \cdot y$.

2.2 Operations over FS Similarly to the construction of classical sets, consider in the following definitions and examples of operations defined over the FS $A, B \subseteq \mathcal{X}$.

Definition 4 *[12] The **complement of** A with respect to $\mathcal{X}$, is a FS $A' = \{(x, f_{A'}) : x \in \mathcal{X}\}$, with $f_{A'} : \mathcal{X} \to [0,1]$ is given by:*

$$f_{A'}(x) = N_S(f_A(x)) = 1 - f_A(x), \quad \forall x \in \mathcal{X}. \tag{98}$$

Definition 5 *Let $T : [0,1]^2 \to [0,1]$ be a t-norm. The **intersection** between the FS A and B, both defined with respect to X, results in a FS $A \cap B = \{(x, f_{A\cap B}(x)) : x \in \mathcal{X}\}$, whose a membership function $f_{A\cap B}(x) : \mathcal{X} \to [0,1]$ is*

$$f_{A\cap B}(x) = T(f_A(x), f_B(x)), \forall x \in \mathcal{X}. \tag{99}$$

The **canonical intersection** is an intersection where the t-norm is the algebraic product T_P. Thus, Eq. (99) can be expressed as:

$$f_{A\cap B}(x) = f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \tag{100}$$

Definition 6 *Let $S : [0,1]^2 \to [0,1]$ be a t-conorm and A and B be FS both defined with respect to $\mathcal{X}$. A **union between** A **and** B results in a FS $A \cup B = \{(x, f_{A\cup B}(x)) : x \in \mathcal{X}\}$, whose a membership function $f_{A\cup B}(x) : \mathcal{X} \to [0,1]$ is*

$$f_{A\cup B}(x) = S(f_A(x), f_B(x)), \forall x \in \mathcal{X}. \tag{101}$$

The **canonical union** is a union where the t-conorm is the algebraic sum S_P. Thus, Eq. (101) can be expressed as:

$$f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \tag{102}$$

Extending the classical equivalence $\neg(p \to q) \Leftrightarrow p \wedge \neg q$, we obtain the difference operator considering a strong fuzzy negation and an S-implication.

Definition 7 *Let S be t-conorm, N be a strong fuzzy negation and I be an S-implication, A and B be FS related to $\mathcal{X}$. A **difference between** A **and** B, both defined with respect to X, results in a FS $A - B = \{(x, f_{A-B}(x)) : x \in \mathcal{X}\}$, whose membership function $f_{A-B} : \mathcal{X} \to [0,1]$ is*

$$f_{A-B}(x) = N(S(f_A(x), f_B(x)))\forall x \in \mathcal{X}. \tag{103}$$

An example of the difference $A - B$ is given by the standard fuzzy negation of the Reichenbach's implication given by Eq. (97), expressed as:

$$\begin{aligned} f_{A-B}(x) = N_S(S_P(N_S(f_A(x)), f_B(x)) = \\ f_A(x) - f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \end{aligned} \tag{104}$$

3. Foundations on *QC*

In *QC* the qubit is the basic information unit, being the simplest quantum system, defined by a unitary and bi-dimensional state vector. Qubits are generally described, in Dirac's notation [21], by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

The coefficients α and β are complex numbers for the amplitudes of the corresponding states in the computational basis (state space), respecting the condition $|\alpha|^2+|\beta|^2 = 1$, which guarantees the unitarity of the state vectors of the quantum system, represented by $(\alpha, \beta)^t$.

The state space of a quantum system with multiple *qubits* is generated (span) by the tensor product of the state space of its subsystems. Considering a two *qubits* quantum system, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$, its tensor product $|\psi\rangle \otimes |\varphi\rangle$ is described by

$$|\psi\rangle \otimes |\varphi\rangle = |\psi\varphi\rangle = \alpha\cdot\gamma|00\rangle + \alpha\cdot\delta|01\rangle + \beta\cdot\gamma|10\rangle + \beta\cdot\delta|11\rangle. \tag{105}$$

Now, the tensor product space $A \otimes B$ of the Hilbert spaces A and B, is the span of the tensor product of their basis elements. The span of a set is the set of all linear combinations of its members.

The state transition of a quantum system is performed by unitary transformations associated with orthonormalized matrices (M) of order 2^N, with N being the number of *qubits* within the system. The unitarity property holds when the condition expressed by $M \cdot M^\dagger = Id$ is satisfied. All the vectors within M must be orthogonal with one another and its module must be 1.

For instance, the definition of the *Pauly X* transformation and its application over a one-dimensional and two-dimensional quantum systems are presented in the Fig. 1. Furthermore, a Toffoli transformation is also shown in order to describe a controlled operation for a 3 *qubits* system. In this case, the *NOT* operator (*Pauly X*) is applied to the *qubit* $|\sigma\rangle$ when the current states of the first two *qubits* $|\psi\rangle$ and $|\varphi\rangle$ are both $|1\rangle$.

Fig 1. Examples of quantum transformations

In order to obtain information from a quantum system, it is necessary to apply measurement operators, defined by a set of linear operators M_m, called projections. The index m **referees** to the possible measurement results related to the basis vectors $(m \in \{0, 1, \cdots, 2^m - 1\})$. If the state of a quantum system is $|\psi\rangle$ immediately before the measurement, the probability of an outcome occurrence is $p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$[12] and the system state after that is given by

$$p(|\psi\rangle) = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \tag{106}$$

When measuring a *qubit* $|\psi\rangle$ with $\alpha, \beta \neq 0$, the probability of observing $|0\rangle$ and $|1\rangle$ are, respectively

$$p(0) = \langle\phi|M_0^\dagger M_0|\phi\rangle = \langle\phi|M_0|\phi\rangle = |\alpha|^2 \; and$$
$$p(1) = \langle\phi|M_1^\dagger M_1|\phi\rangle = \langle\phi|M_1|\phi\rangle = |\beta|^2.$$

After the measuring process, the quantum state $|\psi\rangle$ has $|\alpha|^2$ as the probability to be in the state $|0\rangle$ and $|\beta|^2$ as the probability to be in the state $|1\rangle$.

[12] $M_m^\dagger$ denotes the transpose complex conjugate of the matrix M_m.

All the quantum transformations and measurement operations are distributed along a quantum circuit, as in Fig. 2. It organizes the qubits in a horizontal line, and at each computing step (from left to right), the quantum transformations to be applied are inserted over the target qubit's wire.

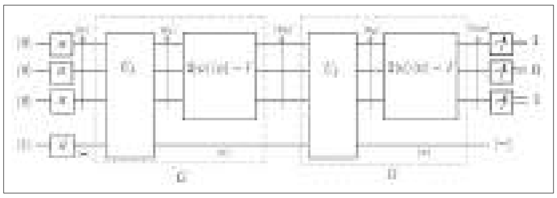


Fig 2. Quantum circuit of Grover's algorithm preformed with 4 qubits and 6 steps of simulation

4. Modeling Fuzzy Sets Through Quantum Computing

The description of FS from the *QC* viewpoint considers a *FS A* given by the membership function $f_A(x)$. Without loss of generality, let $\mathcal{X}$ be a finite subset with cardinality n $(|\mathcal{X}| = n)$. Thus, the definitions can be extended to infinite sets, by considering a quantum computer with an infinite quantum register [21].

4.1 Describing Fuzzy Sets through Quantum Registers As in [12], consider $\mathcal{X} \neq \emptyset, |\mathcal{X}| = n$, $i \in \mathbb{N}_n = \{1, 2, ..., n\}$ and a membership function, $f_A : \mathcal{X} \to [0, 1]$. A n-dimensional quantum register (QR), given by

$$|s_f\rangle = \bigotimes_{1\le i\le n} [\sqrt{1 - f_A(x_i)}|0\rangle + \sqrt{f_A(x_i)}|1\rangle], \tag{107}$$

is called a **classical fuzzy state**(CFS) of n-*qubits* , such as

$$|s_{f_0}\rangle = \begin{cases} \sqrt{1}|0\rangle + \sqrt{0}|1\rangle, & if \; f_0(1) = 0 \\ \sqrt{0}|0\rangle + \sqrt{1}|1\rangle, & if \; f_0(1) = 1 \end{cases} \text{ and}$$

$$|s_{f_1}\rangle = \begin{cases} |00\rangle, & if \; f_1(1) = 0 \; and \; f_1(2) = 0 \\ |01\rangle, & if \; f_1(1) = 0 \; and \; f_1(2) = 1 \\ |10\rangle, & if \; f_1(1) = 1 \; and \; f_1(2) = 0 \\ |11\rangle, & if \; f_1(1) = 1 \; and \; f_1(2) = 1 \end{cases}$$

for a one-dimensional and two-dimensional classical states, respectively. When $f(1) = a$, $f(2) = b$ and $a, b \in]0, 1[$, superpositions of quantum states corresponding to a FS are obtained and expressed as

$$\begin{aligned} |s_f\rangle =& (\sqrt{a}|1\rangle + \sqrt{1-a}|0\rangle) \otimes (\sqrt{b}|1\rangle + \sqrt{1-b}|0\rangle) = \\ & \sqrt{(1-a)(1-b)}|00\rangle + \sqrt{b(1-a)}|01\rangle + \\ & \sqrt{a(1-b)}|10\rangle + \sqrt{ab}|11\rangle. \end{aligned} \tag{108}$$

As it can be seen, the image-set $f[\mathcal{X}]$ defines a QR. In other words, a canonical orthonormal basis in $\otimes^n\mathcal{C}$ denotes a classical-QR of *n-qubits.* Thus, one can describe the classical state of the register $|1100\dots0\rangle$ of n qubits when $f(1) = f(2) = 1$ and $f(i) = 0$ when $i \in \{\mathbb{N}_n - \{1, 2\}\}$.

The generalized expression, described in [12], states that a CFS of $n-qubits$, such as $|s_f\rangle \in [CFS]$, can be expanded in $\mathcal{C}^{2^n}$ by

$$|s_f\rangle = (1-f(1))^{\frac{1}{2}}(1-f(2))^{\frac{1}{2}}\dots(1-f(n))^{\frac{1}{2}}|00\dots00\rangle + f(1)^{\frac{1}{2}}(1-f(2))^{\frac{1}{2}}\dots(1-f(n))^{\frac{1}{2}}|10\dots00\rangle + f(1)^{\frac{1}{2}}f(2)^{\frac{1}{2}}\dots f(n)^{\frac{1}{2}}|11\dots11\rangle. \tag{109}$$

From the perspective of QC, a FS is a superposition of crisp sets. Each $|s_f\rangle$ in [CFS] is a QR described as a superposition of crisp sets and generated by the tensor product of non-entangled QRs [21].

According to [12], it appears that the FS are obtained by overlapping quantum states from a conventional fuzzy quantum register. Moreover, from the set of membership functions representing the fuzzy classical states, we obtain a linear combination, formalizing the notion of a fuzzy quantum register. In this context, we can characterize a **quantum fuzzy set** (QFS) as:
(i) quantum superposition of FS, which have different shapes, simultaneously;
(ii) subsets of entangled superpositions of crisp subsets (or classical FS).

Proposition 13 *[12, Theorem 2] Let $f, g : X \to [0,1]$ be membership functions with respect to $\mathcal{X}$. The classical FS $|s_f\rangle$ and $|s_g\rangle$ are mutually orthonormal CFSs if and only if there exists $x \in \mathcal{X}$ such that either $f(x) = 0$ and $g(x) = 1$ or the converse, $f(x) = 1$ and $g(x) = 0$.*

By Proposition 13, two CFSs $|s_f\rangle$ and $|s_g\rangle$ in $[CFS]$ are mutually orthogonal if and only if there exists $x \in X$ such that $f(x)\cdot g(x) = 0$. In Eq (109), a quantum state $|s_f\rangle$ in $\mathcal{C}^{2^N}$ is characterized as a N-dimensional orthonormal set in $\mathcal{C}^{2^N}$). See details in [21, 22] and [23].

Definition 8 *Consider $f_i : X \to [0,1]$, $i \in \{1,...,k\}$, as a collection of membership function generating fuzzy subsets A_i and $\{|s_{f_1}\rangle,\dots,|s_{f_k}\rangle\} \subseteq [CFS]$, such that their components are two by two orthonormal vectors. Let $\{c_1,\dots,c_k\} \subseteq \mathcal{C}$. A **quantum fuzzy set** (QFS) $|s\rangle$is defined by the linear combination:*

$$|s\rangle = c_1|s_{f_1}\rangle + \dots + c_k|s_{f_k}\rangle. \tag{110}$$

[CFQ] denotes the set of all QFSs. From Def. 8, a fuzzy quantum state of a N-dimensional QR, as described by Eq.(110), can be entangled or not, depending on the family of classical fuzzy states $|s_{f_i}\rangle$ and the set C_i of chosen amplitudes.

5. *FS* Operations from Quantum Transformations

According to [12], FS can be obtained by quantum superposition of CFSs associated with a QR. Additionally, interpretations for fuzzy operations such as complement and intersection are obtained from the NOT and AND quantum transformations. Extending this approach, other operations are introduced, such as union, difference and fuzzy implication, considering the interpretations of OR, DIV and IMP quantum operators.

All the operations here proposed are studied with the aid of the VPE-qGM environment. It provides interpretations for quantum memory, quantum processes and computations related to the simulation of the temporal evolution of quantum states. In order to yield the correct interpretations, consider the membership functions $f, g : \mathcal{X} \to [0,1]$ related to two FS A and B, and a pair $(|s_{f_i}\rangle, |s_{g_i}\rangle)$ of CFSs such that for all $x_i \in \mathcal{X}$, it is given as

$$|s_{f_i}\rangle\sqrt{f(i)}|1\rangle + \sqrt{1-f(i)}|0\rangle,\\ |s_{g_i}\rangle = \sqrt{g(i)}|1\rangle + \sqrt{1-g(i)}|0\rangle. \tag{111}$$

In the following, the MD defined by $f_A(x_i)$, which is related to an element $x_i \in \mathcal{X}$ in the FS A, will be denoted by f_A, in order to simplify the notation.

5.1 Fuzzy Complement Operator

The complement of a FS is performed by the standard negation, which is obtained by the NOT operator, defined as

$$NOT(|s_{f_A}\rangle) = \sqrt{1-f_A}|1\rangle + \sqrt{f_A}|0\rangle \tag{112}$$

The complement operator NOT^n can be applied to the state $|s_f\rangle = \otimes_{1\leq i\leq N}|s_{f_i}\rangle$, resulting in an N-dimensional quantum superposition of 1-*qubit* states, described as $\mathcal{C}^{2^n}$ in the computational basis, represented by $NOT^N|s_f\rangle$ and expressed as

$$NOT^n(|s_{f_A}\rangle) = NOT\left(\otimes_{1\leq i\leq n}(f_A(i)^{\frac{1}{2}}|1\rangle + (1-f_A(i))^{\frac{1}{2}}|0\rangle)\right) = \\ = \otimes_{1\leq i\leq n}\left((1-f_A(i))^{\frac{1}{2}}|1\rangle + f_A(i)^{\frac{1}{2}}|0\rangle\right) \tag{113}$$

Now, Eq. (114 describe other applications related to the NOT transformation acting on the 2nd e 3rd-*qubits* of a quantum system, respectively:

$$NOT_2(|s_{f_1}\rangle|s_{f_2}\rangle) = |s_{f_1}\rangle \otimes NOT|s_{f_2}\rangle\\ NOT_{2,3}(|s_{f_1}\rangle|s_{f_2}\rangle|s_{f_3}\rangle) = |s_{f_1}\rangle \otimes NOT|s_{f_2}\rangle \otimes NOT|s_{f_3}\rangle. \tag{114}$$

In the next sections, these equations will describe other fuzzy operations, such as implications and differences.

5.2 Fuzzy Intersection Operator

Fuzzy intersection operator is modelled by the ***AND* operator**, represented by $AND(|s_{f_i}\rangle, |s_{g_i}\rangle)$ and expressed through the *Toffoli* quantum transformation as

$$AND(|s_{f_i}\rangle, |s_{g_i}\rangle) = T(|s_{f_i}\rangle, |s_{g_i}\rangle, |0\rangle) = \\ \left(\sqrt{f_A}|1\rangle + \sqrt{1-f_A}|0\rangle\right) \otimes \left(\sqrt{f_B}|1\rangle + \sqrt{1-f_B}|0\rangle\right) \otimes \\ \left(\sqrt{f_Af_B}|1\rangle + \sqrt{(1-f_A)f_B}|0\rangle\right). \tag{115}$$

So, by the distributivity of tensor product related to sum, Eq. (115) is given as

$$AND(|s_{f_i}\rangle, |s_{g_i}\rangle) = \sqrt{f_Af_B}|111\rangle + \sqrt{f_A(1-f_B)}|100\rangle + \\ \sqrt{(1-f_A)f_B}|010\rangle + \sqrt{(1-f_A)(1-f_B)}|000\rangle. \tag{116}$$

Thus, a measurement performed over the third *qubit* ($|1\rangle$) in the quantum state expressed by Eq. (116), provides an output $|S_1'\rangle = |111\rangle$, with probability $p = f_A \cdot f_B$. Then, for all $i \in X$, f_A and f_B indicate the probability of $x_i \in \mathcal{X}$ is in the FS defined by $f_A(x) : \mathcal{X} \to U$ and $g_A(x) : \mathcal{X} \to U$, respectively. And then, $f_A \cdot f_B$ indicates the probability of x_i is in the intersection of such FS. Analogously, a measurement of third *qubit* ($|0\rangle$) in the quantum state expressed by Eq. (116), returns an output state given as:

$$|S_2'\rangle = \frac{1}{\sqrt{(1-f_A)f_B}} \Big(\sqrt{f_A(1-f_B)}|100\rangle + \sqrt{(1-f_A)f_B}|010\rangle + \sqrt{(1-f_A)(1-f_B)}|000\rangle \Big)$$

with probability $p = 1 - f_A \cdot f_B$. In this case, an expression of the complement of the intersection between FS A and B is given by $1 - p = f_A \cdot f_B$. This probability indicates the non-MD of x is in the FS $A \cap B$. We also conclude that, by Eq. (116), it corresponds to the standard negation of product t-norm, i.e., the standard negation of algebraic product [17].

By Eq. (117), considering the initial quantum state as $|s_{f_2}\rangle \otimes |s_{f_3}\rangle \otimes |0\rangle$:

$$|S\rangle = \frac{\sqrt{12}}{6}|000\rangle + \frac{\sqrt{6}}{6}|010\rangle + \frac{\sqrt{12}}{6}|100\rangle + \frac{\sqrt{6}}{6}|110\rangle \quad (117)$$

a simulation of the ***AND* operator**, performed in the *VPE-qGM* environment, is illustrated in Fig. 3a. Each box represents an elementary process (*EP*). The first *EP* in the construction models the application of the Toffoli transformation. It is represented as an iteration of EPs, meaning that 8 *EPs* are being executed at the same time. Each *EP* is responsible for calculating one of the memory positions presented in the interface. The computation of each *EP*, in this particular case, is analogous to the execution of one line of the Toffoli's matrix, as presented in Fig. 1.

The EPs surrounded by the vertical lines represents the non deterministic sum (NDS). The *NDS* models the projections associated to the measurement operation. In the simulation, depending on the values in the memory grid, one of the execution paths will be taken. In Fig 3a, the upper path of the *NDS* is highlighted, meaning that this path was executed. Finally, the *Probability* component calculates the final normalized state of the system (from $|000\rangle$ to $111\rangle$) and its corresponding probability (presented in the first gray column − 0.83 or 83%).

The final state of the simulation, represented by the memory grid in the qS' interface, is consistent with the specification of the intersection operation of FS described in Eq. (115) and considering the quantum state $|S\rangle$ in Eq. (117). In this case, after a measurement, two possible situations are held:

- $|S_1'\rangle = |111\rangle$, with probability $p = 17\%$;
- $|S_2'\rangle = \frac{\sqrt{72}}{6\sqrt{5}}|000\rangle + \frac{\sqrt{36}}{6\sqrt{5}}|010\rangle + \frac{\sqrt{72}}{6\sqrt{5}}|100\rangle$, with probability $p = 83\%$.

Such type of interpretation holds the same for all forthcoming examples.

5.3 Fuzzy Union Operator Let $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ be quantum states given by Eqs. (111a) and (111b), respectively. The union of FS is modeled by the ***OR* operator**, based on the expression:

$$\begin{aligned} &OR(|s_{f_i}\rangle, |s_{g_i}\rangle) = NOT^3(AND(NOT|s_{f_i}\rangle, NOT|s_{g_i}\rangle)) = \\ &NOT^3(T(NOT|s_{f_i}\rangle, NOT|s_{g_i}\rangle, |0\rangle)) = \\ &NOT^3(T(\sqrt{f_A f_B}|000\rangle + \sqrt{f_A(1-f_B)}|010\rangle + \\ &\sqrt{(1-f_A)f_B}|100\rangle + \sqrt{(1-f_A)(1-f_B)}|110\rangle)). \end{aligned} \quad (118)$$

In the sequence, applying the Toffoli transformation and the fuzzy standard negation we have that:

$$\begin{aligned} OR(|s_{f_i}\rangle, |s_{g_i}\rangle) &= \sqrt{(1-f_A)(1-f_B)}|000\rangle + \\ &\sqrt{(1-f_A)f_B}|011\rangle + \sqrt{f_A(1-f_B)}|101\rangle + \sqrt{f_A f_B}|111\rangle. \end{aligned} \quad (119)$$

Observe that, a measure performed on third *qubit* of quantum state and defined by Eq. (120) ($|1\rangle$ as the component of quantum basis) results in the final state:

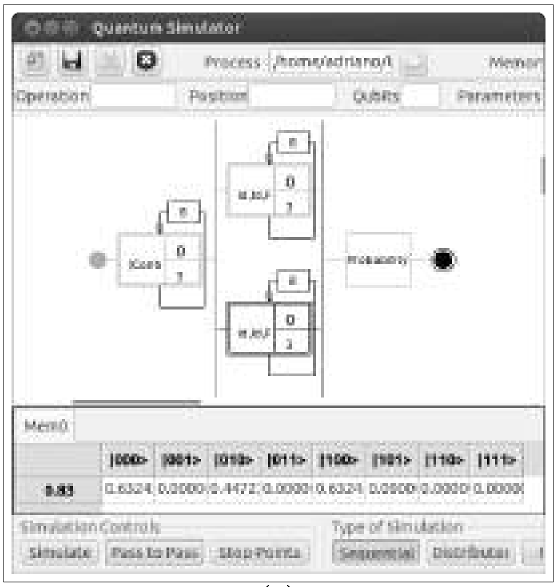
$$|S_1'\rangle = \frac{1}{\sqrt{f_B(1-f_A)+f_A)}} \Big(\sqrt{(1-f_A)f_B}|011\rangle + \sqrt{f_A(1-f_B)}|101\rangle + \sqrt{f_A f_B}|111\rangle \Big),$$

with corresponding probability $p = f_A + f_B - f_A \cdot f_B$ of $x_i \in \mathcal{X}$ is in both FS A e B. The union, expressed by Eq. (120), is therefore defined by the t-conorm product [17]. Additionally, a measure also performed in the third *qubit* (but related to state $|0\rangle$) returns $|S_2'\rangle = |000\rangle$ with probability $p = (1 - f_A) \cdot (1 - f_B)$, indicating that $x_i \in \mathcal{X}$ is not in such FS (neither A nor B).

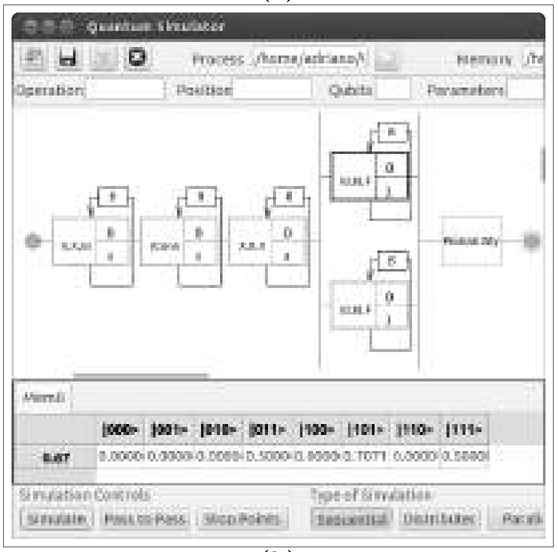
The simulation in *VPE-qGM* in Fig. 3b, presents the first three steps corresponding to the transformations specified by Eq. (118). The *NOT* transformation applied to the first two qubits is followed by the *Toffoli* transformation. Then, the application of the *NOT* gate in all qubits of the system is performed.

The measurement operation, identified by the *NDS* and the *Probability* component yields a final state consistent with the requirements of Eq. (118). After a measurement, one of the following states is reached:

- $|S_1'\rangle = \frac{1}{2}|011\rangle + \frac{\sqrt{2}}{2}|101\rangle + \frac{1}{2}|111\rangle$, with probability $p = 67\%$;
- $|S_2'\rangle = |000\rangle$, with probability $p = 33\%$.



(a)



(b)

Fig 3. Modeling and simulation of fuzzy operations in the VPE-qGM

5.4 Fuzzy Implication Operator Fuzzy implications, as many other fuzzy connectives, can be obtained by a composition of quantum operations applied to QRs. In the following, this paper introduces the expression of the quantum operator denoted by *IMP*, over which an interpretation of Reichenbach implication is obtained.

For that, consider again the pair $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ of quantum states given by Eqs. (111a) and (111b), respectively. The ***IMP*** **operator** is defined by:

$$\begin{aligned} &IMP(|s_{f_i}\rangle, |s_{g_i}\rangle) = NOT_2(AND(|s_{f_i}\rangle, NOT|s_{g_i}\rangle)) = \\ &NOT_2\left(T(|s_{f_i}\rangle, NOT|s_{g_i}\rangle, |0\rangle)\right) = \\ &NOT_2(\sqrt{(1-f_A)f_B}|000\rangle + \sqrt{(1-f_A)(1-f_B)}|010\rangle + \\ &\sqrt{f_A(f_B)}|100\rangle + \sqrt{f_A(1-f_B)}|110\rangle) \end{aligned} \tag{120}$$

By applying the *Toffoli* and NOT transformations in Eq.(120), we have that:

$$\begin{aligned} &IMP(|s_{f_i}\rangle, |s_{g_i}\rangle) = \sqrt{f_A(1-f_B)}|100\rangle + \\ &\sqrt{(1-f_A)f_B}|011\rangle + \sqrt{(1-f_A)(1-f_B)}|001\rangle + \\ &\sqrt{f_A f_B}|111\rangle. \end{aligned} \tag{121}$$

Applying the same procedure, by a measure performed over the third *qubit* in the state given by Eq. (121) we can get the two following quantum states: (1) an output $|S'_1\rangle$ as presented in the following,

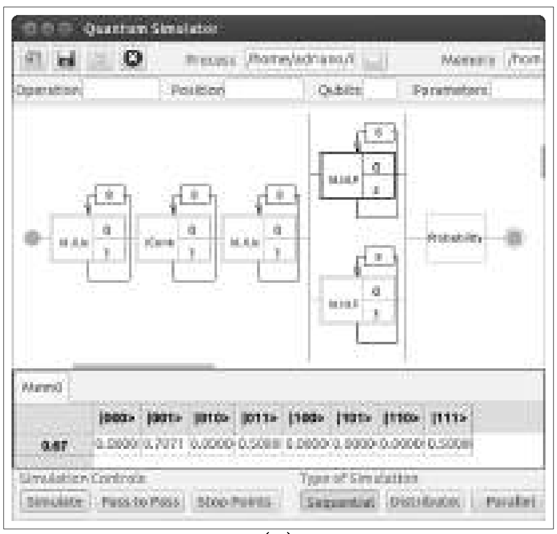
$$\begin{aligned} &|S'_1\rangle = \frac{1}{\sqrt{1-f_A+f_A f_B}}(\sqrt{(1-f_A)(1-f_B)}|001\rangle + \\ &\sqrt{(1-f_A)f_B}|011\rangle + \sqrt{f_A f_B}|111\rangle), \end{aligned} \tag{122}$$

with probability $p_1 = 1 - f_A + f_A \cdot f_B = f_{A \triangleright B}$. Thus, p_1 indicates the MD of an element in the FS $A \triangleright B$ related to I_{RB} fuzzy implication [24], given by Eq. (97).
(2) an output $|S'_2\rangle = |100\rangle$ with probability $p_0 = f_A(1 - f_B)$. In this case, an expression of the complement of the Reichenbach fuzzy implication related to the FS A and B is given by $p_0 = 1 - p_1$. This probability also indicates the non-MD of an element in the FS $A \triangleright B$.

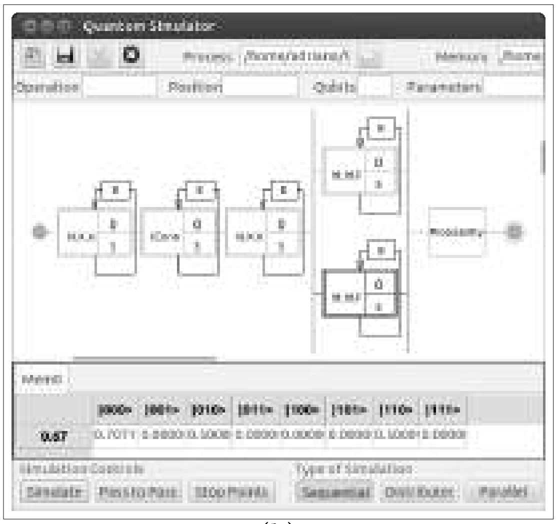
Taking $|s_{f_2}\rangle \otimes |s_{f_3}\rangle \otimes |1\rangle$, according with Eq. (117). The simulation in the *VPE-qGM* is consistent with the definition of the Implication operator of Eq. (120). After a measurement, one of the following states is reached:

- $|P'_1\rangle = \frac{\sqrt{2}}{2}|001\rangle + \frac{1}{2}|011\rangle + \frac{1}{2}|111\rangle$, with probability $p = 67\%$;

- $|P'_2\rangle = |100\rangle$, with probability $p = 33\%$.



(a)



(b)

Fig 4. Simulation of fuzzy operators in the VPE-qGM. Both cases are modeled by three steps containing quantum transformations plus the measurement operation

5.5 Fuzzy Difference Operator The difference operator (DIF) between FS based on QC is modeled by a composition of NOT and IMP quantum transformations, previously presented in Sections 5.1 and 5.4. As initial quantum states, we have $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$, given by Eqs. (111a),(111b), respectively. The ***DIF* quantum operator** is defined as:

$$\begin{aligned}
&DIF(|s_{f_i}\rangle, |s_{g_i}\rangle) = NOT_{2,3}(AND(|s_{f_i}\rangle, NOT|s_{g_i}\rangle)) \\
&= NOT_{2,3}(T(|s_{f_i}\rangle, NOT|s_{g_i}\rangle, |1\rangle)). \\
&= NOT_{2,3}(\sqrt{(1-f_A)f_B}|000\rangle + \sqrt{(1-f_A)(1-f_B)}|010\rangle + \\
&\sqrt{f_A(f_B)}|100\rangle + \sqrt{f_A(1-f_B)}|110\rangle).
\end{aligned} \tag{123}$$

Then, by Eq. (114) and Eq. (123) the DIF operator can be expressed as:

$$\begin{aligned}
&DIF(|\psi\rangle, |\phi\rangle) = \sqrt{(1-f_A)f_B}|01\rangle \otimes |0\rangle + \\
&\sqrt{(1-f_A)(1-f_B)}|00 \otimes |0\rangle + \\
&\sqrt{f_A f_B}|11 \otimes |0\rangle + \sqrt{f_A(1-f_B)}|10\rangle) \otimes |1\rangle.
\end{aligned} \tag{124}$$

After a measure performed over the third *qubit* of the quantum state, given by Eq. (124), it returns one of the two quantum states:
(1) $|S_1'\rangle = |101\rangle$, with the probability $p_1 = f_A - f_A \cdot f_B = f_{A-B}$ related to the MD of an element to the corresponding $FS\ A-B$, see Eq. (103).
(2) the superposition quantum state:

$$|S_2'\rangle = \frac{1}{\sqrt{(1-f_A) + f_A f_B}}(\sqrt{(1-f_A)(1-f_B)}|000\rangle + \sqrt{(1-f_A)f_B}|010\rangle + \sqrt{f_A f_B}|110\rangle),$$

with probability $p_0 = 1 - f_A + f_A f_B = 1 - f_{A-B}$ indicating the MD of an element in the complement of the fuzzy set $A - B$.

The simulation in the *VPE-qGM* is consistent with the definition of the difference operator of Eq. (123). After a measurement, the possible outcomes are:

- $|S_1'\rangle = |101\rangle$, with probability $p = 33\%$;
- $|S_2'\rangle = \frac{\sqrt{2}}{2}|000\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|110\rangle$, with probability $p = 67\%$ obtained by a simulation on VPE-qGM.

6. Conclusion and Final Remarks

This paper analyses the operations of fuzzy complement and fuzzy intersection as described in [12] but it also implements and simulates them in the VPE-qGM presenting an extension of such construction to other important fuzzy operations. This extension considers the modelling of the following fuzzy operations obtained from quantum operators: union, difference and implications, focusing on the class of S-implications named Reichenbach implications. The visual approach of the VPE-qGM environment enables the implementation and validation of such fuzzy operations using QC whose description is based on compositions of controlled and unitary quantum transformations, and the corresponding interpretation of fuzzy operations is obtained by applying operators of projective measurement.

Further work considers the specification of other fuzzy connectives, constructors (e. i. automorphisms and reductions) and the corresponding extension of fuzzy methodology from formal structures provided by QC. Additionally, the results can be used to explore the inner parallelism of QC simulated via GPUs in the VPEq-GM [15] and to model parallel fuzzy inference engine.

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REPORT

Actions of Automorphisms on Some Classes of Fuzzy Bi-implications

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In a previous paper we have studied two classes of fuzzy bi-implications based on t-norms and r-implications, and shown that they constitute increasingly weaker subclasses of the Fodor-Roubens bi-implication. Now we prove that each of these three classes of bi-implications is closed under automorphisms.

1. Introduction

The collection of all automorphisms on a given mathematical object, i.e., isomorphisms from this object into itself, form a group with respect to the composition operator. Automorphisms have played an interesting role with respect to fuzzy connectives, given that when a class of fuzzy connectives is closed under automorphisms the action of the group of automorphisms establishes an equivalence relation among the connectives and therefore determines a partition on this class of connectives. Such partitions, in some cases, have characterized important subclasses of fuzzy connectives. For example, the class of strict t-norms is the equivalence class of the product t-norm [8], the class of nilpotent t-norms coincides with the equivalence class of the Łukasiewicz t-norm [8], the class of strong negations is the same as the equivalence class of the standard negation [9], and the class of implications which are both strong and residual is the equivalence class of the Łukasiewicz implication [1]. Concerning the bi-implication connective, in [5] we have studied the relation between the more well known definition proposed by Fodor and Roubens and other appealing definitions, old or new, of fuzzy operators that extend the interpretation of the classical bi-implication. It seems only reasonable then to study the action of automorphisms on fuzzy bi-implications. In the present paper we prove that each of the three classes of fuzzy bi-implications studied in [5] is closed under automorphisms.

2. Fuzzy extensions of conjunction and implication

All the following definitions concern the totally-ordered unit interval $\mathcal{U} = [0, 1]$.

Definition 9 *A triangular norm (in short, t-norm) is a binary operator T on $\mathcal{U}$ that: agrees with classical conjunction on the boolean inputs $\{0, 1\}$, is commutative, is associative, is increasing on both arguments, and has 1 as neutral element.*

Notions related to continuity are inherited from Analysis. In particular:

Definition 10 *A t-norm T is called left-continuous if for all non-decreasing sequences $(x_n)_{n\in\mathbb{N}}$ we have that $\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y)$.*

It is also opportune to recall that a continuous function preserves both limits and suprema.

Definition 11 *A fuzzy implication is a binary operator I on $\mathcal{U}$ that: agrees with classical implication on boolean inputs, is decreasing on the first argument and is increasing on the second argument.*

Definition 12 *The residuum of a left-continuous t-norm T is the operation I such that $I(x, y) \geq z$ iff $T(z, x) \leq y$.*

It is easy to check that the residuum of a left-continuous t-norm is unique. A particularly interesting class of fuzzy implications is precisely the one based on residua:

Definition 13 *A binary operator I on $\mathcal{U}$ is called an r-implication if there is a t-norm T such that:*

$$I(x, y) = \sup\{z \in \mathcal{U} \mid T(x, z) \leq y\} \quad (125)$$

In such case we may also say that I is an r-implication based on T, and denote it by I^T. We will say that I^T is of type $\mathbb{LC}$ in case T is left-continuous. In the latter situation we also say that (T, I^T) forms an adjoint pair, or that I^T is the adjoint companion of T.

2.1 Automorphisms and their actions on the fuzzy connectives

Definition 14 *An automorphism ρ on $\mathcal{U}$ is a continuous strictly increasing unary function with boundary conditions $\rho(0) = 0$ and $\rho(1) = 1$.*

Recall that the inverse of a strictly increasing function on a totally ordered domain is also strictly increasing, and that continuous strictly increasing functions over closed intervals are bijective. From the above if follows that the inverse of an automorphism on the unit interval is strictly increasing, and in view of the boundary conditions is also follows that automorphisms are bijective. Moreover, since the inverse of an automorphism is also an automorphism and automorphisms are closed under composition, then $Aut(\mathcal{U})$, the set of automorphisms on $\mathcal{U}$, forms a group with respect to the composition operator. Thus, as usual in algebra (see for example [7]), we may entertain the action of members of the group $\langle Aut(\mathcal{U}), \circ\rangle$ on arbitrary representatives of a given collection of n-ary functions on $\mathcal{U}$.

Definition 15 *The action of an automorphism ρ on a function $f : \mathcal{U}^n \to \mathcal{U}$ is the function $f^\rho : \mathcal{U}^n \to \mathcal{U}$ defined by*

$$f^\rho(x_1, \ldots, x_n) = \rho^{-1}(f(\rho(x_1), \ldots, \rho(x_n))) \qquad (126)$$

In such situation we refer to f^ρ as a conjugate of f. A set $\mathcal{F}$ of n-ary functions on $\mathcal{U}$ is said to be closed under automorphisms if it contains the conjugates of each of its elements.

Given a collection $\mathcal{F}$ of functions that turns out to be closed under automorphisms, the relation of 'being a conjugate of' is clearly an equivalence relation on $\mathcal{F}$. Indeed, if g is a conjugate of a function f, then f is a conjugate of g — assuming that $g = f^\rho$, then, given that $f = f^{\rho\circ\rho^{-1}} = (f^\rho)^{\rho^{-1}}$, it follows that $f = g^{\rho^{-1}}$. In addition, if f is conjugate of g and g is conjugate of h then f is a conjugate of h, and clearly each function is conjugate of itself. Consequently, an automorphism allows us to partition the collection $\mathcal{F}$.

It is well known that the sets of t-norms, s-norms, fuzzy negations and implications are each closed under automorphisms (check, e.g., [2, 3, 8]). In the following we will check that the subclasses of left-continuous t-norms and of the r-implications are closed under automorphisms.

Proposition 14 *Let T be a t-norm and ρ be an automorphism. Then T is left-continuous iff T^ρ is a left-continuous t-norm.*

Proof. ($\Rightarrow$) Let $(x_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence. Then, given that ρ is strictly increasing, $(\rho(x_n))_{n\in\mathbb{N}}$ also is a non-decreasing sequence. Thus:

$$\begin{aligned}
&\lim_{n\to\infty} T^\rho(x_n, y) = \lim_{n\to\infty} \rho^{-1}(T(\rho(x_n), \rho(y))) = \\
&\qquad \text{by Eq. (126)} \\
&= \rho^{-1}(\lim_{n\to\infty} T(\rho(x_n), \rho(y))) = \\
&\qquad \text{because } \rho^{-1} \text{ is continuous} \\
&= \rho^{-1}(T(\lim_{n\to\infty} \rho(x_n), \rho(y)) = \\
&\qquad \text{because } T \text{ is left-continuous} \\
&= \rho^{-1}(T(\rho(\lim_{n\to\infty} x_n), \rho(y)) = \\
&\qquad \text{because } \rho \text{ is continuous} \\
&= T^\rho(\lim_{n\to\infty} x_n, y) \\
&\qquad \text{by Eq. (126)}
\end{aligned}$$

($\Leftarrow$) Follows straightforwardly from ($\Rightarrow$) and the fact that $(T^\rho)^{\rho^{-1}} = T$.

Proposition 15 *Let T be a t-norm, (I^T) be its residuum, and let ρ be an automorphism. Then $(I^T)^\rho = I^{(T^\rho)}$.*

Proof. Assume T be a t-norm with residuum (I^T), and notice that:

$$\begin{aligned}
&(I^T)^\rho(x, y) = \rho^{-1}(I^T(\rho(x), \rho(y))) = \\
&\qquad \text{by Eq. (126)} \\
&= \rho^{-1}(\sup\{z \in \mathcal{U} \mid T(\rho(x), z) \leq \rho(y)\}) = \\
&\qquad \text{by Eq. (125)} \\
&= \rho^{-1}(\sup\{z \in \mathcal{U} \mid \rho^{-1}(T(\rho(x), z)) \leq \rho^{-1}(\rho(y))\}) = \\
&\qquad \rho^{-1} \text{ is strictly increasing} \\
&= \rho^{-1}(\sup\{z \in \mathcal{U} \mid \rho^{-1}(T(\rho(x), \rho(\rho^{-1}(z)))) \leq y\}) = \\
&\qquad \rho^{-1} \text{ is the inverse of } \rho \\
&= \sup\{\rho^{-1}(z) \in \mathcal{U} \mid \rho^{-1}(T(\rho(x), \rho(\rho^{-1}(z)))) \leq y\} \\
&\qquad \rho^{-1} \text{ is continuous} \\
&= \sup\{\rho^{-1}(z) \in \mathcal{U} \mid T^\rho(x, \rho^{-1}(z)) \leq y\} \\
&\qquad \text{by Eq. (126)} \\
&= I^{(T^\rho)}(x, y) \\
&\qquad \text{by Eq. (125)}
\end{aligned}$$

Corollary 1 *Consider a mapping $I : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then I is an r-implication of type $\mathbb{LC}$ iff I^ρ is an r-implication of type $\mathbb{LC}$.*

Proof. Straightforward from Propositions 14 and 15.

3. Fuzzy bi-implication and automorphisms

3.1 Automorphisms on an axiomatized class of fuzzy bi-implications Fodor and Roubens have introduced an important class of fuzzy bi-implications [6]:

Definition 16 *The class of f-bi-implications contains all binary operators B on the unit interval $\mathcal{U}$ respecting the following axioms:*

(B1) $B(x, y) = B(y, x)$

(B2) $B(x, x) = 1$

(B3) $B(0, 1) = 0$

(B4) *If $w \leq x \leq y \leq z$, then $B(w, z) \leq B(x, y)$*

In view of (B1), (B2) and (B3), it is easy to see that any f-fuzzy bi-implication is bound to agree with classical bi-implication on boolean inputs.

The following are some examples of f-bi-implications:

Example 14

1. $B_M(x, y) = \begin{cases} 1 & \text{if } x = y \\ \min(x, y) & \text{otherwise} \end{cases}$

2. $B_P(x, y) = \begin{cases} 1 & \text{if } x = y \\ \frac{\min(x,y)}{\max(x,y)} & \text{otherwise} \end{cases}$

3. $B_L(x, y) = 1 - |x - y|$

4. $B_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 1 & \text{otherwise} \end{cases}$

5. $B_B^{TI1}(x, y) = \begin{cases} 1 & \text{if } x = y \text{ or } \max(x, y) \neq 1 \\ 0 & \text{otherwise} \end{cases}$

Proposition 16 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B satisfies (Bi) iff B^ρ satisfies (Bi), for $i = 1, \ldots, 4$.*

Proof. (⇒) If B satisfies (Bi) for $i = 1, 2, 3$ then, from Eq. (126) and the fact that $\rho(1) = 1$ and $\rho(0) = 0$, trivially B^ρ satisfies (Bi). Moreover, if $w \leq x \leq y \leq z$, then because ρ is strictly increasing, $\rho(w) \leq \rho(x) \leq \rho(y) \leq \rho(z)$ and so, since B satisfies (B4), we have that $B(\rho(w), \rho(z)) \leq B(\rho(x), \rho(y))$. Therefore, because ρ^{-1} is strictly increasing, $\rho^{-1}(B(\rho(w), \rho(z))) \leq \rho^{-1}(B(\rho(x), \rho(y)))$, i.e., $B^\rho(w, z) \leq B^\rho(x, y)$ and so B^ρ satisfies (B4).
(⇐) Follows straightforwardly from (⇒) and the fact that $(B^\rho)^{\rho^{-1}} = B$.

Corollary 2 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an f-bi-implication iff B^ρ is also an f-bi-implication.*

Proof. Straightforward from previous proposition.

Example 15 *Let ρ be the following automorphism: $\rho(x) = x^2$ for each $x \in \mathcal{U}$. Then:*

1. $B_M(x, y) = B^\rho_M(x, y)$
2. $B_P(x, y) = B^\rho_P(x, y)$
3. $B^\rho_L(x, y) = \sqrt{1 - |x^2 - y^2|}$
4. $B_D(x, y) = B^\rho_D(x, y)$.
5. $B^{TI1}_B(x, y) = (B^{TI1}_B)^\rho(x, y)$

Note that equations 1, 4 and 5 from the previous example hold good in fact for any choice of automorphism ρ.

Definition 17 *An f-bi-implication B is said to satisfy the diagonal principle if $B(x, y) \neq 1$ whenever $x \neq y$.*

Proposition 17 *Let B be an f-bi-implication and ρ be an automorphism. Then B satisfies the diagonal principle iff its conjugate B^ρ satisfies this same principle.*

Proof. (⇒) If $x \neq y$ then because ρ is injective, $\rho(x) \neq \rho(y)$ and so, because B satisfies the diagonal principle, $B(\rho(x), \rho(y)) \neq 1$. Thus, because ρ^{-1} is injective, then $\rho^{-1}(B(\rho(x), \rho(y))) \neq \rho^{-1}(1)$, i.e., $B^\rho(x, y) \neq 1$.
(⇐) Follows straightforwardly from (⇒) and the fact that $(B^\rho)^{\rho^{-1}} = B$.

As is well known, t-norms, s-norms and fuzzy implications, with the help of the truth constants 0 and 1, induce 'natural' classes of fuzzy negations [2]. In the case of a (f-)bi-implication B, the natural negation is the function $N_B : \mathcal{U} \to \mathcal{U}$ defined by

$$N_B(x) = B(x, 0) \tag{127}$$

In the following result we check that the conjugate of the natural fuzzy negation induced by a fuzzy bi-implication coincides with the natural negation induced by the conjugate (with respect to the same automorphism) of the given bi-implication.

Proposition 18 *Let ρ be an automorphism and B be an f-bi-implication. Then the equation $(N_B)^\rho = N_{B^\rho}$ holds.*

Proof. Let $x \in \mathcal{U}$. Then, by equations (126) and (127), we have that $N^\rho_B(x) = \rho^{-1}(N_B(\rho(x))) = \rho^{-1}(B(\rho(x), 0)) = \rho^{-1}(B(\rho(x), \rho(0))) = B^\rho(x, 0) = N_{B^\rho}(x)$.

3.2 Automorphisms on classes of fuzzy bi-implications based on a defining standard involving t-norms and fuzzy implications In the definitions that follow, we assume fuzzy bi-implication B to be presented through the so-called TI *defining standard*, according to which $B(x, y) = T(I(x, y), I(y, x))$, where T is a t-norm and I an r-implication.

Definition 18 ([5]) *The class of a-bi-implications contains all binary operators B on $\mathcal{U}$ following the TI defining standard and based on an arbitrary t-norm T and on its residuum I^T, that is, operators defined by setting*

$$B(x, y) = T(I^T(x, y), I^T(y, x)) \tag{128}$$

Proposition 19 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an a-bi-implication iff B^ρ is an a-bi-implication.*

Proof. (⇒) Assume B an a-bi-implication, and notice that:

$B^\rho(x, y) = \rho^{-1}(B(\rho(x), \rho(y))) =$
by Eq. (126)
$= \rho^{-1}(T(I^T(\rho(x), \rho(y)), I^T(\rho(y), \rho(x)))) =$
by Eq. (128)
$= \rho^{-1}(T(\rho \circ \rho^{-1}(I^T(\rho(x), \rho(y))), \rho \circ \rho^{-1}(I^T(\rho(y), \rho(x))))) =$
$= T^\rho((I^T)^\rho(x, y), (I^T)^\rho(y, x)) =$
by Eq. (126)
$= T^\rho(I^{(T^\rho)})(x, y), I^{(T^\rho)}(y, x)) =$
by Prop. 15

Therefore, also B^ρ is an a-bi-implication.
(⇐) Follows straightforwardly from (⇒) and the fact that $(B^\rho)^{\rho^{-1}} = B$.

Corollary 3 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an a-bi-implication and not an f-bi-implication iff B^ρ is an a-bi-implication and not an f-bi-implication.*

Proof. Straightforward from Proposition 19, Corollary 2 and the fact that $(B^\rho)^{\rho^{-1}} = B$.

Definition 19 ([5]) *The class of ℓ-bi-implications contains all a-bi-implications based on left-continuous t-norms and their corresponding residua.*

Proposition 20 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an ℓ-bi-implication iff B^ρ is an ℓ-bi-implication.*

Proof. Straightforward from Proposition 19 and Corollary 1.

Corollary 4 *Consider a mapping $B : \mathcal{U}^2 \to \mathcal{U}$ and an automorphism ρ. Then B is an a-bi-implication and not an ℓ-bi-implication iff B^ρ is an a-bi-implication and not an ℓ-bi-implication.*

Proof. Straightforward from Propositions 19 and 20 and the fact that $(B^\rho)^{\rho^{-1}} = B$.

4. Conclusion

In this paper we considered the action of the group of automorphisms on the three classes of fuzzy bi-implications that were studied in [5], namely, f-bi-implications, a-bi-implications and ℓ-bi-implications. More specifically, we proved that all three classes are closed under automorphisms and therefore the actions of automorphisms induce partitions of these classes. For example, the equivalence class of the fuzzy bi-implication B_M is the singleton set $\{B_M\}$, and analogously for the fuzzy bi-implications B_D and B_B^{TI1} and the singleton sets $\{B_D\}$ and $\{B_B^{TI1}\}$, yet the equivalence class of B_L is not a countable set (to see that, in Ex. 15 it is sufficient to replace the automorphism $\rho(x) = x^2$ by $\rho(x) = x^r$, with r a positive real number).

This is a preliminary study, and several other aspects of the actions of automorphisms on bi-implications rest to be investigated. On the one hand, it would seem only natural to extend the present development to cover other classes of fuzzy bi-implications, characterized by other defining standards, as for instance the IST defining standard based on the classical equivalence in between $\alpha \Leftrightarrow \beta$ and $(\alpha \vee \beta) \Rightarrow (\alpha \wedge \beta)$. In general, the classes of fuzzy bi-implications that follow the IST defining standard are not subclasses of the class of f-bi-implications. On the other hand, it would also be interesting to extend the present study to cover other particularly interesting subclasses of the class of f-bi-implications, such as the so-called restricted equivalence functions introduced in [4].

Acknowledgments The research reported in this paper falls within the scope of the EU FP7 Marie Curie PIRSES-GA-2012-318986 project GeTFun: Generalizing Truth-Functionality. The authors acknowledge partial support by Brazilian funding agency CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Edital Universal 14/2011 — Faixa B, under the Proc. No. 480832/2011-0, project ConDE: Conjuntos Difusos Estendidos.

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REPORT

Call for Candidates for the EUSFLAT Board

Dear EUSFLAT members, it is my pleasure to invite you to submit lists of candidates for the new EUSFLAT Board for the period 2013-2015. Please, be aware that the list of candidates needs to:

- consist of EUSFLAT members only;
- consist of at least 6 candidates: president, vice-president, secretary, treasurer and at least two additional Board members;
- be presided by a president candidate who is a paying EUSFLAT member for at least two years;
- be mailed to the EUSFLAT secretary before the deadline specified below;
- be accompanied by letters of acceptance of all the candidates on the list (either signed or sent from emails of the individual candidates).

Sincerely Yours

Javier Montero

Deadline for submissions of lists of candidates: August 10, 2013

Address for the submission of lists of candidates:

Martin Stepnicka
IT4Innovations - IRAFM
University of Ostrava
30. dubna 22
702 00 Ostrava
Czech Republic

Email address for submissions in the electronic form:

Martin.Stepnicka@osu.cz

CONFERENCE REPORT

ISCAMI 2013, 14th International Student Conference on Applied Mathematics and Informatics

14th International Student Conference on Applied Mathematics and Informatics (ISCAMI 2013) was again organized jointly by the University of Ostrava (Centre of Excellence IT4Innovations - Institute for Research and Application of Fuzzy Modeling) and by the Slovak University of Technology (Faculty of Civil Engineering, Department of mathematics and descriptive geometry) in Malenovice - a beautiful village situated on the root of the highest peak in Beskydy mountains near Ostrava. Let us recall, that ISCAMI stems from a rich history of student conferences that were organized by two faculties of the Slovak University of Technology in years 1999-2007. And even after one-year pause in 2008, it still keeps the original idea to bring together graduate students in various areas of mathematics and theoretical informatics relevant for applications.

In 2013, ISCAMI (see `http://irafm.osu.cz/iscami`) was organized jointly with the **2nd Summer School on Applied Mathematics and Informatics**, which means that there were three separate tutorials on distinct topics given by leading European scholars, namely:

- Bernard De Baets (Ghent University, Belgium), A primer on transitivity of reciprocal relations;
- Ladislav Mišík (University of Ostrava, Czech Republic), A short introduction to Ergodic Theory;
- Frank Klawonn (Ostfalia University of Applied Sciences, Germany), Cluster analysis - determining the number of clusters.

Both joined events were strongly supported by A-Math-Net project (reg. nr. CZ.1.07/2.3.00/30.0010 of the Education for Competitiveness Operational Programme) that financially supported all participants who did not have to pay any conference fee and, moreover, covered all the accommodation and foood expenses of the participants. Let us only briefly mention that the overall number of was 84 participants. Among other statistics, let us choose the following:

- number of student talks: 60,
- number of countries: 8 (Czech Republic, Slovak Republic, Turkey, Germany, Latvia, Poland, Belgium, Spain).

Although the scope of the conference is very wide and participants presented talks from distinct areas such as: differential and difference equations, fuzzy logic and fuzzy modeling, evolutionary computing, data-mining, image processing, probablity, algebra or time series analysis, we gratefully acknowledge the support of EUSFLAT which is a single branch society. As every year since 2009, ISCAMI was marked as EUSFLAT endorsed event, which helps the visibility of the conference a lot.

Petra Hodáková (Organizing Chair, University of Ostrava), Jiří Kupka (Programme Chair, University of Ostrava), Martin Štěpnička (Financial Chair, University of Ostrava) and Radko Mesiar (General Chair, Slovak University of Technology).

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CONFERENCE REPORT

MIBISOC 2013, International Conference on Medical Imaging Using Bio-inspired and Soft Computing

The MIBISOC 2013 International Conference was organized as the main dissemination activity and the final event of the MIBISOC Marie Curie Initial Training Network (ITN) (`http://www.mibisoc-itn.eu/`). The conference was held in Brussels, Belgium, at the Campus du Solbosch of the Université Libre de Bruxelles, one of the full partners of the project, from May 15-17, 2013.

The MIBISOC ITN was granted by the European Commission within the Marie Curie Actions of the Seventh Framework Program (FP7-PEOPLE-ITN-2008, Contract Ref. 238819). Its main topic is the application of Soft Computing and Bioinspired Computing (SC & BC) to Medical Imaging (MI). The research project started in October 2009 and involves an interdisciplinary consortium including eight prestigious European institutions in the fields of the BC-SC and MI: European Centre for Soft Computing (Coordinator, Spain, `http://www.softcomputing.es`), Ghent University (Belgium), Université Libre de Bruxelles (Belgium), University of Nottingham (UK), Università degli Studi di Parma (Italy), University of Granada (Spain), Henesis (Italy), and Universitätsklinikum Freiburg (Germany). The main aim of the MIBISOC ITN is to provide research training to 16 early-stage researchers (ESRs) to equip them with the knowledge and skills for developing intelligent systems based on SC-BC techniques for solving real-world MI problems surpassing current state-of-the-art approaches. The ITN will finish by the end of September 2013. Through the intensive and interdisciplinary training programme developed within the MIBISOC network we will provide the European industry and research institutions with highly qualified researchers able to solve complex MI problems. These researchers will promote new scientific knowledge and technological applications in hospitals, healthcare providers, and technological companies.

The MIBISOC 2013 conference provided a forum to foster the discussion of novel problems, the dissemination of advanced research results, and the definition of future directions in the growing interdisciplinary area of MI by means of BC & SC. The support of the European Commission through the MIBISOC ITN allowed us to offer a comprehensive scholarship program to cover partial costs of travel and accommodation for a significant number of external attendees as well as to offer free registration. The conference attracted 75 people, 30 belonging to the member institutions and another 45 being external attendees from 12 European and non European countries: Spain, Belgium, United Kingdom, France, Italy, Czech Republic, Poland, Germany, United States, Mexico, Canada, and India.

As MIBISOC 2013 aimed at fulfilling two conflicting objectives, namely to build a vibrant forum for

exchanging ideas and to facilitate interaction by means of high quality papers, the acceptance of the contributions was treated as a "multi-criteria problem". Taking this into account, the presentation of already published outstanding work was accepted and encouraged to promote discussion and exchange of knowledge and one paper already published in a highly ranked MI journal was presented during the conference.

To guarantee for the quality of the accepted papers, each of the 45 submitted contributions was reviewed by two members of an International Program Committee composed of internationally recognized researchers in the relevant areas. Fifteen of those contributions were authored by the MIBISOC ITN ESRs, for which the MIBISOC2013 conference constitutes the last formation program activity. After this rigorous review process, 38 papers were accepted: 25 for oral presentation (55% of the original submissions) and 13 for poster presentation (29%). Authors of papers accepted for oral presentation were asked to also prepare a poster in order to increase interaction during the conference.

The papers accepted for presentation covered most of the areas within the MIBISOC research theme: MI processing (4 accepted papers), segmentation (10 accepted papers), classification (11 accepted papers), registration (5 accepted papers), motion and tracking (6 accepted papers), as well as advanced SC-BC developments (4 accepted papers).

The MIBISOC 2013 conference was composed of five oral sessions, two poster sessions, two plenary lectures by renowned researchers in the field, and an MI industry session including representation from three different companies. It officially opened on Wednesday morning with an opening ceremony including the interventions of Mr. Sergio Mastropierro, the MIBISOC ITN Project Officer from the Research Executive Agency of the European Commission, Dr. Oscar Cordón, the MIBISOC ITN Coordinator, and Dr. Raúl del Coso, the ECSC General Manager. Just after the welcome address, an exciting plenary talk entitled "Graph-based Methods in Bio-Imaging and Beyond: Setting the state of the Art" was given by Dr. Nikos Paragios. The first conference day also included the two oral sessions on "Medical Image Processing and Registration" and "Bio-inspired and Soft Computing Methods" as well as the first poster session on "Medical Image Processing, Segmentation and Registration". The attendees gathered for the conference gala dinner on the first evening of the symposium, following our original idea to additionally promote interaction by making it easier for them to know each other from the beginning.

Thursday morning started with the intervention of the second plenary speaker, Dr. Isabelle Bloch, who delivered an outstanding talk about "Structural Models and Spatial Reasoning - Application to Segmentation and Recognition of Anatomical Structures in Medical Images". The invited talk was followed by two oral sessions on "Medical Image Segmentation" and "Medical Image Classification", and the second poster session on "Medical Image Classification, Motion and Tracking. Bioinspired and Soft Computing Methods".

Finally, the third conference day, Friday May 17, opened with the MI industry session. Three CEOs from three companies working in the MI area, Harry Hatzakis from Biotronics3D, Dr. Vincenzo Ferrari from ENDOCAS/*e-SPres 3D*, and Dr. Luca Ascari from *Henesis* (another full partner of the MIBISOC ITN). They provided very interesting discussions on different aspects of the industrial side of the MI area, followed by a round table discussion. The last oral session on “Medical Image Motion and Tracking” was held after the industry session and the conference concluded with the closure session.

As MIBISOC ITN Coordinator and MIBISOC 2013 General Co-chair I would like to take this opportunity to thank the Marie Curie Actions Program of the European Commission for its sponsorship of MIBISOC2013 through the MIBISOC ITN. I would also like to thank all those who have been involved in the preparation of the workshop: program chairs, the international program committee, the organizing committee, and the publication chair.

The success of these kinds of events always relies mainly on the dedication and passion of the team involved in their organization. Part of MIBISOC 2013 success is also owed to the work of my good friend Dr. Thomas Stützle, who was the responsible for the local organization and always had a solution for every problem that could arise. I also wish to thank all those who actually substantiate the conference: contributors and participants. Finally, my special thanks go to the MIBISOC ESRs, the young researchers who have traveled this exciting research journey with us along the last three years and who are now just finishing their contracts: Nico, Andrea, Viktor, Luis, Jérémie, Gabriele, Bartosz, Torbjörn, Pablo, Youssef, Michela, Benjamin, Pouya, Lara, David, and Esme. As I said in my speech at the opening session, they have been the core of the MIBISOC ITN and I wish them all the best for their future research career. I will always keep them in my memory.

Oscar Cordón
MIBISOC Initial Training Network Coordinator and MIBISOC 2013 General Co-chair European Centre for Soft Computing, Mieres, Asturias, Spain
Department of Computer Science and Artificial Intelligence and Research Center on Information and Communication Technologies (CITIC-UGR), University of Granada, Granada, Spain

NEWS

EUSFLAT Facebook page is active

http://www.facebook.com/EUSFLAT

- Do not miss any endorsed event information.
- Do not miss parts of the Mathware magazine.
- Do not miss lot of further information.
- Do not lose the chance to be connected.

Just like it, read it and share it.

NEWS

A view of Pedro Burillo, teacher and researcher at the Public University of Navarra

This year, prof. Pedro Burillo retired from the Public University of Navarra.

He was the first dean of this university, and it cannot be understood without his impulse in the very first years, in late eighties. But, apart from his administrative career, which led him to be even Education counselor in the government of Navarra, prof. Burillo has also developed an outstanding researcher profile. He was one of the four founders of the Computer Sciences area in Spain, and he was also responsible for the introduction of fuzzy sets theory in the Spanish scientific community in such a way that nowadays some of the most relevant international researchers in this field are precisely from Spain. Moreover, he has also been author or coauthor of some crucial Works that are nowadays unavoidable references

for the work in fuzzy sets. But if there is some special characteristic that needs to be highlighted is that of transmitting to students his passion for science and research. Every of his students remember him as one of the best teachers in their life. He was always able to guide Ph. D. students out from apparently irresoluble stands-by in their work, giving back the fortitude and the illusion to continue researching, working, advancing in the study. More than 20 successful Ph.D. students are the living proof of his capacity. And I have the great honor of being the first one here in the Public University of Navarra.

Thanks a lot, Pedro.

Humberto Bustince Professor of the Public University of Navarra

NEWS

Your Thesis in Three Minutes (T3M)

"Your Thesis in Three Minutes (T3M)", is a competition organized by the Spanish Network for the Advance and Transfer of Applied Computational Intelligence (ÁTICA) addressed to doctoral students in Spain whose thesis involve any aspect of computational intelligence.

The competition aims to bring closer the area of Computational Intelligence to society, and also to stimulate the communication skills of young researchers, trying to raise their awareness on the importance of properly disseminating their scientific work.

Contestants have to explain their research in 3 minutes time and do it using in a language that is understandable by the general public. They are allowed to use a single microphone and a static slide as only support. Equipped with the word as their only communication tool, applicants must show their best skills to convey the contributions of their work to the audience (and the society) in a simple and didactic manner. All presentations should be made in spanish.

The first phase of the competition began on 15th May and will last until September 10, 2013. In this first phase, contestants must upload a video with their presentation through the Platform ÁTICA has set up for the competition. A voting will be made until September 30th among the participants, in order to select the best 10 presentations. The selected participants will participate in the final presential phase, which will be held in Madrid in late November (during the ÁTICA Anual Seminar, SEMÁTICA), where they will have to perform a viva presentation before a jury of experts in scientific and technical communication. Three prizes will be awarded by the jury, all of them including a financial support for a one month's research stay in ÁTICA groups or in the R+D section of sponsoring companies.

Further information on the competition, registration form, and the complete basis of the competition can be reached at the ATICA Website:

`atica.lsi.upc.edu/t3m`

NEWS

Ph.D. Thesis defended by Rosa M. Rodríguez Domínguez

Department of Computer Science, University of Jaen, Spain

Rosa M. Rodríguez Domínguez defended her PhD Thesis, entitled "Using Comparative Linguistic Preferences in Decision Making under Uncertainty" the last 29th April. Her supervisor is Luis Martínez López from the University of Jaen.

Usually, the complexity of decision making problems is because of the uncertainty that surrounds the alternatives and the knowledge of the experts. One of the main topics in decision making is the modelling and treatment of the uncertainty of the different types of information that can be used by experts to provide their preferences. Our research focuses on decision making problems in qualitative contexts with high degree of uncertainty, where experts usually use linguistic information to express their preferences. The use of linguistic information has provided reliable and successful results in this type of decision making problems, but sometimes it is limited because the linguistic models use linguistic terms defined a priori and restrict to experts for providing their preferences by using just one term, which may not reflect exactly what they want. To overcome these limitations, different linguistic models have been introduced in the literature to provide richer expressions than single linguistic terms, but they are either far away from the common language used by experts in decision making problems and/or do not have defined any formalization to generate the linguistic expressions.

Therefore, the main aim of this thesis focused on trying to enrich the vocabulary of expression for experts provide their assessments in a more flexible way in decision making problems. To do so, it is presented a new representation model of linguistic information entitled Hesitant Fuzzy Linguistic Term Sets (HFLTS), which allows experts express their preferences with more than one linguistic term by comparative linguistic expressions close to common language used by human beings in decision problems. Besides, different models of group decision making, multicriteria decision making and multiexpert multicriteria decision making, where experts can provide their preferences by using single linguistic terms or comparative linguistic expressions based on HFLTS have been proposed.

To show the useful of the HFLTS, a real problem of nuclear safeguards evaluation has been solved by using the multiexpert multicriteria decision making model defined in this research memory. Finally, a new module has been developed to manage the HFLTS and comparative linguistic expressions in a decision support system so-called Multicriteria Decision Analysis with Computing with Words.

Some results of this thesis have been published in international journals such as, IEEE Transactions on Fuzzy Systems, Information Sciences, International Journal of General Systems and International Journal of Uncertainty, Fuzziness and knowledge-based Systems.

NEWS

Ph.D. Thesis defended by Aránzazu Jurío

Department of Automatic and Computation, Universidad Pública de Navarra, Spain

Aránzazu Jurío defended her PhD Thesis, entitled "Numerical measures for image processing. Magnification and thresholding" the last 23rd May. Her supervisors are Humberto Bustince and Miguel Pagola from the Universidad Pública de Navarra.

Digital image processing consists of a set of techniques that are applied to digital images for solving two problems: to improve the visual quality of the image and to process the information that it contains, in such a way that a machine is able to understand it by itself.

Some of the most common procedures to be carried out over digital images are those of magnification and segmentation. In this Ph. D. thesis we have presented methods for both of them

Image segmentation consists in separating each of the objects that are in the image. To do so, each pixel is analyzed, in such a way that all of them sharing some characteristics are considered to be part of the same object. Digital image segmentation is used nowadays to solve many problems, as teledetection, pattern recognition or medical image analysis. On the other hand, image magnification consists in increasing the spatial resolution of an image, that is, obtaining a larger image (with more pixels), which represents the same scene and keeps the details and clarity of the original one. Image magnification techniques are very useful when we transmit images from one device to another or when we upload them to the web, since for a faster transmission we often use a reduced version of the image which must be magnified to its original size. Magnification is also widely used for low resolution images, as it is the case of those obtained from video-vigilance.

In this Ph. D. thesis we have presented two new magnification methods. One for magnifying grayscale images and another for color images. These methods were developed to solve a problem for an infography company, where, apart from the quality of the results, it is also necessary to use fast algorithms. To do so, we have studied the way of storing the relevant information about each pixel and its neighbors by means of intervalar techniques, as well as different methods to get the values of the new pixels from those intervals. In this way, our magnification algorithms outstand not only because of the obtained quality but also because of its temporal efficiency, since they obtain images of similar quality 700 faster than other existing algorithms.

Along this dissertation we have also presented two image segmentation algorithms. The first one is adapted to work with digital fingerprint images. One of the first steps to make a personal identification through fingerprints is to separate the fingerprint itself from the background efficiently. To do so, in this thesis we have proposed a method to measure the homogeneity of each region of an image, that is, to measure how similar the pixels on a given region are. From this measure we have developed an algorithm which is able to segment fingerprints in a correct way.

The second segmentation algorithm that we have proposed is oriented towards brain magnetic resonance images segmentation. One of the most important features of our method is that we make different approaches to such segmentation by means of fuzzy techniques and we get a consensus between all of them, in such a way that the final segmentation is very precise.

Some results of this thesis have been published in international journals such as, IEEE Transactions on Image Processing, Fuzzy Sets and Systems, European Journal of Operational Research and Soft Computing.

NEWS

Ph.D. Thesis defended by Daniel Paternain

Department of Automatic and Computation, Universidad Pública de Navarra, Spain

Daniel Paternain defended his PhD Thesis, entitled "Optimization of image reduction and restoration algorithms based on penalty functions and aggregation techniques" the last 31st May. His supervisors are Humberto Bustince and Javier Fernandez from the Universidad Pública de Navarra and Gleb Beliakov from Deakin University

Digital image processing has widely developed during the last 30-40 years. It has made possible a huge improvement in the quality of digital images. This is partly due to the fact that images have nowadays a larger spatial resolution, that is, a larger amount of pixels. However, high resolution images suffer from two main problems: the cost for storing or transmitting them (by internet, for instance), and the high computational cost of their processing (specially the time it takes). To solve both problems simultaneously, in this thesis we propose several image reduction algorithms (both for grayscale and for color images) whose aim is to reduce de amount of pixels in the image keeping all (or as much as possible) the information and properties in the original image

The main idea of the reduction algorithms that we have developed consists in dividing the image in small regions that we are going to treat individually. For each region, the goal is to find a value for the resulting pixel which is the least dissimilar to all the values of the intensities of the pixels in that region. With this methodology, we are able to design very time-efficient algorithms which are able to adapt to the local properties of each region of the image.

First of all, in the thesis we have developed two reduction algorithms in grayscale where we apply aggregation functions to solve the reduction problem. Aggregation functions are widely studied these last years due to their applicability to any research field, since they provide a way to fuse or combine several information sources (homogeneous or heterogeneous) into a single representing value. This is a key point in any decision making (experts' or juries' valuations) procedure, in approximate reasoning, etc. In the thesis we have studied the properties a large class of aggregation functions, analyzing the advantages of each of them as, for instance, stability in the presence of noise in the images.

However, the application of such functions in color images is much more complicate, since such images contain much more information than grayscale ones (each pixel contains several sources of information). For this reason we have studied penalty functions, a mathematical tool that, by means of optimization algorithms, has allowed us to develop a color image reduction algorithm where the most appropriate aggregation function is used for each region of the image.

Finally, we have applied the concept of reduction algorithm to one of the most difficult problems in image processing: digital images restoration. Assume that, due to a transmission or image processing problem we have lost a lot of pixels from an image. The goal of a restoration algorithm is to estimate the values of the lost pixels in such a way that we recover an image as similar to the original one as possible. To create this new information we follow these steps: first, we store a reduced version of the original image that keeps as many properties as possible. This reduced version cannot be very big as we don't want to increase the storage cost of the original image (reduced images must contain between 1 and 10% of the information of the original one). Secondly, from the damaged image we generate an optimization algorithm which is able to estimate the value of the lost pixels using the information which is kept both in the damaged image and in the reduced version.

Some results of this thesis have been published in international journals as Information Sciences, IEEE Transactions on Image Processing, Fuzzy Sets and Systems and Optics Express.

NEWS

Report on IFSA Awards and IFSA Fellowships, 2013

IFSA Fellowship

This year, the IFSA Commission for IFSA Fellowships (Ronald Yager acting as Chairman, Michio Sugeno, Henri Prade, Janusz Kacprzyk and George Klir) acknowledged the following colleagues as new IFSA Fellows:

- Krassimir Atanassov
- Miguel Delgado
- Antonio Di Nola
- János Fodor
- Lluis Godo
- Francisco Herrera
- Donald Kraft
- Chin-Teng Lin

IFSA Award

The IFSA Award is presented biannually, at the IFSA World Congresses, to one person (although eventually, considering the circumstances, more than one award may be offered).

The primary base of decision about awardees is their life time academic achievement. The IFSA award is defined consisting of a bronze plaque, trip reimbursement to attend the awards ceremony, and free registration for IFSA conferences.

In 2011, the IFSA Award Committee named by the IFSA President and the Council of IFSA (Kaoru Hirota as President, Oscar Castillo as President Elect, Janusz Kacprzyk as Immediate Past President, Laszlo T. Koczy as Past President and Chairman of this Committee, and Luis Magdalena as Vice President for Awards) assigned two awards, to Ronald Yager and Enric Trillas.

In 2013, considering the candidatures from each IFSA institutional member, the Committee named by the IFSA President (Oscar Castillo as President, Kaoru Hirota as Inmediate Past President, Christer Carlsson as President Elect, and Javier Montero as Vice President for Awards) assigned this award to:

Janusz Kacprzyk

(Janusz Kacprzyk, as IFSA Past President, had been previously excluded from the Committee, once he was named by an IFSA institutional member).

IFSA Award for Outstanding Applications of Fuzzy Technology 2011

The IFSA Award for Outstanding Applications of Fuzzy Technology is awarded to engineers and scientists who have made significant contributions to the transfer of fuzzy logic technology from the research environment to successful commercial, industrial, or medical applications.

In 2011, the appointed Committee (Luis Magdalena as Vice President for Awards, Oscar Castillo, Fernando Gomide, Christer Carlsson and Sungshin Kim) distinguished as the winner of the Outstanding Applications of Fuzzy Technology Award to "Forensic Identification System Using Craniofacial Superimposition Based on Fuzzy Sets and Evolutionary Algorithms", by O. Cordón, S. Damas, O. Ibáñez, J. Santamaría, I. Alemán, M. Botella, and F. Navarro.

In 2013, no Outstanding Application of Fuzzy Technology Award was acknowledged.

L.A. Zadeh Prize (Best Paper Award)

This is an award mostly supported by Prof. Zadeh's donation. It recognizes a paper published within the previous two years. Nominations come from institutional members. The paper should have been published within the previous two years, to be selected from FSS, IJAR, IJCIS, IJUFKBS, JACI, or any official journal published by institutional members, and should be written in English. This prize should encourage young researchers.

The first L.A. Zadeh was acknowledged in 2011 to the paper by A. Fernandez, M.J. del Jesus and F. Herrera: "Hierarchical fuzzy rule based classification systems with genetic rule selection for imbalanced data-sets" (International Journal of Approximate Reasoning, 50:561-577, 2009).

In 2013, considering all the nominations from each IFSA institutional member among papers published in those journals during 2011 and 2012, the Committee (Janusz Kacprzyk acting as Chairman, Piero Bonissone, Chin-Teng Lin and Burhan Turksen) decided to award the paper:

P. Bosc, O. Pivert and G. Smits: "An Approach to Database Preference Queries Based on an Outranking Relation", by P. Bosc, O. Pivert, G. Smits (International Journal of Computational Intelligence Systems 5:789-804, 2012).

IFSA newsletter: `http://isdlab.ie.ntnu.edu.tw/ntust/ifsa/files/newsletter/Vol10/IFSAnewsletterV10N1.pdf`

NEWS

On Fuzziness. A Homage to Lotfi A. Zadeh

Recently a new book of two volumes appeared:

On Fuzziness. A Homage to Lotfi A. Zadeh.

The editors of this voluminous collection of many scientists, researchers, academics in the disciplines of Soft Computing/Computational Intelligence are Rudolf Seising, Enric Trillas, Settimo Termini and Claudio Moraga.

The edition of the book was a longer-than-a-year project of the Unit Fundamentals of Soft Computing in the European Centre for Soft Computing (ECSC) in Mieres, Asturias (Spain). The editors succeeded to convince over 170 authors from all over the world, to write on their experiences with Fuzziness and Soft Computing as well as, possibly on their personal meetings with Lotfi Zadeh.

- Seising, Rudolf; Trillas, Enric; Termini, Settimo and Moraga, Claudio (eds.): On Fuzziness. A Homage to Lotfi A. Zadeh - volume I, Berlin, Heidelberg et al.: Springer-Verlag (Studies in Fuzziness and Soft Computing, vol. 298), 2013.
- Seising, Rudolf; Trillas, Enric; Termini, Settimo and Moraga, Claudio (eds.): On Fuzziness. A Homage to Lotfi A. Zadeh - volume II, Berlin, Heidelberg et al.: Springer-Verlag (Studies in Fuzziness and Soft Computing, vol. 299), 2013.

For more information see:
`http://www.springer.com/engineering/computational+intelligence+and+complexity/book/978-3-642-35640-7` and
`http://www.springer.com/engineering/computational+intelligence+and+complexity/book/978-3-642-35643-8`

NEWS

Fuzziness and Medicine: Philosophical Reflections and Application Systems in Health Care

More than two years ago (in March 23-25, 2011) the First International Symposium on Fuzzy Logic, Philosophy and Medicine and the First International Open Works-hop on Fuzziness and Medicine have been organized at the European Centre for Soft Computing (ECSC) in Mieres, Asturias (Spain) with the support of an anonymous donator, the International Fuzzy Systems Association (IFSA), the European Society for Fuzzy Logic and Technology (EUSFLAT), the Hospital Universitario Central de Asturias and Springer-publisher.

The first part of this bipartite scientific event was devoted to the conception of the Handbook of Analytical Philosophy of Medicine that the Iranian-German physician and philosopher Kazem Sadegh-Zadeh was publishing in the year 2012. With Sadegh-Zadeh's consent, the organizer Rudolf Seising assembled a group of philosophers and scientists from different disciplines, such as mathematics, logics, social sciences, computer sciences, linguistics, to comment and discuss various parts, subjects and propositions from the Handbook.

A lively discussion of such topics was held immediately followed by the I. International Open Workshop on Fuzziness and Medicine, in which further theories, and methods, developments and challenges, application systems, tools and case studies were presented and debated. The beneficial exchange continued in the following months, more scientists were added to the roster, and all the written contributions to the debate were collected in the volume edited by Rudolf Seising (ECSC, Spain) and Marco Tabacchi (Palermo, Italy) as a "Companion volume" to the Handbook of Analytic Philosophy of Medicine that now appeared:

- Seising, Rudolf and Tabacchi, Marco (eds.): Fuzziness and Medicine: Philosophical Reflections and Application Systems in Health Care. A Companion Volume to Sadegh-Zadeh's "Handbook on Analytical Philosophy of Medicine", (Studies in Fuzziness and Soft Computing, Vol. 302), Berlin [u.a.]: Springer 2013.

For more information see:
http://www.springer.com/engineering/computational+intelligence+and+complexity/book/978-3-642-36526-3

NEWS

Archives for the Philosophy and History of Soft Computing

In 2013 a new online journal was founded in the European Cente for Soft Computing in Mieres, Asturias (Spain) ECSC. The editorial team consists

- as editors in chief: Rudolf Seising and Alejandro Sobrino
- as assistant editors: Martin Pereira Fariña and Marco Elio Tabacchi

There are more than 70 academics and scientists in the Editorial Board and we have seven members in the Advisory Board.

See for more information the online journal's web page:

http://www.aphsc.org/index.php/aphsc

We founded this Online journal because we are concerned that there are certain difficulties for philosophers and other academics in humanities and social sciences to publish articles that concern fuzziness and Soft Computing/ Computational Intelligence in science and technology.

Almost all journals that cover these topics restrict their acceptance to papers that are written in a highly standardized "scientific" style. The publication of philosophical thoughts and reflections has often failed because of these obstacles. Conversely scientists and engineers suffer from analogue problems when trying to publish reflection papers in humanistic/philosophical journals.

We believe, however, that these negative experiences point to something more fundamental, namely the fact that the new notions involved require opportunities for joint reflection by scholars of different fields on the problems posed by the interaction between conceptual and technical/formal aspects.

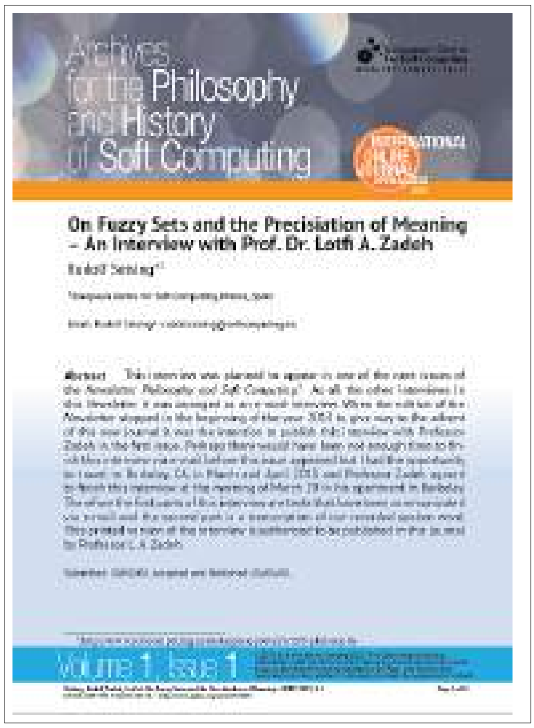
The journal seeks to provide a "neutral" forum in which these new questions could be frankly debated.

- The *Archives of Philosophy and History of Soft Computing* that we established provides a forum for reflections on Fuzziness and Soft Computing/Computational Intelligence from both science/technology and humanities/social sciences.
- The *Archives of Philosophy and History of Soft Computing* is devoted to philosophical, historical, sociological, and other humanistic approaches to reflect on the role of Fuzzy Logic, Fuzzy Sets and Systems, Artificial Neural Networks, Evolutionary Strategies and other methods of "Soft Computing/Computational Intelligence" in science and technology. It will be neither restricted to articles with a humanistic style nor with a scientific one.
- The *Archives of Philosophy and History of Soft Computing* will publish articles, interviews, discussions, and reviews. We foresee that each issue will include three or four refereed articles.

Subjects of the Archives of Philosophy and History of Soft Computing include (but are not limited to):

- Interaction between conceptual aspects and technical developments of the notion of fuzziness and satellite notions
- Logical, epistemological, anthropological, ethical, and aesthetical aspects of Soft Computing/Computational Intelligence science and technologies
- Historical, sociological, psychological and linguistic aspects of Soft Computing/Computational Intelligence science and technologies
- Soft Computing/Computational Intelligence and arts, music, painting, sculpture and architecture
- Soft Computing/Computational Intelligence in science, technology, and medicine
- Didactics and teaching of Soft Computing/Computational Intelligence in schools, high schools and universities
- Soft Computing/Computational Intelligence and Law, Politics, archaeology, criminology, economics, education, communication studies

The first issue of the new online journal will appear in August 2013. In this issue we publish a new interview with Professor Dr. Lotfi A. Zadeh:



CALLS

AGOP 2013 7th International Summer School on Aggregation Operators

Pamplona (Spain) 16-19 July 2013

AGOP 2013 will be held in Condestable's Palace (16th century) located in the centre of Pamplona.

http://giara.unavarra.es/agop2013/

Registration:

- Full: 400 eur
  Includes accommodation (with breakfast) + lunches + coffee breaks + material + social event
- Reduced Type I: 300 eur
  Includes accommodation (with breakfast) + coffee breaks + material
- Reduced Type II: 150 eur
  Includes coffee breaks + material

More than 50 world's top researchers including

- Bernard De Baets (Belgium)
- Fabrizio Durante (Italy)
- Jozo Dujmović (USA)
- MichałBaczyński (Poland)
- Anna Kolesárová (Slovakia)
- Ana Pradera (Spain)
- Salvatore Greco (Italy)
- Gabriella Pigozzi (France)
- József Dombi (Hungary)
- Radko Mesiar (Slovakia)

Topics including, but not restricted to:

Theoretical aspects

- Properties of aggregation functions
- New forms of aggregation functions
- Copulas and triangular norms
- Fuzzy measures and integrals
- Averaging aggregation operators
- Aggregation with ordinal and nominal scales
- Aggregation functions for extensions of fuzzy sets
- Implication operators

Practical aspects

- Security intelligence, analysis and decision support
- Evaluation problems
- Medical decision problems
- Hybrid intelligent systems and computational intelligence
- Approximate reasoning
- Image processing
- Model identification and parameterization
- Diagnostics and prognostics
- Data mining

Program Committee:

- Humberto Bustince (Spain)
- Javier Fernández (Spain)
- Tomasa Calvo (Spain)
- Radko Mesiar (Slovakia)

CALLS

EUROFUSE 2013 Uncertainty and Imprecision Modelling in Decision Making

Oviedo (Spain) 2-4 December 2013

The UNIMODE Research Unit of the University of Oviedo and EUROFUSE invite you to participate in the EUROFUSE 2013 workshop that will be held in Oviedo, Spain from 2 to 4 December 2013.

http://eurofuse2013.uniovi.es

AIMS AND SCOPE

The goal of this workshop is to bring together researchers and practitioners developing and applying fuzzy techniques in preference modeling and decision making. Topics of interest include but are not limited to:

- Preference representation and modeling
- Properties and semantics of preferences
- Decision Making
- Imprecise Probabilities
- Aggregation Operators

The invited lectures will be given by:

- Janós Fodor (Óbuda University, Hungary).
- Esteban Induráin (Universidad Pública de Navarra, Spain).
- Serafín Moral (Universidad de Granada, Spain).

DEADLINES

- Paper submission: June 17, 2013
- Notification of acceptance/rejection: July 31, 2013
- Camera-ready papers due: October 3, 2013
- Conference: December 2-4, 2013

In addition to the main track, the following special sessions will be held:

- Preference Modeling and Preference Learning using Imprecise Probabilistic Approaches
- Measures of similarity
- Applications of aggregation functions
- New educational techniques in the context of decision making with incomplete and imprecise information

REGISTRATION FEES AND STUDENT GRANTS

Information about the registration fees can be found at http://eurofuse2013.uniovi.es/?Registration

Students will profit from special discounts in the registration fees. In addition, EUSFLAT will sponsor 3 student grants.

Accepted papers will be published in the EUROFUSE 2013 proceedings (with ISBN). In addition, extended versions of selected papers will be published in a special issue of Fuzzy Sets and Systems.

All the information about the conference can be found at http://eurofuse2013.uniovi.es/

Looking forward to meeting you in Oviedo,

Susana Montes