Mathware & Soft Computing

The magazine of the European Society for Fuzzy Logic and Technology

In memory of Lofti A. Zadeh
by Enric Trillas
by Rudolf Seising
by Martin Stepnicka

Selected papers from the Brazilian Conference of Fuzzy Systems 2016

News and calls

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World Echoes

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Message from the Editor-in-Chief (December 2017)

HUMBERTO BUSTINCE

Here it is at your disposal the new issue of our Mathware&Soft Computing online magazine. As usual, it arrives with the closure of the year. This is time to think about the changes in the past and to prepare the future. And this is specially true because this year. First of all, because as all of us already know, Lofti Zadeh, the father of fuzzy theory, passed away last September, after a whole life devoted to research and science.

It is difficult to underestimate such a great scientific personality. Zadeh was undoubtfully one of the mot relevant figures in science in the last century and his ideas have had and will have a large influence in the development of Computer Science. Moreover, his willingness to collaborate and to share his developments have helped to create a strong community as well as to give raise to many different, original orientations from his seminar ideas.

For all these reasons and for many more this is not a usual issue, because it could not be so. This pages intend to be, to a large extent, an hommage to that great man called Lofti Zadeh. For this reason we include three main articles: One from Enric Trillas, with the personal view on Zadeh of somebody who has been a witness of the whole development of fuzzy theory; another one by Rudolf Seising, as an expert on the history of the theory of fuzzy sets; and the last one by Martin Stepnicka, the new president of our EUSFLAT society. I think that, among the three, we get a complete view on Zadeh's achievements and significance.

This year has also brought a new board to head our EUSFLAT association. Let me first of all thank from here the great work carried out in the last years by Gabriella Pasi, which have helped to boost our society. And let me also welcome Matin Stepnicka as the new chair. All of us know his huge capacity, so I have no doubt about the great future that we can expect for our society in the forthcoming years.

What this issue does not include, due to its special character, is an interview. But next issue will include the talk intended for this issue.

We also publish in this issue several selected works from the Brazilian Conference on Fuzzy Systems. And, of course, you have here news, conference reports, etc... So, in brief, all the things that show how alive the community which originated in Zadeh's work is! So it's time to enjoy your new issue of the Mathware&Soft Computing magazine. Have a nice reading!

Humberto Bustince
Editor-in-chief

Message from the President (December 2017)

MARTIN ŠTĚPNIČKA

Dear EUSFLAT members,

These are the first lines of my very first "Message from the President" and to tell the truth, I have never thought how difficult it is going to be to start writing such a letter. The reason for my nervousness is clear, I am writing to a Society that I appreciate as the one that deserves the highest respect, I am writing in a position that was held by so great Presidents before, and I still cannot stop thinking on how to keep the quality level set-up by my predecessors. Indeed, Francesc Esteva, Luis Magdalena, Ulrich Bodenhofer, Javier Montero and Gabriella Pasi, nothing has to be added, just a big thanks to all of them for their great contribution. On the other hand, looking back what all has been done by previous Presidents and previous Boards, one necessarily feels a gigantic duty that is the best motivation to work. And I have a fantastic advantage - the Board behind me. I know I have many very experienced members including two last Presidents there and I have also many new members with innovative ideas, freshness and enthusiasm, who will serve as wells of ideas and initiatives. There is a group of friends one may trust and one may lean on their shoulders, and this is the biggest support I may have and which makes me fully sure that at the end of this mission, EUSFLAT Society will be still strong and respectable Society as it is, and possibly offering even more services and advantages to the current state. The Board members, who decided to accept the offer and to devote their time and efforts to a volunteer job for the Society, are: **Vladik Kreinovich** (VicePresident), **José Maria Alonso** (Secretary), **Susana Montes** (Tresurer), **Eulalia Szmidt** (Awards), **Javier Montero** (Link to IFSA and Other Societies), **Christophe Marsala** (Grants), **Humberto Bustince** (Magazine), **Gabriella Pasi** (Publications), **S.P. Tiwari** (Promotion in Asia), **Daniel Sánchez** (Web and Social Networks), **Luis Martínez López** (Atlantis Press), **Guy De Tré** (Working groups), **Sebastià Massanet Massanet** (Recruitement), and all of them deserve maximal respect and my deepest gratitude. There are many challenges in front of us and we have already started to work on them.

But at this moment, allow me to look back a bit. It is still 2017 and many things have happened in the recent months. First of all, and most importantly, I would like to **recall EUSFLAT 2017** conference `http://www.eusflat2017.ibspan.waw.pl/` held in **Warsaw in September**. It was a jubilee 10th conference of our Society and I am personally very happy, that it was organized by our friends from the **System Research Institute of the Polish Academy of Sciences**, many thanks for the great job has to be expressed to local organizers, **to Janusz Kacprzyk, to Sławomir Zadrożny, to Eulalia Szmidt and to many others**. I have been participating on all EUSFLAT events since 2005 in Barcelona and I have took a big part of the organization duties related to EUSFLAT 2007 in Ostrava so, I know the job behind, I know those millions invisible tasks that have to be done, all the pressure and I also do have a comparison and I can frankly say, that the efforts made in Warsaw led to an unforgettable experience with a huge amount of services and a friendly atmosphere that could compete with any of the previous EUSFLAT conferences. If there is something that could be called a "flagship" of the Society, it is undoubtedly the EUSFLAT conference and we have to grateful to the local organizers for this privilege. I am sure, that organizing such fantastic events with significantly cheaper price for members is the best way how to attract members that will benefit from their membership.

EUSFLAT 2017 was important also because of another reason. As usually it was an event on which EUSFLAT officially delivered several awards. Namely, **Janusz Kacprzyk** obtained Honorary membership, **Bernadette Bouchon-Meunier** and **Radko Mesiar** both obtained EUSLAT Scientific Excellence Awards, **Amanda Vidal** Wandelmer obtained EUSFLAT 2015 Best PhD Award and **Alejandro Ramos Soto** obtained EUSFLAT 2016 Best PhD Award, and finally, **Laura De Miguel** obtained Best Student Paper Award for her conference paper presented in Warsaw. Congratulations to all awardies.

When recalling the awards delivered during the EUSFLAT 2017, it makes me really proud observing how significant the contribution of EUSFLAT members to the world science is. This fact is often confirmed by other awards distributed to EUSFLAT members by organizations different from EUSFLAT. From the recent valuable records, allow me to recall, for example, **HAFSA (Hispanic-American Fuzzy Systems Association) Pioneering Awards**. These awards are given for "pioneering contributions to the areas of Fuzzy Logic and Systems" and for helping to "promote the advancement of these areas in Mexico and Hispanic America via participation in the activities of HAFSA". This year, five people received this award, and two out of these awardies were EUSFLAT members, namely **Janusz Kacprzyk**, our Honorary members, and **Vladik Kreinovich**, our vice-president. Congratulations.

Another crucial event organized under the wings of EUSFLAT in 2017 was the **European Summer School on Fuzzy Logic and Applications SFLA 2017** `https://eventos.citius.usc.es/sfla2017/index` that was organized in **Santiago de Compostela by Centro Singular de Investigación en Tecnoloxías da Información (CiTIUS), University of Santiago de Compostela**, and I am very proud of the job made by local organizers, especially **José María Alonso** and **Félix Díaz Hermida**. I very appreciate their efforts that led to a fabulous summer school with very positive feedback from participants and we owe the organizers our gratitude for overtaking the hard job of organizing this beautiful yet very recent event. The SFLA summer school is one of our very recent initiatives focusing on attracting new young members and on supporting students and postdocs. And I am very happy that this idea, introduced by the previous President Gabriella Pasi, keeps is fruitful and successful existence, which is a great promise for the future. In **2018**, the SFLA event will be again organized as a separate self-content event and it is my great pleasure to uncover the very first details. The location is **Bari**, Italy, the general chair is **Corrado Mencar**, the quality and the attractiveness of the event will be, as always, impressive, and it is only up to you to do your best not to miss or, not let your students to miss such an opportunity. Stay with us and follow the updates.

Last, but not least, allow me to point your interest also to EUSFLAT 2019 Conference http://eusflat2019.cz/. I know, it is very early time, still nearly two years due to the event. But EUSFLAT biannual conference is our major event and we have to be proud of it. Already ten such events have been organized, and all of them were great. This is a huge obligation we have to meet in the future. Therefore, there is no time to rest on our laurels, it is time to follow the message from Warsaw and from other previous EUSFLAT conference organizers. Follow our Society websites http://eusflat.org/, follow the websites of the EUSFLAT 2019 Conference http://eusflat2019.cz/, follow our Facebook page https://www.facebook.com/EUSFLAT/, we will be doing our best and we will be keeping you informed.

Martin Štěpnička
President of EUSFLAT

RECOGNITION

Obituary for a visionary scientist: Lotfi Aliasker Zadeh (1921-2017)

Rudolf Seising

Lotfi A. Zadeh, the founder of the theory of fuzzy sets, passed away on 6th September 2017. “Soft computing” is an umbrella term Zadeh coined to denote “a collection of methodologies...Its principal constituents are fuzzy logic, neurocomputing, and probabilistic reasoning...with the latter subsuming genetic algorithms, belief networks, chaotic systems, and parts of learning theory”[1].

Lotfi A. Zadeh, undated photo around 1950, Photo credit: Fuzzy Archive Rudolf Seising.

We had known each other since the beginning of the 21st century, when I was planning to write a science-historical book on the genesis and development of fuzzy set theory and its first applications[2]. Later I continued writing historical papers on allied fields[3]. In various face-to-face interviews -the first took place during the Zittau East-West Fuzzy Colloquium in 1999 and the last began in 2012, but we continued our correspondences via e-mail until they were published in 2013[4]- I became familiar with his way of thinking. Since the year 2000, I visited Lotfi Zadeh many times to talk with him in his office in Soda Hall at UCB, in Asian restaurants around Euclid and Hearst Avenue, and later in his house in Mendocino Avenue over tea. In the last decade, he allowed me to copy or scan any documents in his office and house that seemed to be important for my historical research work, and I have exhibited this “virtual collection” as posters at conferences in recent years[5].

I travelled again to Berkeley to see him on 2nd September of this year, my previous visit having been in February 2017. I was working in his house the day before he died; and, two days later, during the EECS DepartmentâĂŹs commemoration, his son gave me permission to continue scanning documents, letters, photographs, etc. There was so much historical material to survey, but I had to leave the US a few days later. Hopefully, my digital collection contains the most important sources, but there may have been papers, photographs, movies, etc. that I did not see. We will remain unaware of what escaped me; hence, the historiography of Lotfi Zadeh’s life and work will perhaps remain fuzzy!

Fuzzy Mathware

“Math” stands for mathematics and “ware” means products, goods, and methods. Thus, “mathware” – the first word in the title of this journal – means something like mathematical products, goods and methods. In 1999, when Lotfi Zadeh gave me the first interview, he said, “I was always interested in mathematics, even when I was in Iran, in Teheran, but I was not sufficiently interested to become a pure mathematician. In other words, I never felt that I should pursue pure mathematics or even applied mathematics. So, this mixture of an engineer was perfectly suited for me. So, essentially, I’m sort of a mathematical engineer; that’s the way I would characterize myself. But I’m not a mathematician. I was somewhat critical of the fact that mathematics has gone away from the real world. ...I criticized the fact that mathematics has gone too far away from the real world.”[6]

To my mind, Lotfi Zadeh was a creator of mathware, especially of the fuzzy mathware. He invented it in the mid 1960s as the concept of “imprecisely defined «classes»”, which “play

[1]Zadeh LA, Soft computing and fuzzy logic, IEEE Software 11(6), 1994, 48–56:48

[2]Seising R, The Fuzzification of Systems. The Genesis of Fuzzy Set Theory and Its Initial Applications – Its Development to the 1970s, Berlin [et al.]: Springer (Studies in Fuzziness and Soft Computing 216) 2007

[3]See e.g., Seising R, From Vagueness in Medical Thought to the Foundations of Fuzzy Reasoning in Medical Diagnosis. *Artificial Intelligence in Medicine* 38, 2006, 237-256; – On the Absence of Strict Boundaries – Vagueness, Haziness, and Fuzziness in Philosophy, Medicine, and Science, Applied Soft Computing 8(3), 2008, 1232-1242; – Cybernetics, System(s) Theory, Information Theory and Fuzzy Sets and Systems in the 1950s and 1960, Information Sciences, 180, 2010, 459-476; – When Computer Science emerged and Fuzzy Sets appeared. The Contributions of Lotfi A. Zadeh and Other Pioneers, IEEE Systems, Man, and Cybernetics Magazine, July 2015.

[4]On Fuzzy Sets and the Precisiation of Meaning. An Interview with Prof. Dr. Lotfi A. Zadeh, University of California at Berkeley, USA, Archives of the History and Philosophy of Soft Computing, 1, 2013, 1-18. http://aphsc.org/index.php/aphsc/article/view/1

[5]“50 Years ago: The Genesis of Fuzzy Sets”, poster exhibition at the 6th World Congress of the International Fuzzy Systems Association (IFSA) and the 9th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT) in Gijon, Spanien, June 30. Juni – July 3, 2015. Enlarged for the 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2015), Kadir Has University, Istanbul, Turkey, August 3-4, 2015. Further enlarged for the 2016 Conference “Glorification of Professor Zadeh”, University of Tehran, Iran, March 8, 2016

[6]Zadeh LA in an Interview with Rudolf Seising on September 8, 1999 in Zittau, at the margin of the 7th Zittau Fuzzy Colloquium at the University Zittau/Görlitz.

an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction”.[7] Next to abstraction we can list the domain of “generalization” because Zadeh wrote in a “memorandum” one month earlier that these “two basic operations: abstraction and generalization appear under various guises in most of the schemes employed for classifying patterns into a finite number of categories.”[8] He completed his argument as follows: “Although abstraction and generalization can be defined in terms of operations on sets of patterns, a more natural as well as more general framework for dealing with these concepts can be constructed around the notion of a “fuzzy” set – a notion which extends the concept of membership in a set to situations in which there are many, possibly a continuum of, grades of membership.”[9] In a later article, he wrote that these “fuzzy sets”, as he named these “classes”, “do not constitute classes or sets in the usual mathematical sense of these terms”.[10]

In that paper, Zadeh presented the framework for the theory of fuzzy sets. He defined fuzzy sets, empty fuzzy sets, equal fuzzy sets, the complement and the containment of a fuzzy set. He also defined the union and intersection of fuzzy sets as the fuzzy sets that have membership functions that are the maximum or minimum, resp., of their membership values. He gave a historically interesting interpretation of the union and intersection of fuzzy sets that makes explicit that his concept of fuzzy sets arose from the conventional synthesis techniques for switching circuits as a “network of sieves $S_1(x), \dots, S_n(x)$“.[11] He proved that the distributivity laws and De Morgan’s laws are valid for fuzzy sets with these definitions of union and intersection. In addition, he defined other ways of forming combinations of fuzzy sets and relating them to one another, such as, the *algebraic sum*, the *absolute difference* and the *convex combination* of fuzzy sets. After a discussion of two definitions of *convexity* for fuzzy sets and the definition of *bounded* fuzzy sets, he defined *strictly* and *strongly convex* fuzzy sets. Finally, he proved the separation theorem for bounded convex fuzzy sets, which was relevant to the solution of the problem of pattern discrimination and classification that he perhaps[12] presented in the previous summer when he was invited to talk on pattern recognition in the Wright–Patterson Air Force Base, Dayton, Ohio: “For example, suppose that we are concerned with devising a test for differentiating between handwritten letters O and D. One approach to this problem would be to give a set of handwritten letters and indicate their grades of membership in the fuzzy sets O and D. On performing abstraction on these samples, one obtains the estimates $\tilde{\mu}_O$ and $\tilde{\mu}_D$ of μ_O and μ_D, respectively. Then given a letter x that is not one of the given samples, one can calculate its grades of membership in O and D, and, if O and D have no overlap, classify x in O or D.”[13]

In this summer of 1964, Zadeh and his close friend at the RAND Corporation Richard E. Bellman planned to do some research together. Before that there was the trip to Dayton, Ohio, but, at any rate, within a short space of time, he had further developed his little theory of “gradual membership” into an appropriately modified set theory: “Essentially the whole thing, let’s walk this way, it didn’t take me more than two, three, four weeks, it was not long.”[14] When he finally met with Bellman in Santa Monica, he had already worked out the entire theoretical basis for his theory of fuzzy sets: “His immediate reaction was highly encouraging and he has been my strong supporter and a source of inspiration ever since.”[15]

In 1943, Lotfi Aliaker Zadeh, already qualified as a Bachelor of Electrical Engineering, boarded a ship bound for Cairo, Egypt, where he had to wait some months for the necessary documents to emigrate to the United States of America. He was born in 1921 in Baku, the capital of Azerbaijan, as the son of Rasim Aleasgarzadeh, an Azerbaijani with Iranian roots, who worked as a foreign correspondent in Iran, and Fanya Korenmann, a Russian paediatrician born in Odessa. The family chose to move back to Tehran in 1931 due to Stalin’s harsh immigration policies.

Lotfi A. Zadeh, 18 years old in his room, Photo credit: Fuzzy Archive Rudolf Seising.

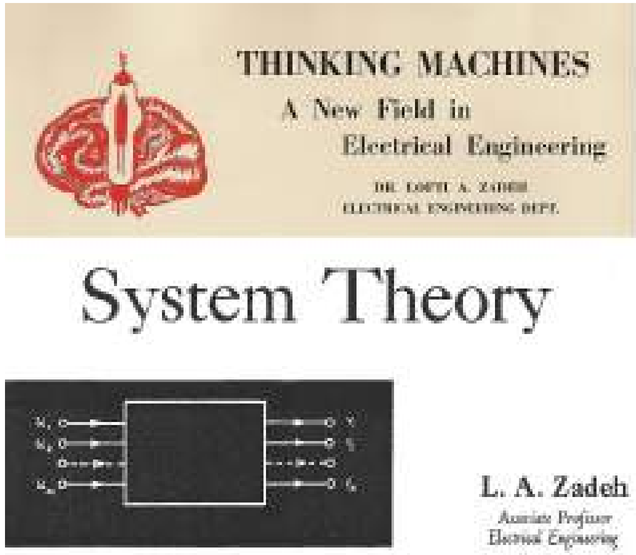
Until 1939 Lotfi Zadeh continued his education at a private Presbyterian missionary school run by the USA. He studied electrical engineering at the University of Tehran, and the University awarded him the BSc. degree in 1942. The following year, he emigrated to the USA. For some time, he worked for the International Electronic Laboratories in New

[7] Zadeh LA, Fuzzy sets, Information and Control, 8(3), June 1965, 338–353:338.
[8] Bellman R, Kalaba R, Zadeh LA, Abstraction and Pattern Classification, Memorandum RM–4307-PR, United States Air Force Project Rand, The Rand Corporation, Santa Monica, California, October 1964, 1.
[9] Bellman R, Kalaba R, Zadeh LA, Abstraction and Pattern Classification, 1.
[10] Zadeh LA, Fuzzy sets, Information and Control, 8(3), June 1965, 338-353:338.
[11] Zadeh LA, Fuzzy sets, 343
[12] Neither a manuscript nor any other sources exist. Zadeh did not want to either confirm or rule out this detail in my interviews in 1999, 2001 and 2002
[13] Bellman R, Kalaba R, Zadeh LA, Abstraction and Pattern Classification, 30.
[14] Zadeh LA in an Interview with the author on June 19, 2001, UC Berkeley, Soda Hall.
[15] Zadeh LA “Autobiographical Note 1” – undated two-page typewritten manuscript, written after 1978.

York. In 1944, he went to Boston to continue his studies at the Massachusetts Institute of Technology (MIT). Amongst others, he attended the lectures of Norbert Wiener (1894-1964) and Ernst A. Guillemin (1898-1970). Two years later, in 1946, Zadeh had earned a Masters of Science degree. His thesis was "An Investigation of Current Distribution and Radiation Field of a Solenoidal Antenna", which he began writing under the supervision of Robert Fano (1917-2016) but finished with Parry Hiram Moon (1898-1988). He had planned to study for his doctorate at MIT, but because his parents had also immigrated to New York in 1945, he changed to the Columbia University in New York. He acquired a position as an instructor in the Department of Electrical Engineering and was responsible for teaching the theories of circuits and electromagnetism. He was awarded a PhD for his thesis "Frequency Analysis of Variable Networks".[16] In 1950, he was appointed assistant professor. He was extremely excited by Wiener's cybernetics[17], and he became interested in the theory of ideal and optimal filtering. Together with his supervisor John Ralph Ragazzini (1912-1988) he published some joint research work[18]. One of their papers led to the widely-used method of Z-transformation[19], and another paper "An Extension of Wiener's Theory of Prediction"[20], resulting from project work he had published on his own, was an important milestone in the development of network synthesis.

Lotfi A. Zadeh, undated photo, around 1950, Photo credit: Fuzzy Archive Rudolf Seising.

In 1949, when he received his PhD, he turned his attention to other problems. In addition to Wiener's lectures on cybernetics, Zadeh had also attended a lecture by Claude E. Shannon in New York in 1946, two years before Shannon's Mathematical Theory of Communication appeared[21]. The new era of digital computers had started in World War II with the ENIAC (Electronic Numerical Integrator And Computer, dedicated on 15th February 1946), and it continued after the War with the EDVAC (Electronic Discrete Variable Computer, delivered 1949). Both computers were designed by John P. Eckert and John W. Mauchly, and in 1950 Zadeh wrote an article entitled "Thinking Machines – A New Field in Electrical Engineering" for the Columbia Engineering Quarterly.[22]



Zadeh's two Articles' headlines in the Columbia Engineering Quarterly in the 1950s. Photo credit: Fuzzy Archive Rudolf Seising.

In the same year Alan M. Turing published his famous Mind-article "Computing Machinery and Intelligence", in which he proposed the "Imitation game", later named "Turing Test."[23] Unaware of Turing's article in that philosophical journal, Zadeh was interested in "the principles and organization of machines which behave like a human brain." Such machines were then variously referred to as "thinking machines", "electronic brains", "thinking robots", and similar names. He mentioned that the "same names are frequently ascribed to devices which are not «thinking machines» in the sense used in this article"; he therefore made the following distinction: "The distinguishing characteristic of thinking machines is the ability to make logical decisions and to follow these, if necessary, by executive action."[24]

Before the advent of the era of digital computers that was the beginning of today's digitization, there was a period in the 1950s, in which electrical engineering was dominated by a rising discipline called system theory. Zadeh's research was also concerned with system theory, which was devoted "to the study of systems per se, regardless of their physical structure". His first article on this subject, published in 1954 again in the Columbia Engineering Quarterly, was headlined

[16] Zadeh LA, Frequency Analysis of Variable networks, Proceedings of the IRE, 38, March 1950, 291-198

[17] Wiener N, Cybernetics or Control and Communications in the Animal and the Machine, Cambridge, Massachusetts: MIT Press, 1948.

[18] Ragazzini, JR, Zadeh LA, Probability criterion for the design of servomechanisms, Journal of Applied Physics, 20, 1949,141-144; – A wide band audio phasemeter, Review of Scientific Instruments, 21(2), 1950, 145-148.

[19] Ragazzini, JR, Zadeh LA, The analysis of sampled-data systems, Applications and Industry (AIEE) 1, 1952, 224–234.

[20] Zadeh LA, An Extension of Wiener's Theory of Prediction, Research and Development Report âĂŞ SPD 243, The M W Kellogg Company, Special Projects Department, Jersey City, New Jersey, August 1, 1949; Zadeh LA, Ragazzini JR, An Extension of Wiener's Theory of Prediction, Journal of Applied Physics, 21, 1950, S. 645-655.

[21] Shannon CE, The Mathematical Theory of Communication, Bell System Technical Journal 27(3 & 4) 1948, 379-423 & 623-656.

[22] Zadeh LA, Thinking Machines – A New Field in Electrical Engineering, Columbia Engineering Quarterly, Jan. 1950, 12-13, 30–31.

[23] Turing AM, Computing Machinery and Intelligence, Mind, LIX (236), October 1950, 433-460: 433.

[24] Zadeh LA, Thinking Machines, 12

"System Theory". The illustration of the headline characterized systems as "black boxes" with inputs and outputs, and the dynamic behaviour of the system can be studied mathematically as the systemâĂŹs input-output-relationship if the inputs and outputs are describable as time dependent functions.[25]

Eight years later, in May 1962, Zadeh contributed the paper "From Circuit Theory to System Theory" to the anniversary edition of the Proceedings of the IRE to mark the 50th anniversary of the Institute of Radio Engineers (IRE). In it we find the following famous paragraph that motivated his later creation of fuzzy set theory: "In fact, there is a fairly wide gap between what might be regarded as «animate» system theorists and «inanimate» system theorists at the present time, and it is not at all certain that this gap will be narrowed, much less closed, in the near future. There are some who feel that this gap reflects the fundamental inadequacy of the conventional mathematics – the mathematics of precisely defined points, functions, sets, probability measures, etc.– for coping with the analysis of biological systems, and that to deal effectively with such systems, which are generally orders of magnitude more complex than man-made systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions."[26] About two or three years later, he tried to bridge this gap by introducing the new mathematical theory of fuzzy sets. [27]

In 1959 he had changed from Columbia University to the University of California, Berkeley, where he became professor in the Department of Electrical Engineering. With two colleagues, he published two well-known books: *Linear System Theory* together with Charles A. Desoer (1926-2010) in 1963,[28] and *System Theory* with Elijah Polak (born 1931).[29] His own contribution in the latter book was "The Concept of State in System Theory".[30] This general notion of state was the "new view" on system theory that Zadeh also presented in April 1963 during the Second Systems Symposium at the Case Institute of Technology in Cleveland, Ohio. In 1965 he offered the electrical engineering community a further "new view" on system theory. Within the theory of fuzzy sets, he was able to start establishing a theory of fuzzy systems: Given a system S with input $u(t)$, output $y(t)$ and state $x(t)$, it is a fuzzy system if input or output or state or any combination of them includes fuzzy sets, and, in Zadeh's view, only fuzzy systems are can adequately cope with complex man–made systems and living systems.

In 1963 he had become Chairman of the Department of Electrical Engineering (EE) at Berkeley, and he recalled these years as follows: "System theory became grown up but then computers came along, and computers then took over. In other words, the center of attention shifted. [...] So, before that, there were some universities that started departments of system sciences, departments of system engineering, something like that, but then they all went down. They all went down because computer science took over."[31]

During Zadeh's chairmanship, the Department was renamed to Department of Electrical Engineering and Computer Science (EECS).[32] "There wasn't much activity in the computer field at that time, but there was significant", he recalled many years later, "there was a Computer Center in Cory Hall that was run by the EE Department. The principal and only figures in computer science and engineering in EE at that time were Paul Morton and Harry Huskey. They can be rightly regarded as the progenitors of computer science and engineering at Berkeley. [...] When I was appointed as Chairman in 1963, I was not a computer person and I am not a computer user to this day, I regret to say. But I was always a very strong believer in the importance of computers and digital technology. My first action as Chairman was to send a memo to the faculty in which I suggested that we assign the highest priority to the development of computer science in EE. But what is obvious today was not so obvious then. The reaction to my memo was mixed and some influential faculty members objected strongly to my proposal."[33] However, finally Zadeh was successful in changing the name of the Department to EECS.[34]

In an article, he presented the new EE curriculum at Berkeley that "reflects the fact that, today (in the mid-1960s), electrical engineering is no longer an aggregation of a small number of subject areas sharing a large common body of concepts and techniques –as it was in the thirties, forties, and to a lesser extent, in the fifties. Rather, it is an assemblage of a wide range of subjects, falling into three major areas which have a relatively small common core. [...] If this premise is accepted, then the only logical conclusion is that the student must be provided with a choice of several basic programs, which could permit him to focus his studies in one of the major areas falling within or nearest to his main field of interest."[35]

Zadeh was very engaged in the development of education in the new discipline of computer science. He was responsible for bringing about the "Berkeley solution", which ultimately led to the formation of a department for computer science in the College of Letters and Science and a programme in computer science in the College of Engineering within his Department.[36] It was during this period that he submitted

[25] Zadeh LA, System Theory, Columbia Engineering Quarterly, Nov. 1954, 16-19, and 34

[26] Zadeh LA, From Circuit Theory to System Theory, Proceedings of the IRE, 50(5), 1962, 856âĂŞ865

[27] Zadeh LA, Fuzzy Sets, Information and Control, 8, 1965, 338-353.

[28] Zadeh LA, Desoer ChA, Linear System Theory: The State Space Approach, New York; San Francisco, Toronto, London: McGraw–Hill Book Company 1963.

[29] Zadeh LA, Polak E (eds.), System Theory, Bombay and New Delhi: McGraw–Hill 1969.

[30] Zadeh LA, The Concept of State in System Theory, in: Zadeh LA, Polak E. (eds.), System Theory, Bombay and New Delhi: McGraw-Hill 1969, 9-42.

[31] Zadeh LA in an interview with the author on July 26, 2000, UC Berkeley, Soda Hall

[32] Zadeh LA in an interview with the author on July 26, 2000, UC Berkeley, Soda Hall.

[33] Zadeh LA, History of Computer Science at Berkeley, Manuscript from Nov. 1, 1992, Fuzzy Archive Rudolf Seising, p. 2.

[34] For more details in the change of the department's name, see: Seising R, When Computer Science emerged and Fuzzy Sets appeared. The Contributions of Lotfi A. Zadeh and Other Pioneers, IEEE Systems, Man, and Cybernetics Magazine, July 2015

[35] Zadeh LA, Electrical Engineering at the Crossroads, 31.

[36] For more details on these aspects see: R. Seising, When Computer Science emerged and Fuzzy Sets appeared. The Contributions of Lotfi A. Zadeh and Other Pioneers, IEEE Systems, Man, and Cybernetics Magazine, 4, 2016.

his seminal paper "Fuzzy Sets" to the journal *Information and Control*[37] and in the following paragraphs we will see how Zadeh's reflections on and activities in the scientific discipline of computer science and his theory of fuzzy sets were interlinked.

Lotfi A. Zadeh as a young professor, around 1960. Photo credit: Fuzzy Archive Rudolf Seising .

In 1968, he gave a talk on "Education in Computer Science" at Israel's 4th National Conference on Data Processing that took place in the Hebrew University, Jerusalem. Initially, he claimed that computer science is "a collection of concepts and techniques which serve to systematize the employment of the means with which modern technology provides us for purposes of stage, representation and processing of information."[38] He affirmed that "computer science cuts across the boundaries of many established fields. It is glamorous; it draws a large number of students – many of them from other departments; it is hitched to the bandwagon of computers and the information revolution."[39] In this paper he emphasized "the main premise of Berkeley's «solution» [...] that computer science is not a homogeneous and unified field –at least not at this time– and that, in paraphrased words of Professor A. Oettinger of Harvard, "it has some components which are the purest of mathematics and some that are the dirtiest of engineering." This split personality of computer science makes it very difficult to create a single academic unit within the university structure where mathematically oriented automata theorists, formal language experts, numerical analysts and logicians could establish a comfortable modus vivendi with non–mathematical oriented hardware designers, systems programmers and computer architects. [...] In essence, the Berkeley "solution" provides a partial answer to the dilemma by dividing computer science not into two non-overlapping parts but into two overlapping parts which differ from one another mainly in the degrees of emphasis each places on various subject areas."[40]

In another article, "Computer Science as a Discipline", which appeared in the same month, he brought into focus that CS "cuts across the boundaries of many established fields" and that the parts of CS differ from one another "in degrees of emphasis". Here he linked his reflection on CS education with fuzzy sets: "Specifically, let us regard computer science as a name for a fuzzy set of subjects and attempt to concretize its meaning by associating with various subjects their respective degrees of containment (ranging from 0 to 1) in the fuzzy set of computer science. For example, a subject such as «programming languages» which plays a central role in computer science will have a degree of containment equal to unity. On the other hand, a peripheral subject such as «mathematical logic» will have a degree of containment of, say, 0.6."[41]

TABLE 1
Containment Table for Computer Science

Subject	Degree of Containment in Computer Science
Programming languages	1
Computer design and organization	1
Data structures	1
Models of computation	1
Operating systems	1
Programming systems	1
Formal languages and grammars	0.9
Computational linguistics	0.8
Automata theory	0.8
Finite-state systems	0.8
Theory of algorithms	0.9
Discrete mathematics	0.8
Mathematical logic	0.6
Combinatorics and graph theory	0.8
Dynamic programming	0.7
Mathematical programming	0.7
Numerical methods	0.8
Switching theory	0.8
Analog and hybrid computers	0.7
Computer graphics	0.7
Digital devices and circuits	0.7
Artificial intelligence and heuristic programming	0.9
Information retrieval	0.7
Information theory and coding	0.6
Pattern recognition and learning systems	0.6

Elements of the fuzzy set "computer science" and their grades of membership. Photo credit: Fuzzy Archive Rudolf Seising.

Soft Computing

Under the heading "Containment Table for Computer Science", he arranged the most relevant "subjects in question and their degrees of containment in computer science". He explained: "Clearly, such numerical values of degrees of

[37] L. A. Zadeh, Fuzzy Sets, Information and Control, 8, 1965, pp. 338-353.
[38] Zadeh LA, Education in Computer Science, Proceedings of the National Conference on Data Processing, Jerusalem, Information Processing Association of Israel, 1968, E157-E167:E157.
[39] Zadeh LA, Education in Computer Science, E158.
[40] Zadeh LA, Education in Computer Science, E164f. Citation of Oettinger in: Oettinger AG, The Harvard-Software-Complementarity, Communications of the ACM, vol. 10, October 1967, pp. 604–606.
[41] Zadeh LA, Computer Science as a Discipline, Journal of Engineering Education, 58(8), 1968, 913-916: 913.
[42] Zadeh LA, Computer Science as a Discipline, 913.

containment represent merely this writer's subjective assessment, expressed in quantitative terms, of the current consensus regarding the degrees of inclusion of various subjects in computer science."[42] He also emphasized "that a high degree of containment of a particular subject in computer science does not imply that it cannot have a similar high or even higher grade of containment in some other field. For example «automata theory» has the degree of containment of 0.8 in computer science; it also has the same, or nearly the same, degree of containment in system theory. Also, the subjects listed in the table may have substantial overlaps with one another. This is true, for example, of «automata theory» and «finite state systems»."[43]

In 1969, Zadeh proposed his new theory of fuzzy sets to biologists: "The great complexity of biological systems may well prove to be an insuperable block to the achievement of a significant measure of success in the application of conventional mathematical techniques to the analysis of systems. ... By «conventional mathematical techniques» in this statement, we mean mathematical approaches for which we expect precise answers to well-chosen precise questions concerning a biological system that has a high degree of relevance to its observed behaviour. Indeed, the complexity of biological systems may force us to alter in radical ways our traditional approaches to the analysis of such systems. Thus, we may have to accept as unavoidable a substantial degree of fuzziness in the description of the behaviour of biological systems as well as in their characterization."[44]

We find great complexity not only in biological systems but also in the social sciences and humanities. At the end of the 1960s and for a greater audience two years later, Zadeh wrote more generally: "What we still lack, and lack rather acutely, are methods for dealing with systems which are too complex or too ill–defined to admit of precise analysis. Such systems pervade life sciences, social sciences, philosophy, economics, psychology and many other 'soft' fields."[45] When asked in an interview in 1994, "how did you think fuzzy logic would be used at first?", he answered, "I expected people in the social sciences, economics, psychology, philosophy, linguistics, politics, sociology, religion and numerous other areas to pick up on it. It's been somewhat of a mystery to me why even to this day, so few social scientists have discovered how useful it could be. Instead, fuzzy logic was first embraced by engineers and used in industrial process controls and in 'smart' consumer products such as hand-held camcorders that cancel out jittering and microwaves that cook your food perfectly at the touch of a single button. I didn't expect it to play out this way back in 1965."[46]

Lotfi Zadeh was inspired by the "remarkable human capability to perform a wide variety of physical and mental tasks without any measurements and any computations", e.g. parking a car, playing golf, deciphering sloppy handwriting, and summarizing a story. He distinguished between mechanical (or inanimate or man-made) systems, on the one hand, and humanistic systems, on the other hand, and he saw the following state of the art in computer technology: "Unquestionably, computers have proved to be highly effective in dealing with mechanistic systems, that is, with inanimate systems whose behavior is governed by the laws of mechanics, physics, chemistry and electromagnetism. Unfortunately, the same cannot be said about humanistic systems, which âĂŞ so far at least âĂŞ have proved to be rather impervious to mathematical analysis and computer simulation." He defined a "humanistic system"[47] to be "a system whose behaviour is strongly influenced by human judgment, perception or emotions. Examples of humanistic systems are: economic systems, political systems, legal systems, educational systems, etc. A single individual and his thought processes may also be viewed as a humanistic system." Zadeh summarized "that the use of computers has not shed much light on the basic issues arising in philosophy, literature, law, politics, sociology and other human oriented fields. Nor have computers added significantly to our understanding of human thought processes - excepting, perhaps, some examples to the contrary that can be drawn from artificial intelligence and related fields."[48]

In 1978, when the first issue of the *International Journal of Fuzzy Sets and Systems* was launched, he wrote in the editorial that "it has become increasingly clear [that] classical mathematics – based as it is on set theory and two-valued logic – is much too restrictive and much too rigid to serve as an effective tool for the understanding of the behavior of humanistic systems, that is, systems in which human judgement, perceptions and emotions play an important role."[49]

After 17 years of a very successful circulation of this journal, Hans Jürgen Zimmermann, who was still Editor in Chief at the time, foresaw in an editorial that the development of hybrid systems which combined fuzzy sets with neurocomputing, evolutionary computing, probabilistic computing and other methodologies would continue in the future. Consequently, he deliberated over a suitable name for the common field of research, which could then also become the subtitle of Fuzzy Sets and Systems: "Soft computing, biological computing and computational intelligence have been suggested so far." These terms were attractive in different ways and varied with respect to their expressive power. Zimmermann suggested calling the field – and thus also the new subtitle of the journal – "soft computing and intelligence" since the other concepts seemed to place too much emphasis on "computing", "which is certainly not appropriate at least for certain areas of fuzzy set theory." The name "soft computing and intelligence" is more appropriate than "artificial intelligence," but both have in common the word "intelligence," which Zimmermann found defined in a dictionary as follows:

[43] Zadeh LA, Computer Science as a Discipline, 914.

[44] Zadeh LA, Biological Application of the Theory of Fuzzy Sets and Systems, in: Proctor LD (ed.), Proceedings of the International Symposium on Biocybernetics of the Central Nervous System, 199âĂŞ206. Little, Brown and Comp., London (1969).

[45] Zadeh LA, Towards a theory of fuzzy systems, in: Kalman RE, DeClaris N (eds.), Aspects of Network and System Theory, 469âĂŞ490. Holt, Rinehart and Winston, New York 1971.

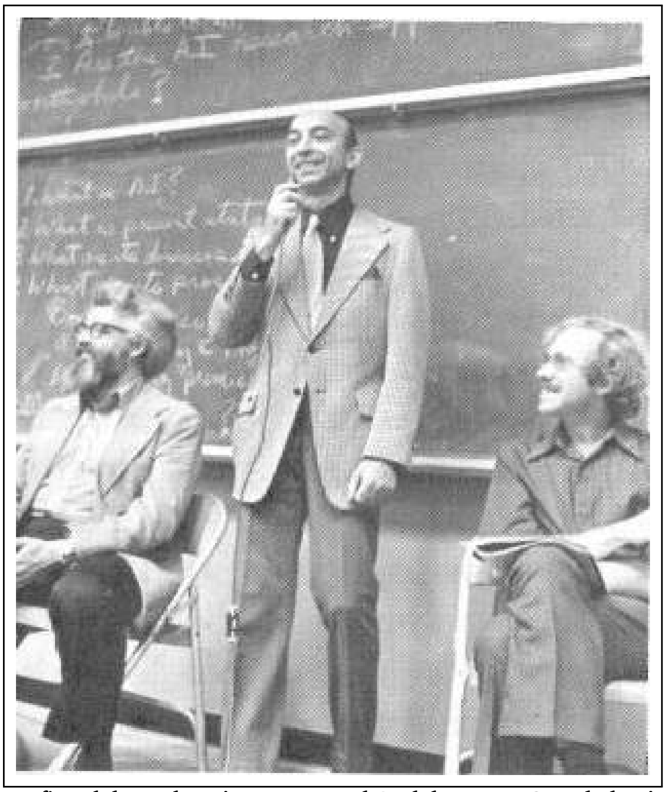
[46] Blair B, Lotfi Zadeh, creator of fuzzy logic, Interview by Betty Blair, Azerbaijada International 2(4), 1994

[47] Zadeh LA, The concept of a linguistic variable and its application to approximate reasoningâĂŤI, Information Sciences, 8, 1975, 199-249: 200

[48] Zadeh LA, The concept of a linguistic variable and its application to approximate reasoning, 200

[49] Zadeh LA, An Editorial Perspective, International Journal for Fuzzy Sets and Systems, 1(1), 1978, 1; founding co-editors of the then new journal were C. V. Negoita and H.-J. Zimmermann. Zadeh formulated the draft that has not been changed.

"Capacity for reasoning, understanding and for similar forms of mental activity." This was exactly what the editors of this journal had considered to be central to fuzzy set theory in the first issue.[50] Thus, since the first issue of 1995 *Fuzzy Sets and Systems* has appeared with the subtitle *International Journal for Soft Computing and Intelligence.*



Lotfi Zadeh moderating an annual AI debate at UC Berkeley in the 1980s; fltr. John McCarthy, Lotfi A. Zadeh, Hubert Dreyfus. Photo credit: Fuzzy Archive Rudolf Seising.

"The concept of soft computing crystallized in my mind during the waning months of 1990",[51] wrote Lotfi Zadeh in a retrospective foreword to the journal *Applied Soft Computing* founded in 1990. In 1994 in Barcelona at the *Universitat Politècnica de Catalunya* the journal *Mathware & Soft Computing* was launched. Founding editor-in-chief, Joan Jacas wrote in his editorial that "we have witnessed the growth and expansion of a wide type of computing techniques in fields like Logic, Approximate Reasoning, Applied Functional Equations, Possibility Theory, etc. that are included in the term introduced by Prof. L. A. Zadeh as Soft Computing. Therefore, it is time to open a new forum where relevant theoretical contributions to the fields mentioned could be presented. In this sense, Mathware and Soft Computing will basically include theoretical contributions that use mathematical tools and models that could be relevant in applications for Cognitive Sciences, pure or applied Logic and Artificial Intelligence."[52] For the first issue there was already financial support given by the Spanish Association for Logic and Fuzzy Technologies (FLAT), which later became the European Society for Fuzzy Logic and Technology (EUSFLAT), and, from 1998 to 2010, it was the official journal of EUSFLAT.[53] Since 2011 the *International Journal of Computational Intelligence Systems* became the official EUSFLAT-journal, whereas *Mathware & Soft Computing* moved to be the Magazine of EUSFLAT.

: Lotfi Zadeh with the Scientific Committee of the ECSC on April 27, 2006; fltr: Henri Prade, Enric Trillas, Lotfi A. Zadeh, Rudolf Kruse, Janusz Kacprzyk, Abe Mamdani, Luis Magdalena, Crister Carlson, and Maria Angeles Gil, Photo credit: Fuzzy Archive Rudolf Seising.

A decade earlier, in 2001, the Berkeley Initiative in Soft Computing (BISC) had already been launched, and, five years on, the *Foundation for the Advancement of Soft Computing* started the *European Centre for Soft Computing* (ECSC) in Mieres, Asturias, Spain.

On 19th May 2009, Zadeh repeated his former suggestion made eight years earlier in the Foreword,[54] already cited above, to incorporate the research field of Soft Computing to the BISC mailing list: "As we move further into the age of intelligent systems, the problems that we are faced with become more complex and harder to solve. To address the problems, we have an array of methodologies – principally fuzzy logic, neurocomputing, evolutionary computing and probabilistic computing. In large measure, the methodologies are complementary; and yet, there is an element of competition among them. In this setting, what makes sense is the formation of a coalition. It is this perception that motivated the genesis of soft computing – a coalition of fuzzy logic, neurocomputing, evolutionary computing, probabilistic computing and other methodologies."[55]

Today, soft computing is almost equivalent to computational intelligence, a label that was first used in 1985 by a Canadian journal.[56] The founding editorial board chose this name "to reflect the fact that AI is distinct from other studies of intelligence in its emphasis on computational models", as the editors recalled about 10 years later, and they continued: "The name was also short enough to be catchy but general

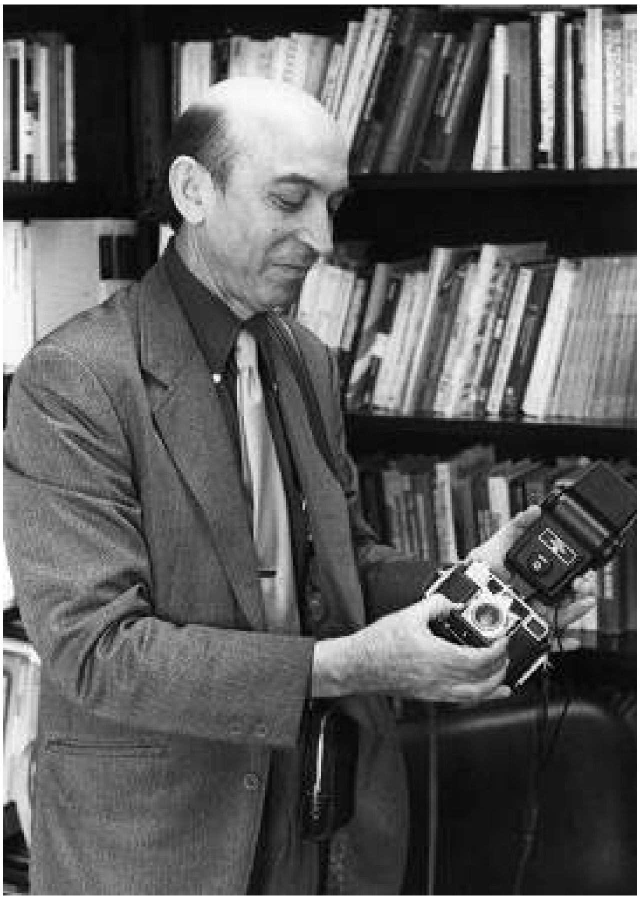
[50] Zimmermann H.-J, Editorial, Fuzzy Sets and Systems, 69(1), 1995, 1–2.
[51] Zadeh LA, Foreword. Applied Soft Computing, 1(1), June 2001, pp. 1–2.
[52] Jacaa J, Editorial, *Mathware & Soft Computing*, 1(1), 1994.
[53] The predesessor of EUSFLAT was the National Spanish Fuzzy Logic Society with the national congresses ESTYLF (Español sobre Tecnologías y Lógica Fuzzy). To open the society to members from other European countries in 1998 it became EUSFLAT.
[54] Zadeh LA, Foreword, Applied Soft Computing 1(1), June 2001, 1–2.
[55] As the "backdrop" for his suggestion, Zadeh used almost the same words that he wrote in 2001 in Ref. 5.
[56] http://eu.wiley.com/WileyCDA/WileyTitle/productCd-COIN.html

enough to reflect our purpose and attract submissions from all areas of AI." [57] The word "computational" should refer to subsymbolic problem representation, knowledge aggregation and information processing.

Advertisement for McCorduckâĂŹs book, Photo credit: Fuzzy Archive Rudolf Seising.

Computational intelligence (CI) became a collection of methods, but in the meantime there have been attempts to characterize this research area explicitly. Considering the problems it is concerned with, the computer scientist Włodziław Duch wrote in 2007: "CI studies problems for which there are no effective algorithms, either because it is not possible to formulate them or because they are NP-hard and thus not effective in real life applications!"[58] As opposed to artificial (or inanimate) systems, animate systems like living brains are able to solve problems for which there are no effective computational algorithms: "extracting meaning from perception, understanding language, solving ill-defined computational vision problems thanks to evolutionary adaption of the brain to the environment, survival in a hostile environment."[59]

Another view on CI arrives at a different relationship between AI and CI; this view emerged from James C. BezdekâĂŹs reflections in "On the Relationship between Neural Networks, Pattern Recognition and Intelligence", 1992,[60] that led him to a first definition of CI. "A system is computationally intelligent when it: deals with only numerical (low-level) data, has pattern recognition components, does not use knowledge in the AI sense; and additionally when it (begins to) exhibit 1) computational adaptivity, 2) computational fault tolerance, 3) speed approaching human-like turnaround and 4) error rates that approximate human performance." [61]



The photographer Lotfi Zadeh, Photo credit: Fuzzy Archive Rudolf Seising.

Since his introduction of fuzzy sets, Lotfi Zadeh has often compared the strategies of problem solving by computers, on the one hand, and by humans, on the other hand. In a conference paper in 1969, he called it a paradox that the human brain is always solving problems by manipulating "fuzzy concepts" and "multidimensional fuzzy sensory inputs", whereas "the computing power of the most powerful, the most sophisticated digital computer in existence" is not able to do this. Therefore, he stated, "in many instances, the solution to a problem need not be exact", so that a considerable measure of fuzziness in its formulation and results may be tolerable. The human brain is designed to take advantage of this tol-

[57] Cercone N, McCalla G: Ten Years of Computational Intelligence, Computational Intelligence, 10(4), 1994, I.

[58] Duch W, What is Computational Intelligence

[59] Duch W, What is Computational Intelligence

[60] Bezdek JC, On the Relationship between Neural Networks, Pattern Recognition and Intelligence, International Journal of Approximate Reasoning, 6, 1992, 85-107.

[61] Bezdek JC, What is Computational intelligence? In: Zurada JM, Marks II RJ, Robinson ChJ, Computational Intelligence Imitating Life, IEEE Press, 1994, 1-12.

erance for imprecision whereas a digital computer, with its need for precise data and instructions, is not."[62] He continued: "Although present-day computers are not designed to accept fuzzy data or execute fuzzy instructions, they can be programmed to do so indirectly by treating a fuzzy set as a data-type which can be encoded as an array [. . .]. Granted that this is not a fully satisfactory approach to the endowment of a computer with an ability to manipulate fuzzy concepts, it is at least a step in the direction of enhancing the ability of machines to emulate human thought processes. It is quite possible, however, that truly significant advances in artificial intelligence will have to await the development of machines that can reason in fuzzy and non-quantitative terms in much the same manner as a human being."[63]

In 1979, after a whole slew of interviews and conversations with many researchers in the field of AI, Marvin Minsky, Herbert Simon and Alan Newell and Lotfi Zadeh included, Pamela McCurdock published her book *Machines Who Think*.[64] During one of these meetings Lotfi Zadeh, who was a very good amateur photographer, took a portrait picture of McCorduck that was later placed in an advertisement for the book.

New York Times Magazine **(December 7, 1980), frontpage with the portrait of Marvin Minsky, Photo credit: Fuzzy Archive Rudolf Seising.**

At the end of the next year, Marvin MinskyâĂŹs portrait appeared on the frontpage of the *New York Times Magazine* and the headline next to him was "Creating Computers To Think Like Humans".[65]

To affiliate his own thoughts to this area of research, Lotfi Zadeh then wrote the article "Making Computers Think like People", which was printed in 1984.[66] In his view, computers–"thinking machines"–do not think like humans. For this purpose, the machine's ability "to compute with numbers" should be supplemented by an additional ability that is similar to human thinking: computing with words and perceptions.

Making computers think like people

Lotfi Zadeh's article "Making computers think like people" (1984), Photo credit: Fuzzy Archive Rudolf Seising.

In the 1990s, he sketched new theories to achieve this goal. Based on the methodology of fuzzy logic and as its "main contribution" he outlined a scheme for "Computing with Words" (CW) instead of exact computing with numbers.[67] Based on the methodology of CW, he delineated a "Computational Theory of Perceptions" (CTP) in which "words play the role of labels of perceptions and, more generally, perceptions are expressed as propositions in natural language."[68] With these approaches he intended to establish a new dimension of research in artificial intelligence.

[62]Zadeh LA, Fuzzy Languages and their Relation to Human and Machine Intelligence, in: Marois M (ed.), Man and Computer, Proceedings of the International Conference, Bordeaux June 22-26, 1970, 132.

[63]Zadeh LA, Fuzzy Languages and their Relation to Human and Machine Intelligence, 132.

[64]McCurdock P, Machines Who Think. A Personal Inquiry into the History and Prospects of Artificial Intelligence, Natick, Mass: Peters, 1979. The book appeared in 2004 in a new 25th anniversary edition.

[65]Stockton W, Creating Computers to Think, New York Times Magazine, December 7, 1980, 41.

[66]Zadeh LA, Making Computers Think like People, IEEE Spectrum, 8, 1984, 26-32.

[67]Zadeh LA, Fuzzy Logic, Neural Networks, and Soft Computing, Communications of the ACM, 37(3), 1994, 77-84; –, Fuzzy Logic = Computing with Words, IEEE Transactions on Fuzzy Systems, 4(2), 1996, 103-111.

[68]Zadeh LA, Fuzzy Logic = Computing with Words, IEEE Transactions on Fuzzy Systems, 103.

His thesis was "that progress has been, and continues to be, slow in those areas where a methodology is needed in which the objects of computation are perceptions – perceptions of time, distance, form, direction, color, shape, truth, likelihood, intend, and other attributes of physical and mental objects." The creation of a "perception-based system modelling", where the input, the output and the states are assumed to be perceptions was Zadeh's third "new view" on system theory.[69] He received an opportunity to propose these considerations concerning "A New Direction in AI" to the AI community, when his article appeared in the spring issue of *AI Magazine* 2001.[70]

In 2011, the new series of World Conferences on Soft Computing started with the conference at the San Francisco State University from 23rd to 26th May. This conference celebrated Lotfi Zadeh's 90th birthday, and the Ministry of Communications and Information Technologies of the Republic of Azerbaijan honoured it with its support. After Zadeh's plenary talk[71] the second plenary speech was given by Ali M. Abbasov, the Minister of Communications and Information Technologies.[72] Lotfi Zadeh felt very honoured, and since its presentation by the President of Azerbaijan (see figure), he used to wear the High State Award "Friendship Order". In 2012, he was honoured with the fifth edition of the BBVA Foundation Frontiers of Knowledge Award in the Information and Communication Technologies (ICT) category "for the invention and development of fuzzy sets and fuzzy logic, a revolutionary concept and methodology that created a new field of research, which proved powerful in many application domains."[73] This "revolutionary" breakthrough, according to the jury in its citation, "has enabled machines to work with imprecise concepts, in the same way humans do, and thus secured results that are more efficiently aligned with reality. In the last fifty years, this methodology has generated over 50,000 patents in Japan and the U.S. alone."[74]

Zadeh giving his speech after receiving the BBVA Award 2012 in Madrid, Spain, Photo credit: Fuzzy Archive Rudolf Seising.

To celebrate his 95th birthday and the 50th anniversary of his introduction of fuzzy sets, the University of Tehran and the Iranian Association for Fuzzy Systems organized a "glorification ceremony" to honour his scientific achievements on 8th March 2016 in a congress at Shahid Chamran Hall. On the campus and in the museum, two busts were revealed and in the next month, April 2016, Azerbaijan's National Academy of Sciences awarded him the Nizami Ganjavi Gold Medal of Azerbaijan.

One of the busts of Lotfi Zadeh in Tehran, Iran, Photo credit: Fuzzy Archive Rudolf Seising.

On 9th August 2017, the mass media informed us of the worsening state of Lotfi Zadeh's health. In an exclusive statement with AzVision, his son, Norm (originally Norman) Zada, was quoted with the sentence: "My father is pretty ill. We are supposed to transport him to Azerbaijan when that horrible day comes. He specifically asked me to have him buried in Azerbaijan."[75] Norman had signed a letter saying that it was Lotfi Zadeh's last will to be buried in Azerbaijan, which was delivered to Prof. Shahnaz Shahbazova of the Azerbaijan Technical University. According to this letter it was also his will that all his awards, orders, medals and books be transferred to Professor Shakhbazova.

Perhaps it was this news that resulted in the erroneous message on a website of the University of Tehran announcing his death on 12th August already. The premature obituary was later withdrawn with apologies. [76]

In the presence of the President of Azerbaijan, Ilham Aliyev, Lotfi Zadeh was laid to rest on 29th September 2017 in Baku's Alley of Honour. Most members of the fuzzy community living outside of Azerbaijan received the news of the funeral after the event.

[69]Zadeh LA, The Birth and Evolution of Fuzzy Logic–A Personal Perspective, Journal of Japan Society for Fuzzy Theory and Systems, 11(6), 1999, 891-905.

[70]Zadeh LA, A New Direction in AI. Toward a Computational Theory of Perceptions, AI-Magazine, 22(1), 2001, 73-84.

[71]Zadeh LA, "The Concept of a Z-number - Toward a Higher Level of Generality in Uncertain Computation", plenary at the 1st World Conference on Soft Computing, San Francisco State University, May 23-26, 2011.

[72]Abbasov AM, "Information Boom: New Trends and Expectations", plenary at the 1st World Conference on Soft Computing, San Francisco State University, May 23-26, 2011.

[73]FundaciÃşn BBVA, Frontiers of Knowledge Awards, Lotfi A. Zadeh, Frontiers of Knowledge Laureate, Information and Communication Technologies, 5th edition, https://www.frontiersofknowledgeawards-fbbva.es/galardonado/lotfi-a-zadeh-2/

[74]BBVA, press release, http://newsroom.bbvacompass.com/2013-01-15-Lotfi-Zadeh-inventor-of-fuzzy-logic-wins-the-BBVA-Foundation-Frontiers-of-Knowledge-Award-for-enabling-computers-and-machines-to-behave-and-decide-like-human-beings

[75]https://en.azvision.az/news/70395/lotfi-zadehs-will-bury-me-in-azerbaijan-exclusive.html

[76]http://www.presstv.com/Detail/2017/08/12/531553/iran-lofti-zadeh-fuzzy-logic

RECOGNITION

In Memory of L.A. Zadeh

Enric Trillas

Zadeh, a great man

Professor Lotfi Aliasker Zadeh (Baku, 1921- Berkeley, 2017) is one of the biggest scientists in the second half of the Twenty Century and, even more, he is among this epoch more relevant creative minds. But, essentially, he was a great and generous man whose intelligence, kind personality and warm character attracted many people around the world towards accepting his ideas on imprecision, not random uncertainty and fuzzy sets, in both the directions of studying and applying them to an automation close to the human mind. The big amount of people that have been showing with sweet words their deep sorry when he recently passed away, is a good test of it.

Lotfi A. Zadeh was a warm person to whom a lot of people, like myself, is indebted. In addition and due to his especially influential personality the worldwide community of researchers and practitioners of fuzzy logic, inherited a surprising friendly relationship among his members and organizations.

Professor Zadeh did move between science and engineering, but I always did see him as a researcher in the field of Technology; a word that, coming from the old Greek "Tekhne", and perhaps through the German "Technologie", was actually coined in the American English in the second half of the XIX Century. It was when the name of the famous MIT, the "Massachusetts Institute of Technology", was given to such establishment for showing the then already necessary mixing of engineering with science and measuring; a word without the difference, existing in other languages, with the word "Technics".

Zadeh is, in fact, an intellectual son of the marvelous even if short period of time in which Cybernetics appeared and disappeared since, due to its genetic creative potential in association with automated computing, exploded in many branches, and the digital computer began to spread all over the world. I always did see Zadeh's ideas placed in between the analogical and the digital, and I doubt that the idea of a fuzzy set could appear by the hands of an engineer after the personal computer start its predominance. In fact, its immediate predecessors, the 1951 Menger's "hazy sets" and the 1963 Black's "profile functions", were introduced by, respectively, a geometer and a philosopher that, contrarily to what Zadeh did, just presented these concepts but neither continued them, nor studied their possible algebraic structure.

But the fact is that fuzzy sets appeared around 1965, in the context of automation, and when computer mainframes still were very big devices placed in a special room and entering into which was like entering into a Hospital; fuzzy sets, at the end an analogical idea, appeared just a minute before the digital epoch began. Beside the trials for constructing "fuzzy computers" and, at least, "fuzzy chips", fuzzy logic is currently and computationally well enough managed by the today powerful digital binary computers. Currently, fuzzy logic is not yet challenging the computer's hardware.

Zadeh was not a mere engineer, he was also a scientist always trying to translate his ideas into a mathematical formalism, like it was done in Cybernetics by following its founder Norbert Wiener; to see it, a glance at the today old book "Linear System Theory: The State Space Approach", Zadeh coauthored with Charles Desoer in 1963, suffices. Such book was a well known textbook at its time; Zadeh was a reputed professor and scientist before moving from Columbia University to Berkeley University, and before introducing fuzzy sets.

Actually, from its very beginning fuzzy sets theory follows the 1958 John von Neumann's claim on the necessity of introducing Mathematical Analysis in Logic, and towards not only avoiding to be just forced considering "yes" and "not" questions, but for widening the spectrum of problems to be taken into account and like they are, for instance, those permeated by imprecision and that fuzzy sets allow to manage pretty well. The theory of fuzzy sets opened the door to the entrance of Mathematical Analysis into the study of reasoning.

Zadeh's fuzzy sets

Professor Zadeh's starting point for the theory of fuzzy sets was his clever observation that when trying to be absolutely precise, only scarcely significant answers can be obtained; meaning appeared from the very beginning in his idea on what a fuzzy set is. Only posing problems in precise terms is not what people usually does, and hence the scientific domestication of imprecision is basic to manage it in favor of reaching, based on commonsense reasoning, useful solutions approaching many real problems. The idea of constructing machines thinking like people was constant in Zadeh's early thought.

Semantics is essential for capturing problems described in natural language; a fuzzy set is but the name of a "linguistic collective" in the universe of discourse (its linguistic label) and is a cloudy entity when such name is an imprecise word, but is a crisp subset just when it is precise and, hence, should obey the Cantor-Zermelo's axiom of specification. Membership functions represent, at their turn, numerical states measuring the meaning of the fuzzy set's linguistic label, and the same fuzzy set appears in several states depending on the context and the purpose for the use of its linguistic label; a membership function cannot be seen as the characteristic function of a fuzzy set or linguistic label, like it happens with the characteristic function of a crisp set that individuates it.

It should be added that if linguistic collectives or fuzzy sets are cloudy entities, they are notwithstanding well anchored in natural language; for example, everybody recognizes what are the collectives of “young Londoners”, “tall Berliners”, etc. Hence, a fuzzy set is but a purely linguistic entity that can be contextually managed once a membership function is fixed. Zadeh, contrarily to Menger and Black, provided membership functions with an initial algebraic structure translating the connectives “and”, “or”, and “not”, that further on was extended by his followers. In this way, Zadeh enlarged the possibility of representing linguistic statements into mathematical formulas, something that George Boole did start in 1848 for precise ones. Zadeh’s work meant a new step, towards the very ambitious goal expressed by Wilhem Leibniz by his famous “Calculemus!”, and even if he lacked to consider that many laws in the mathematical models can’t always be presumed in natural language and commonsense reasoning.

Professor Zadeh felt, as it is typical of a creative mind, such a passion for his newborn fuzzy sets that he did not only introduce them and the first ideas on its algebraic structure, but further developed its theory by introducing the main concepts of fuzzy logic like they are, for instance, those of a Linguistic Variable, and the “Compositional Rule of Fuzzy Inference” allowing to deduce an imprecise conclusion from a set of imprecise premises and an also imprecise observation.

His new ideas first permitted to develop the so-called field of “Approximate Reasoning”, and latter on a new field Zadeh himself Christened by “Computing with Words and Perceptions” that, in my view, is the true future of fuzzy logic. Before others like Marvin Minsky, whose idea at the respect was more restricted, Zadeh also introduced the idea, opposed to the classical “Hard” Computing, of hybrid computer systems mixing fuzzy logic, neural nets, probabilistic computing, and evolutionary computing, the “Soft Computing”, for sub-optimally affronting some problems that, permeated by imprecision and not always random uncertainty, were not possible to be well posed and solved. In addition, his 1978 paper on the theory of possibility and necessity, jointly with the 1972 one on the probability of fuzzy events, opened the door to study the uncertainty associated to imprecision that, latter on, he extended with the idea of a fuzzy valued probability.

Each year and in the pursuit of his ideas, Zadeh travelled around the world to sequentially attend a lot of conferences, speak at many countries, help several iniciatives at different places, always trying to advise the young researchers that approached him, and never refusing the controversy with others. More than twenty Doctorates Honoris Causa by universities around the world, many Awards, Honors and Decorations, surrounded the life of a true genius who, during his large life, knew from the refusal of his theory to its theoretical success, as well as the big number of commercially successful industrial applications following from it. If only seen from this point of view, Zadeh is a very singular researcher in the full history of science and technology, who is in the Hall of Fame in the Sillicon Valley. It should not be avoided that his plea in favor of imprecision was coincidental with the “Space Race” confronting the USA and the old USSR, a time in which the need of precise computations seemed reinforce precision as the big goal of science. Perhaps by this historical fact, in the USA were many scientists placing themselves against Zadeh’s ideas.

Zadeh and Spain

I was personally acquainted with fuzzy sets in August, 1975 and by chance. In that troublesome time in Spain, I often read French newspapers and in one of them I found an interview with professor Arnauld Kaufmann, of whom I knew a nice textbook, that revealed me his last book whose title was “Ensembles flous”. It called my attention since knowing the 1951 paper in French “Ensembles flous et fonctions aléatoires”, written by Karl Menger, I thought Kaufmann was dealing with hazy sets. I bought such book and was a little bit disappointed by the fact that fuzzy sets were no related at all with the concept of probability; Menger’s hazy sets were like a cloud of elements belonging, with a positive probability, to a crisp set and, since I did my Ph.D thesis in a Department of Statistics and, already full professor, was working in Probabilistic and Generalized Metric Spaces, I was unable to imagine how fuzzy sets were out the domain of probability. There is often some slavery coming from what we learnt!

Once I read the 1965 paper “Fuzzy Sets”, I understood why Zadeh escapes in it from probability, jointly with the fact that fuzzy sets are not related to something necessarily physical but are, essentially, linked to natural language and to qualitative qualifications in commonsense reasoning. I discovered the possibility of representing some aspects of language by non-probabilistic entities related with the meaning of words; I realized that not all in language has a random character. All this interested me at the extreme of wishing to do research in fuzzy sets theory; a wish I kept up to today, and that moved me to correspond with Zadeh and, some time later, with Aldo de Luca and Settimo Termini whose 1972 paper on the non- probabilistic concept of fuzzy entropy also kept my interest.

Since I was organizing in Barcelona the “First World Conference on Mathematics at the Service of Man” for the Spring of 1977, I invited Zadeh to deliver a plenary lecture; he accepted the invitation, travelled to Barcelona with his wife Fay Zadeh, and it represented a nice opportunity to know Professor Zadeh for the then few and young Spaniards who were trying to do research in the then new fuzzy field. The first group devoted to Fuzzy Logic was under me in Barcelona, and it was soon followed by another in Granada under Miguel Delgado; after 1983 it was an explosion of such groups around Spain. Zadeh returned to Spain a lot of times up to his last visit in 2014 for receiving the prestigious BBVA Award in Computer Science; in between 1977 and 2014 he was honored with doctorates Honoris Causa by the universities of Oviedo, Granada and Technical of Madrid. Zadeh was essential for establishing in Spain the, today and unfortunately disappeared, “European Centre for Soft Computing” that lasted for ten years, and whose creation was, indeed, an idea of his own. For me, the 1977 Conference meant the beginning of a long-standing friendship with Zadeh with whom I corresponded up to the Summer of this year, two months before passing away. Zadeh left very good seeds in Spain growing up to rent as income the Gospel’s “hundred per one”, as it is shown by its worldâĂŹs place for scientific production in Fuzzy Logic and Soft Computing. Certainly, Spain is in a scientific and non reimbursable debt with Professor Zadeh.

Let me remember something that most members of

Eusflat probably donâĂŹt know. This Magazine is the third order derivation of the old journal “Stochastica” that, once the editors decided to change its name to “Mathware”, preferred to previously ask Zadeh at the respect in a meeting with him at a seminar. He suggested, even if the editors understood it as a command, to add “Soft Computing”, and the new journal took the title “Mathware and Soft Computing”. Finally, at the end of its “life”, such journal was transformed in the current on-line magazine of the European Society for Fuzzy Logic. Hence, also this Magazine and like many other European iniciatives, is indebted with Zadeh.

No fuzzy researcher will never forget ZadehâĂŹs sense of friendship, his upmost degree of creativity, his warm attitude towards young researchers, his generous advise and help for any possible iniciative aiming at developing fuzzy logic, his gentle social behavior, and his openness towards views different from his own. By my part, I will never forget the many conversations we had on everything, as well as the good advise I received from him. Zadeh, for me an unforgettable master, was a practitioner of what Einstein did express with the words, “Creativity is contagious, pass it!”.

RECOGNITION

In the memory of Lotfi Aliasker Zadeh

Martin Štěpnička

On September 6, 2017, a sad news about the decease of prof. Lotfi Aliasker Zadeh ran around the world and we all felt, that unlike the previous premature news, this is was sadly right. That day, we have lost the founder of the Fuzzy Set Theory, the "Father of Fuzzy Logic", our teacher, our mentor, our colleague, our friend. We can recall all his fabulous research contributions and his scientific excellence with all the appropriate adverbs strongly stressing his significant superiority. And it would be very appropriate. Indeed, Lotfi Zadeh was a man who influenced our lives probably more than any other scientist. We owe him a lot and his memory deserves our attention and our esteem. This common feeling is strengthened by the uniqueness of the situation. As Lotfi Zadeh was the founder, we have never had the situation without him, this is the first time, we are "alone", the evolvement of the fuzzy community could be split into the phase with Lotfi Zadeh and after Lotfi Zadeh and the latter one is just being opened.

Lotfi Zadeh and his colleagues, IFSA 1997, Prague. Photo credit: Fuzzy Archive Rudolf Seising.

But we do not have to be sad, the legacy of Lotfi Zadeh is still alive and due to his legacy, the number of researchers in the field of modeling under uncertainty is now establishing a strong community significantly contributing to the world science, and that is, in my opinion, a reason to be grateful, proud, and happy. We do not need to list all the fabulous achievements and superiorities of Lotfi Zadeh and I personally do not want to do so. Every single one of you would do it better from your personal perspective and I am no intending to intervene your personal silent memories. I want to add only a small fragment of my memory to Lotfi Zadeh recalling his non-scientific character.

Lotfi Zadeh and Irina Perfilieva, IFSA 1997, Prague. Photo credit: Fuzzy Archive Rudolf Seising.

I have not spend as much time with Lotfi Zadeh as many of you, my contact with him was limited to short meetings on conferences and similar events and my first meeting with him dates to 2004. It was in Dortmund, the conference was called 8th Fuzzy Days and during a coffee break I was participating on a scientific discussion in a small group consisting of, apart from me, Vilém Novák, Irina Perfilieva, and Szilveszter Kovács, who was carefully explaining his previous conference talk in details. Suddenly, Lotfi Zadeh came. He came alone, silently, and politely asked me, whether he can join us or not. I was a non-experienced student and used to show so much respect to all recognized scholars and somehow having a (luckily) fake feeling that the enormous respect is something really expected by all of them. Suddenly, the situation was different, the founder himself did come and talked to me in such a calming way that I felt extremely comfortably and I could continue listening to Szilveszter's explanation. And Lotfi did not interrupt anyone during the discussion. He was listening carefully too. The new knowledge he could gain was more important than sharing his wisdom with the others. Learning new things played a more important role than demonstrating his knowledge. Letting a young PhD student listening to a talk of a colleague and discussing in a group was more important than interrupting the group by his own wisdom. And later on, in order to break the ice, Lotfi offered us to bring some tea, coffee or sweets, as he had some, while we were deep in the discussion and have and time to pick up some refreshment before the discussion started.

Lotfi Zadeh in New York, 1950s, Prague. Photo credit: Fuzzy Archive Rudolf Seising.

t first sight, it is a story of a minor importance, actually a story about nothing. A recognized scholar came to us, addressed me, behaved politely and did not require any special respect. But for me, it is a very important story. This is how I remember Lotfi. As a wise and mainly as a friendly, polite man. Lotfi Zadeh never pushed anyone to listen to him, and he was listened naturally more than anyone else. His respect never came from his own intention. Vice-versa, his respect was natural, it came from his real work, from his personal character, from his charisma. And this feeling of mine, from this truly a unique man, was only emphasized by every single meeting between two of us in the succeeding years.

Lotfi Zadeh, Martin Štěpnička, Irina Perfilieva, Szilveszter Kovács, 8th Fuzzy Days in Dortmund, 2004. This is how I remember Lotfi Aliasker Zadeh.

This is my Lotfi Aliasker Zadeh, smart, educated, wise and creative âĂę but mainly a very friendly person. This is why ho got my full respect. This is why I will never forget him. This is how I will always remember him. And I am sure, you all have your own Lotfi Aliasker Zadeh in your hearts too.

Rest in peace, Lotfi.

Ostrava, Czech Republic, December 4, 2017

Martin Štěpnička

SCIENTIFIC REPORT

Brazilian Conference of Fuzzy Systems (CBSF 2016)

Marcos Eduardo Valle and Graçaliz Pereira Dimuro

The Brazilian Conference of Fuzzy Systems (CBSF, in Portuguese Congresso Brasileiro de Sistemas Fuzzy) is the major South America event dedicated solely to fuzzy systems and their applications in areas such as mathematics, physics, engineering, health, and social sciences. Besides being an attractive forum for fuzzy systems and their applications, the CBSF aims to aggregate the South America community around this subject, favoring exchange of ideas as well as increasing national and international collaborations. The CBSF is held every two years since 2010. The IV CBSF took place at the University of Campinas, Campinas - Brazil, in November 2016.

The IV CBSF brought together 121 participants, including researchers, engineers, students, and professionals interested on fuzzy systems and their applications. The congress counted on 6 plenaries, 4 tutorials, and several technical sessions in which the works approved by the scientific committee were presented either orally or by poster. The plenary sessions featured the internationally prominent researchers Bernard De Baets (Ghent University, Belgium), Barnabas Bede (DigiPen Institute of Technology, USA), and Hao Ying (Wayne State University, USA) as well as Heriberto Flores (Universidad de Tarapacá, Chile), Regivan Santiago (Universidade Federal do Rio Grande do Norte, Brazil), and Geraldo Silva (Universidade Estadual Paulista, Brazil) from South America. The full program, with links to all contributions presented at the technical sessions, is available at `https://www.ime.unicamp.br/~cbsf4/Program_EN.php`

The contributions submitted to the IV CBSF, which are either full papers or two-page abstracts, have been all judged by at least two anonymous referees. Only those full papers that received a positive evaluation from all reviewers have been accepted to be presented orally. Similarly, the abstracts with a positive average grade where accepted for presentation by poster. The 90 accepted contributions represent a view of the Brazilian contributions to the area of fuzzy systems. In particular, the 45 full papers have been organized as chapters of the book "Recent Trends on Fuzzy Systems", which is freely available for download at `https://www.ime.unicamp.br/~cbsf4/Papers_IVCBSF/ProceedingsIVCBSF.pdf` The authors of the best full papers have been invited to submit an extended version of their contribution to this current issue of Mathware and Soft Computing Magazine. All the invited manuscripts have been judged by anonymous referees, allowing us to select 6 papers for this special issue.

During the conference dinner, recognition was given to José Arnaldo Roveda, Regivan Santiago, and Ronei Moraes as the chairs of the previous three CBSF. Also, Fernando Gomide has been awarded for outstanding contributions and teaching excellence, in Brazil and abroad, in the area of fuzzy systems.

We hope the participants had a great time during the IV CBSF, with many interesting and fruitful discussions and novel collaborations among fuzzy systems researchers. We would like to express our gratitude towards all individuals, including participants and members of the scientific and organizing committees, who have contributed in making the IV CBSF a success.

Analysing Fuzzy Entropy via Generalized Atanassov's Intuitionistic Fuzzy Indexes

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Abstract. This extended study of fuzzy entropy measures considers aggregations of Generalized Atanassov's Intuitionistic Fuzzy Index, which are obtained from conjugate and dual (co)implications. Main properties of fuzzy entropy are discussed and a numerical example illustrates an application based on this method.

Keywords: Intuitionistic Fuzzy Sets, Fuzzy Entropy, Conjugation, Duality

1 Introduction

The Atanassov-intuitionistic fuzzy index (A-IFIx), also called as hesitancy or indeterminance degree of an element in an Atanassov-intuitionitic fuzzy set (A-IFS), allows the expression related to the expert uncertainty to identify a particular membership function. In applications in which experts do not have precise knowledge, the A-IFIx provides a measure of the lack of information for or against a given proposition based on Atanassov-intuitionistic fuzzy logic (A-IFL).

The Generalized Atanassov's Intuitionistic Fuzzy Index associated with a strong intuitionistic fuzzy negation N_I (A-GIFIx(N_I)) [1] is characterized in terms of fuzzy implication operators which is described by a construction method with automorphisms. In [2], by means of special aggregation functions applied to the A-GIFIx, the Atanassov's intuitionistic fuzzy entropy is introduced. In [3], the A-GIFIx, can be generated based on the concept of conjugate and dual fuzzy (co)implications, mainly interested in the class of (S, N)-(co)implications. Additionally, A-GIFIx associated with the standard negation together with the well known fuzzy implications are considered: Łukaziewicz, I_0 Reichenbach and Gaines-Rescher [4].

This article extends the above results, introducing new results concerned with the duality and conjugation of A-GIFIx. Additionally, main properties of fuzzy entropy are discussed and a numerical example illustrates an application based on this method.

The preliminaries describe the basic properties of fuzzy connectives and basic concepts of A-IFL. The study of the A-GIFIx(N_I) and general results in the analysis of its properties are stated in Section 3 and 4. Section 5 expresses fuzzy measures based on A-GIFIx. Final remarks are reported in the conclusion.

2 Preliminaries

We firstly give a brief account on FL, keeping this paper self-contained by reporting basic concepts of automorphisms, fuzzy negations on $U = [0, 1]$ and main properties of fuzzy implications.

2.1 Fuzzy connectives

By [5, Def. 4.1], an **automorphism** $\phi : U \to U$ is a bijective, strictly increasing function (SIF) satisfying the monotonicity property:

A1: $x \leq y$ iff $\phi(x) \leq \phi(y), \forall x, y \in U$.

Analogously, in [6], $\phi : U \to U$ is a SIF satisfying the continuity property and the boundary conditions:

A2: $\phi(0) = 0$ and $\phi(1) = 1$.

Both notions of automorphisms are equivalent to definition stated in [7]. The set $Aut(U)$ of all automorphisms are closed under composition, meaning that $\phi \circ \phi' \in Aut(U)$, $\forall \phi, \phi' \in Aut(U)$. In addition, there exists the inverse $\phi^{-1} \in U$, such that $\phi \circ \phi^{-1} = id_U$, $\forall \phi \in Aut(U)$. Thus, $(Aut(U), \circ)$ is a group, with the identity function being the neutral element. The action of an automorphism $\phi : U \to U$ on a function $f : U^n \to U$ is called the **conjugate** of f and given by

$$f^{\phi}(x_1, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \ldots, \phi(x_n))). \tag{1}$$

A function $N : U \to U$ is a *fuzzy negation* (FN) if

N1: $N(0) = 1$ and $N(1) = 0$; **N2**: If $x \geq y$ then $N(x) \leq N(y), \forall x, y \in U$.

FNs satisfying the involutive property **N3** are called *strong* fuzzy negations [6]:

N3: $N(N(x)) = x, \forall x \in U$.

Among several definitions, see [8–10], an aggregation is a function $A : U^n \to U$ demanding, for all $\boldsymbol{x}, \boldsymbol{y} \in U^n$, the following conditions:

Ag1: $A(\mathbf{0}) = A(0, 0, \ldots, 0) = 0$ and $A(\mathbf{1}) = A(1, 1, \ldots 1) = 1$;
Ag2: If $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \leq \boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ then $A(\boldsymbol{x}) \leq A(\boldsymbol{y})$;

Moreover, when $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ is a σ-permutation, a symmetric aggregation A verifies:

Ag3: $A(\overrightarrow{x_\sigma}) = A(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) = A(x_1, x_2, \ldots, x_n) = A(\boldsymbol{x})$.

A **triangular-(co)norm** (t-(co)norm) $T(S)$:$U^2 \to U$ is a binary aggregation with the identity element $T(1, x) = x$ ($S(0, x) = x$), for all $x \in U$.

By [11], a **fuzzy (co)implication** $I(J) : U^2 \to U$ satisfies the conditions:

I1: $x \leq z \Rightarrow I(x, y) \geq I(z, y)$;	**J1**: $x \leq z \Rightarrow J(x, y) \geq J(z, y)$;
I2: If $y \leq z$ then $I(x, y) \leq I(x, z)$;	**J2**: If $y \leq z$ then $J(x, y) \leq J(x, z)$;
I3: $I(0, x) = 1$;	**J3**: $J(1, x) = 0$
I4: $I(x, 1) = 1$;	**J4**: $J(x, 0) = 0$
I5: $I(1, 0) = 0$;	**J5**: $J(0, 1) = 1$.

Several reasonable properties may be required for fuzzy (co)implications:

I6: $I(1, x) = x$; **J6**: $J(0, x) = x$;
I7: $I(x, I(y, z)) = I(y, I(x, z))$; **J7**: $J(x, J(y, z)) = J(y, J(x, z))$;
I8: $I(x, y) = 1 \Leftrightarrow x \leq y$; **J8**: $J(x, y) = 0 \Leftrightarrow x \geq y$;
I9: $I(x, y) = I(N(y), N(x))$, N is a SFN; **J9**: $J(x, y) = J(N(y), N(x))$, N is a SFN;
I10: $I(x, y) = 0 \Leftrightarrow x = 1$ and $y = 0$; **J10**: $J(x, y) = 1 \Leftrightarrow x = 0$ and $y = 1$.

If $I(J) : U^2 \to U$ is a fuzzy (co)implication satisfying **I1** (**J1**), then the function $N_I : U \to U$ defined by

$$N_I(x) = I(x, 0) \text{ and } N_I(x) = J(x, 1) \tag{2}$$

is a fuzzy negation [12, Lemma 2.1].

Let $T(S)$ be a t-(co)norm and N be a SFN. Based on [6,11,12], an $(S, N)-$implication $((T, N)-$coimplication) is a fuzzy (co)implication $I_{S,N}(J_{T,N}) : U^2 \to U$ defined by

$$I_{S,N}(x, y) = S(N(x), y); \quad J_{T,N}(x, y) = T(N(x), y). \tag{3}$$

In this paper, such implications are called *strong S-implications*. In [13, Theorem 3.2] $I : U^2 \to U$ is a *strong S-implication* iff it satisfies **I1** – **I4**, and **I10**. Analogously, it can be stated by its N-dual construction, a *strong T-coimplication*.

2.2 Intuitionistic Fuzzy Connectives

Based on [14], an **intuitionistic fuzzy set** (IFS) A_I in a universe $\chi \neq \emptyset$, is given as

$$A = \{(x, \mu_A(x), \nu_A(x)) : \mathrm{x} \in \chi\} \in \mathcal{A_I}, \tag{4}$$

whenever $\mu_A(x) + \nu_A(x)) \leq 1$ and $\mathcal{A}_I$ denotes the set of all Atanassov's intuitionistic fuzzy sets on U. So, an intuitionistic fuzzy truth value of an element $x \in \chi$ is related to an ordered pair $(\mu_A(x), \nu_A(x))$. Moreover, an IFS A_I generalizes a FS A given as

$$A = \{(x, \mu_A(x)) : x \in \chi\}.$$

When $\mathcal{A}$ denotes the set of all fuzzy sets on χ, $\mathcal{A} \subset \mathcal{A}_I$ since $\nu_A(x) = 1 - \mu_A(x)$, meaning that the non-membership degree of an element x, is less than or equal to the complement of its membership degree $\mu_A(x)$.

Let $\tilde{U} = \{(x_1, x_2) \in U^2 \colon x_1 \leq 1 - x_2\}$ be the set of all intuitionistic fuzzy values and $l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \to U$ be the projection functions on $\tilde{U}$, which are given by $l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1$ and $r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2$, respectively.

By [14], the usual partial order $\leq_{\tilde{U}}$ is given as $\tilde{x} \leq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, for $\tilde{x}, \tilde{y} \in \tilde{U}$ such that $\tilde{0} = (0, 1) \leq_{\tilde{U}} \tilde{x}$ and $\tilde{1} = (1, 0) \geq_{\tilde{U}} \tilde{x}$.

In this paper, we also consider the partial order $\preceq_{\tilde{U}}$ which is given as

$$\tilde{x} \preceq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2, \forall \tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in \tilde{U}. \tag{5}$$

A function $\Phi : \tilde{U} \to \tilde{U}$ is an **intuitionistic automorphism** on $\tilde{U}$ if it is bijective and $\tilde{x} \leq_{\tilde{U}} \tilde{y}$ iff $\Phi(\tilde{x}) \leq_{\tilde{U}} \Phi(\tilde{y})$. The action of $\Phi : \tilde{U} \to \tilde{U}$ on $f_I : \mathbb{U}^n \to \mathbb{U}$ is a function $f_I^{\Phi} : \tilde{U} \to \tilde{U}$, called conjugate function f_I, defined as follows

$$f_I^{\Phi}(\tilde{\mathbf{x}}) = \Phi^{-1}(f_I(\Phi(\tilde{x}_1), \Phi(\tilde{x}_2), \ldots, \Phi(\tilde{x}_n))). \tag{6}$$

According with [15, Theorem 17], let $\phi : U \to U$ be an automorphism on U. Then, for all $x \in U$, a **ϕ-representable automorphism** $\Phi : \tilde{U} \to \tilde{U}$ is defined by

$$\Phi(\tilde{x}) = (\phi(l_{\tilde{U}}(\tilde{x})), 1 - \phi(1 - r_{\tilde{U}}(\tilde{x}))). \tag{7}$$

See, $\Phi_2(\tilde{x}) = \left(x_1^2, 1-(1-x_2)^2\right)$, which is related to $\Phi_n(\tilde{x}) = (\phi_n(x_1), 1-\phi_n(1-x_2))$ by taking $n = 2$ and $\phi_n(x) = x^n$.

An **intuitionistic fuzzy negation** (IFN) $N_I : \tilde{U} \to \tilde{U}$ satisfies, for all $\tilde{x}, \tilde{y} \in \tilde{U}$, the following properties:

$\mathbf{N}_I$ 1: $N_I(\tilde{0}) = N_I(0,1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1,0) = \tilde{0}$;
$\mathbf{N}_I$ 2: If $\tilde{x} \geq_{\tilde{U}} \tilde{y}$ then $N_I(\tilde{x}) \leq_{\tilde{U}} N_I(\tilde{y})$.

Additionally, N_I is a strong intuitionistic fuzzy negation (SIFN) verifying the condition $\mathbf{N_I 3} : N_I(N_I(\tilde{x})) = \tilde{x}, \forall \tilde{x} \in \tilde{U}$. Additionally, If N_I as IFN, the N_I-dual intuitionistic function $\tilde{f}_{N_I} : \tilde{U}^n \to \tilde{U}$ is given by:

$$\tilde{f}_{N_I}(\tilde{\mathbf{x}}) = N_I(\tilde{f}(N_I(\tilde{x}_1), \ldots, N_I(\tilde{x}_n))), \forall \tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n. \tag{8}$$

Moreover, by [16], taking a SFN $N : U \to U$, a IFN $N_I : \tilde{U} \to \tilde{U}$ such that

$$N_I(\tilde{x}) = (N(N_S(x_2)), N_S(N(x_1))), \tag{9}$$

is a SIFN called **N-representable** IFN. Additionally, if $N = N_S$, Eq. (9) can be reduced to $N_I(\tilde{x}) = (x_2, x_1)$. Thus, we consider the complement of an IFS A given as $A' = \{(x, N(N_S(\nu_A(x)), N_S(N(\mu_A(x)))) : x \in \chi\} \subseteq \mathcal{A}_I$.

3 (Co)Generalized Atanassov's Intuitionistic Fuzzy Index

In [2], the concept of generalized Atanassovs intuitionistic fuzzy index is characterized in terms of fuzzy implication operators and a construction method with automorphisms is also proposed in [1], together with some special properties of a GIFIx. In [3], this concept studies its dual and conjugate constructions.

Definition 1. *[2, Def. 1], A function $\Pi : \tilde{U} \to U$ is called a* ***generalized intuitionistic fuzzy index associated with a SIFN*** *N_I $(A - GIFIx(N_I))$ if it holds that:*

Π1*: $\Pi(x_1, x_2) = 1$ if, and only if, $x_1 = x_2 = 0$;*
Π2*: $\Pi(x_1, x_2) = 0$ if, and only if, $x_1 + x_2 = 1$;*
Π3*: If $(y_1, y_2) \preceq_{\tilde{U}} (x_1, x_2)$ then $\Pi(x_1, x_2) \leq \Pi(y_1, y_2)$*
Π4*: $\Pi(x_1, x_2) = \Pi(N_I(x_1, x_2))$, for all $x_1, x_2, y_1, y_2 \in U$.*

Proposition 1. *[2, Theorem 3] [3, Prop. 1] Let N_I be an N-representable IFN obtained by a SFN N. A function $\Pi : \tilde{U} \to U$ is a $A - GIFIx(N_I)$ iff there exists a function $(J)I : U^2 \to U$ verifying* **I1** *(***J2***),* **I8** *(***J8***),* **I9** *(***J9***) and* **I10** *(***J10***) such that*

$$\Pi_I(\tilde{x}) = N(I(N_S(x_2), x_1)); \ \Pi_J(\tilde{x}) = J(x_1, N_S(x_2)), \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \tag{10}$$

Theorem 1. *Based on conditions of Prop. 1, when $I = J_N$ the following holds:*

$$\Pi_I(\tilde{x}) = \Pi_J(\tilde{x}), \ \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \tag{11}$$

Proof. Straightforward Proposition1.

3.1 Atanassov's Intuitionistic Fuzzy Index - Dual and Conjugate Operators

In this section we study the duality and conjugation properties related to A-GIFIx.

Theorem 2. *Let N be a SFN and N_I be its corresponding N-representable SIFN. For a A-GIFIx(N) $\Pi : \tilde{U} \to U$ the following holds:*

$$(\Pi)_N(\tilde{x}) = N(\Pi(\tilde{x})), \forall \tilde{x} \in \tilde{U}. \tag{12}$$

Proof. By Eq.(9) and Property **Π4**, $(\Pi)_N(\tilde{x}) = N(\Pi(N_I(\tilde{x}))) = N(\Pi(\tilde{x}))$.

Proposition 2. *[3, Prop. 2] Let N be a SFN and N_I be its corresponding N-representable SIFN. For a A-GIFIx(N) $\Pi_I : \tilde{U} \to U$ the following holds:*

$$(\Pi_I)_N(\tilde{x}) = I(N_S(x_2), x_1); \tag{13}$$

$$(\Pi_J)_N(\tilde{x}) = N(J(x_1, N_S(x_2))), \forall \tilde{x} = (x_1, x_2) \in \tilde{U} \tag{14}$$

Consequently, one can describe hesitance and accuracy in terms of A-GIFIx:

Corollary 1. *[3, Corollary. 1] Let $I_{LK} : U \to U$ be the Lukaziewicz fuzzy implication. A function $\pi : \tilde{U} \to U$, called an **Atanassov's intuitionistic fuzzy index** (A-IFIx) of an element $\mathrm{x} \in \chi$ related to an IFS A_I, is an $(A - GIFIx(N_{SI}))$ given as*

$$\Pi_{I_{LK}}(\tilde{x}) = \pi(\tilde{x}) = N_S(\mu_A(\mathrm{x}) + \nu_A(\mathrm{x})) \tag{15}$$

*providing the hesitancy (indeterminance) degree of $\mathrm{x} \in \chi$ in A. Dually, the **accuracy function** $h : \tilde{U} \to U$ providing the accuracy degree of $\mathrm{x} \in \chi$ related to A_I, is given as:*

$$(\Pi_{I_{LK}})_{N_S}(\tilde{x}) = h(\tilde{x}) = \mu_A(\mathrm{x}) + \nu_A(\mathrm{x}). \tag{16}$$

By Corollary 1, $h(\tilde{x}) + \pi(\tilde{x}) = 1$ meaning that the largest hesitance $\pi(\tilde{x})$, the higher accuracy degree $h(\tilde{x})$ of $\mathrm{x} \in \chi$ related to IFS A_I. Moreover, by considering $\tilde{x} = (x, y) \in \tilde{U}$, $x + y \leq 1$, Table 1 not only illustrates Proposition 1, but also presents additional examples of $A - GIFIx(N_{SI})$ associated with the following fuzzy implications: Lukaziewicz (detailed in Corollary1), I_0, Reichenbach and Gaines-Rescher.

Proposition 3. *[3, Prop. 3] Let $\Phi \in Aut(\tilde{U})$ be a ϕ-representable automorphism, $N^\phi : U \to U$ be the ϕ-conjugate of a SFN N. A function ${\Pi_G}^{\Phi} : \tilde{U} \to U$ given by*

$$\Pi^{\Phi}(x_1, x_2) = (\phi^{-1}(\Pi(\phi(x_1)), 1 - \phi(1 - x_2)), \tag{17}$$

is a $A - GIFIx(N_I)$ whenever $\Pi_G : \tilde{U} \to \tilde{U}$ is also a $A - GIFIx(N_I)$.

Proposition 4. *Based on conditions of Proposition 3, the following holds:*

$$\Pi_{I_N^\phi}(\tilde{x}) = N_S \circ \phi \circ N_S(\Pi_I(\tilde{x})). \tag{18}$$

Proof. Straightforward from Proposition 3.

See Table 2, $A - GIFIx$ associated to ϕ-conjugate implications in Table 1.

Table 1. Generalized intuitionistic fuzzy index associated with the standard negation.

Pairs of N_S-Dual Fuzzy (Co)Implications	N_{SI}-Dual A-GIFIx
$I_{LK}(x,y)=\begin{cases}1, & \text{if } x \le y,\\ 1-x+y, & \text{otherwise;}\end{cases}$ $J_{LK}(x,y)=\begin{cases}0, & \text{if } x \ge y,\\ y-x, & \text{otherwise;}\end{cases}$	$\mathit{\Pi}_{LK}(x,y)=1-x-y$ $(\mathit{\Pi}_{LK})_{N_{SI}}(x,y)=x+y$
$I_0(x,y)=\begin{cases}1, & \text{if } x \le y,\\ \max(1-x,y), & \text{otherwise;}\end{cases}$ $J_0(x,y)=\begin{cases}0, & \text{if } x \ge y,\\ \min(1-x,y), & \text{otherwise;}\end{cases}$	$\mathit{\Pi}_0(x,y)=1-\max(x,y)$ $(\mathit{\Pi}_0)_{N_{SI}}(x,y)=\max(x,y)$
$I_{RB}(x,y)=\begin{cases}1, & \text{if } x \le y,\\ 1-x+xy, & \text{otherwise;}\end{cases}$ $J_{RB}(x,y)=\begin{cases}0, & \text{if } x \ge y,\\ 1-x-y+xy, & \text{otherwise;}\end{cases}$	$\mathit{\Pi}_{RB}(x,y)=1-x-y+xy$ $(\mathit{\Pi}_{RB})_{N_{SI}}(x,y)=y-xy$
$I_{GR}(x,y)=\begin{cases}1, & \text{if } x \le y,\\ 0, & \text{otherwise;}\end{cases}$ $J_{GR}(x,y)=\begin{cases}0, & \text{if } x \ge y,\\ 1, & \text{otherwise;}\end{cases}$	$\mathit{\Pi}_{GR}(x,y)=1$ $(\mathit{\Pi}_{GR})_{N_{SI}}(x,y)=0$

3.2 A-GIFIx (S, N)-implications and (T, N)-coimplication

Now, $S(T)$-(co)implicationsare considered to express a new A-GIFIx.

Proposition 5. *[3, Prop. 6] Let N be a SFN. The function $\mathit{\Pi} : \tilde{U} \to U$ is a A-GIFIx(N) iff there exists an S-implication (T-coimplication) $I_{S,N}(J_{T,N}) : U^2 \to U$ and it verifies:*

$$\mathit{\Pi}_{I_{S,N}}(x_1, x_2) = S_N(N_S(x_2), N(x_1)); \tag{19}$$
$$\mathit{\Pi}_{J_{T,N}}(x_1, x_2) = T(N_S(x_2), N(x_1)). \tag{20}$$

Proof. ($\Leftarrow$) Let N be a SFN and $I_{S,N}(J_{T,N})$ be an $S(T)$-(co)implication, by Eqs.(10),(3).

$$\mathit{\Pi}_{I_{S,N}}(x_1,x_2)=N\left(I_{S,N}(N_S(x_2),x_1)\right)=N\left(S(N(N_S(x_2)),x_1)\right)=S_N\left(N_S(x_2),N(x_1)\right)$$
$$\mathit{\Pi}_{J_{T,N}}(x_1,x_2)=N\left(J_{T,N}(N_S(x_2),x_1)\right)=N\left(T(N(N_S(x_2)),x_1)\right)=T_N\left(N_S(x_2),N(x_1)\right)$$

($\Rightarrow$) A strong S-implication satisfies **I1** – **I2** and **I10**. Additionally, **I3** and **I4** imply **I8** and **I9**. Then, by Proposition 1, $S_N(N_S(x_2), N(x_1))$ is an A-GIFIx(N).

Corollary 2. *When $N = N_S$, Eqs.(19) and (20) can be expressed as*

$$\mathit{\Pi}_{I_{S,N_S}}(x_1, x_2) = N_S(S(x_1, x_2));\ \ \mathit{\Pi}_{J_{T,N_S}}(x_1, x_2) = N_S(T_{N_S}(x_1, x_2)). \tag{21}$$

4 Generation of Atanassov's Intuitionistic Fuzzy Entropy

This study of Atanassov's intuitionistic fuzzy entropy follows results stated in [2].

Definition 2. *[2, Def. 2] A real function $E : \mathcal{A}_I \to U$ is called an Atanassov's intuitionistic fuzzy entropy (A-IFE) if the following properties are verified:*
E1: $E(A) = 0$ *iff* $A \in \mathcal{A}$, **E2**: $E(A) = 1$ *iff* $\mu_A(x) = \nu_A(x)$, $\forall x \in \chi$,
E3: $E(A) = E(A_N)$, **E4**: *if* $A \preceq B$ *then* $E(A) \ge E(B), \forall A, B \in \mathcal{A}_I$.

Table 2. $A - GIFIx(N_{SI})$ associated with the automorphisms $\phi(x) = x^2$ *and* $\phi^{-1} = \sqrt{x}$.

Fuzzy (Co)Implications	$A - GIFIx(N_{SI})$
$I_0^\phi(x,y) = \begin{cases}1, \text{ if } x \leq y,\\ \sqrt{\max(1-x^2, y^2)}, \text{otherwise;}\end{cases}$ $J_0^\phi(x,y) = \begin{cases}0, \text{ if } x \geq y,\\ \sqrt{\min((1-x)^2, y^2)}, \text{otherwise;}\end{cases}$	$\Pi_{I_0^\phi}(x,y) = 1 - \sqrt{\max(x^2, 1-(1-y)^2)}$ $\Pi_{I_0^\phi}(x,y) = \sqrt{1 - \max(x^2, 1-(1-y)^2))}$
$I_{LK}^\phi(x,y) = \begin{cases}1, \text{ if } x \leq y,\\ \sqrt{1-x^2+y^2}, \text{ otherwise;}\end{cases}$ $J_{LK}^\phi(x,y) = \begin{cases}0, \text{ if } x \geq y,\\ \sqrt{1-x^2+y^2}, \text{ otherwise;}\end{cases}$	$\Pi_{I_{LK}^\phi}(x,y) = 1 - \sqrt{1+x^2-(1-y)^2}$ $\Pi_{I_0^\phi}(x,y) = \sqrt{x^2-(1-y)^2}$
$I_{RH}^\phi(x,y) = \begin{cases}1, \text{ if } x \leq y,\\ \sqrt{1-x^2+x^2y^2}, \text{otherwise;}\end{cases}$ $J_{RH}^\phi(x,y) = \begin{cases}0, \text{ if } x \geq y,\\ \sqrt{y^2-x^2y^2}, \text{otherwise;}\end{cases}$	$\Pi_{I_{RH}^\phi}(x,y) = 1 - \sqrt{1-(1-y)^2(1-x^2)}$ $\Pi_{I_0^\phi}(x,y) = \sqrt{(1-y)^2(1-x^2)}$
$I_{GR}^\phi(x,y) = \begin{cases}1, \text{ if } x \leq y,\\ 0, \text{otherwise;}\end{cases}$ $J_{GR}^\phi(x,y) = \begin{cases}0, \text{ if } x \geq y,\\ 1, \text{otherwise;}\end{cases}$	$\Pi_{I_{RH}^\phi}(x,y) = 0$ $\Pi_{I_0^\phi}(x,y) = 1$

Proposition 6. *Let Φ be a ϕ-representable automorphism and E be an A-IFE. Then, the Φ-conjungate function $E^\Phi : \mathcal{A} \to U$ is an A-IFE.*

Proof. (**E1**)If $E^\Phi(A) = 0$, $\phi^{-1}(E(\phi(A))) = 0 \Leftrightarrow E(\phi(A)) = 0 \Leftrightarrow \phi(A) \in \mathcal{A} \Leftrightarrow A \in \mathcal{A}$.
(**E2**) If $E^\Phi = 1$, $\phi^{-1}(E(\phi(A))) = 1$ iff $\phi(\mu_A(\mathrm{x}_i)) = 0$ and $1 - \phi(1 - \nu_A(\mathrm{x}_i) = 1$ iff $\mu_A(\mathrm{x}_i) = 0$ and $1 - \phi(\nu_A(\mathrm{x}_i)) = 1$. Therefore, $\mu_A(\mathrm{x}_i) = \nu_A(\mathrm{x}_i) = 0$.
(**E3**) $E^\Phi(A') = \phi^{-1}(E(\phi(A'))) = \phi^{-1}(E(\phi(A')')) = \phi^{-1}(E(\phi(A))) = E^\Phi(A)$.
(**E4**) If $A \preceq B$, $\Phi(A) \preceq \Phi(B)$ then $E^\Phi(A) = \phi^{-1}(E(\phi(A)) \geq \phi^{-1}(E(\phi(B)) = E^\Phi(B)$.

Consider $\chi = \{x_1, \ldots, x_n\}$ to discuss properties of A-IFEs:

Proposition 7. *[2, Prop. 4] Let Ag be an aggregation on U, N be a SFN, Π_G be an A-GIFIx(N). Then, the mappings $E, E^\Phi : \mathcal{A} \to U$ defines an Atanassov's intuitionistic fuzzy entropy (A-IFE) respectively expressed by*

$$E(A) = Ag_{i=1}^n \Pi_G(A(\mathrm{x}_i)), \tag{22}$$
$$E^\Phi(A) = Ag_{i=1}^n (\Pi_G)^\Phi(A(\mathrm{x}_i)). \tag{23}$$

Now, an A-IFE obtained is extended their dual and conjugate constructions.

Proposition 8. *[3, Prop. 9] Consider $\phi \in Aut(U)$. Let $N : U \to U$ be a SFN, $Ag : U^n \to U$ be an aggregation function and $I_N : U^2 \to U$ be a N-dual operator of an implication $I : U^2 \to U$ which satisfies properties* **I1**, **I8**, **I9** *and* **I10**, *as discussed in Prop. 1. Then, the mappings $E_I, E_{I^\Phi} : \mathcal{A} \to U$ defined by*

$$E_I(A) = Ag_{i=1}^n N(I(1 - \nu_A(\mathrm{x}_i), \mu_A(\mathrm{x}_i))), \tag{24}$$
$$E_{I^\phi}(A) = Ag_{i=1}^n N^\phi(I^\phi(1 - \nu_A(\mathrm{x}_i), \mu_A(\mathrm{x}_i))), \tag{25}$$

providing new expressions of A-IFEs obtained from an A-GIFIx(N_I).

Proposition 9. *Consider $\phi \in Aut(U)$. Let $N : U \to U$ be a SFN, $Ag : U^n \to U$ be an aggregation function and $J_N : U^2 \to U$ be a N-dual operator of a coimplication $J : U^2 \to U$ satisfying properties* **J2**, **J8**, **J9** *and* **J10**, *according with Eq. (8). Then, the mappings $E_J, E_{J^\phi} : \mathcal{A} \to U$ defined by*

$$E_J(A) = Ag_{i=1}^n J\left(N(1-\nu_A(\mathrm{x}_i)), N(\mu_A(\mathrm{x}_i))\right), \tag{26}$$

$$E_{J^\phi}(A) = Ag_{i=1}^n J^\phi\left(N^\phi(1-\nu_A(\mathrm{x}_i)), N^\phi(\mu_A(\mathrm{x}_i))\right). \tag{27}$$

are also Atanassov's intuitionistic fuzzy entropies.

Proposition 10. *[3, Prop. 10] Let $E_J, E_{J_N} : \mathcal{A} \to U$ be A-IFEs according with Propositions 8 and 9. Then, for all $A \in \mathcal{A}$, the following holds:*

$$E_{J_N}(A) = E_J(A) \text{ and } E_{I_N}(A) = E_I(A) \tag{28}$$

5 Expressing fuzzy measures based on A-GIFIx

In this section, several IFEMs discussed by M.Liu and H.Ren [17] are reported:

$$[18]\ E_1(A) = -\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{2}(2\mu_A(\mathrm{x}_i) + \mathit{\Pi}_A(\mathrm{x}_i)) \cdot \log_2 \frac{1}{2}(2\mu_A(\mathrm{x}_i) + \mathit{\Pi}_A(\mathrm{x}_i)) + \frac{1}{2}(2\nu_A(\mathrm{x}_i) + \mathit{\Pi}_A(\mathrm{x}_i)) \cdot \log_2 \frac{1}{2}(2\nu_A(\mathrm{x}_i) + \mathit{\Pi}_A(\mathrm{x}_i))\right] \tag{29}$$

$$[18]\ E_2(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\sqrt{2}cos(\mu_A(\mathrm{x}_i) - \nu_A(\mathrm{x}_i))\frac{\pi}{4} - 1\right)\frac{1}{\sqrt{2}-1}\right] \tag{30}$$

$$[19]\ E_3(A) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\left[\frac{1}{2}\left(2\mu_A(\mathrm{x}_i) + \mathit{\Pi}_A(\mathrm{x}_i)) \cdot e^{1-\frac{1}{2}(2\mu_A(\mathrm{x}_i)+\mathit{\Pi}_A(\mathrm{x}_i))} + \frac{1}{2}(2\nu_A(\mathrm{x}_i) + \mathit{\Pi}_A(\mathrm{x}_i))e^{1-\frac{1}{2}(2\nu_A(\mathrm{x}_i)+\mathit{\Pi}_A(\mathrm{x}_i))}\right) - 1\right] \tag{31}$$

$$[20]\ E_4(A) = \frac{1}{n}\sum_{i=1}^{n} cos\left(\frac{\mu_A(\mathrm{x}_i) - \nu_A(\mathrm{x}_i)}{(1+\mathit{\Pi}_A(x_i))}\frac{\pi}{4}\right) \tag{32}$$

$$[21]\ E_5(A) = \frac{1}{n}\sum_{i=1}^{n} cot\left(\frac{\pi}{4} + \frac{|\mu_A(\mathrm{x}_i) - \nu_A(\mathrm{x}_i)|}{(1+\mathit{\Pi}_A(\mathrm{x}_i))}\frac{\pi}{4}\right) \tag{33}$$

$$[17]\ E_6(A) = \frac{1}{n}\sum_{i=1}^{n} cot\left(\frac{\pi}{4} + (|\mu_A(\mathrm{x}_i) - \nu_A(\mathrm{x}_i)| * (1 - \mathit{\Pi}_A(\mathrm{x}_i)))\frac{\pi}{4}\right) \tag{34}$$

Moreover, E_7 is obtained from Eq.(10)a by taking the arithmetic mean:

$$[8]\ E_7\,(A) = \sum_{i=1}^{4} \mathit{\Pi}_{I_{S,N_S}}(\mu_A(\mathrm{x}_i), \nu_A(\mathrm{x}_i)). \tag{35}$$

For any positive real number n, consider $\chi = \{x_1, x_2, ..., x_n\}$ to express the A-IFS $A^n = \{(\mathrm{x}_i, (\mu_A(\mathrm{x}_i)^n, 1-(1-\nu_A(\mathrm{x}_i))^n) : \mathrm{x}_i \in \chi\}$. Thus, when $\chi = \{6, 7, 8, 9, 10\}$, let $A = \{(6, 0.1, 0.8), (7, 0.3, 0.5), (8, 0.6, 0.2), (9, 0.9, 0.0), (10, 1.0, 0.0)\}$ as the characterization of linguistic variables treated as follows: $A^{1/2}$ is "rather large"; A is "large"; A^2 is "quite large"; A^3 is"very large"; A^4 is "extremely large".

The above entropy measures are compared based on four expressions of an A-GIFIx in Table 1. They are also preserved by the proposed methodology related to the ordered set of fuzzy implications [22] $\{I_{GR}, I_0, I_{RB}, I_{LK}\}$ as presented from E_6 in Tables from 1 to 4, in Figure 1, respectively. It may be also mentioned that from a logical consideration, the related entropies of these IFSs also follow the pattern proposed in [17], taking Π_{LK} from E_1 from E_6 in Table 1, respectively: If $A^{1/2} < A < A^2 < A^3 < A^4$ then $E(A^{1/2}) > E(A) > E(A^2) > E(A^3) > E(A^4)$.

Π_{LK}	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A^{\frac{1}{2}}$	0,5499	0,5051	0,5105	0,8659	0,3645	0,3685	0,0923
A^1	0,5496	0,4939	0,5054	0,8685	0,3564	0,3632	0,12
A^2	0,4588	0,3952	0,4064	0,8436	0,3338	0,3407	0,132
A^3	0,3895	0,3330	0,3437	0,8262	0,2512	0,2643	0,1344
A^4	0,3466	0,2937	0,3044	0,8146	0,2141	0,2313	0,1359

Π_{RB}	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A^{\frac{1}{2}}$	0,5095	0,4669	0,8474	0,8701	0,3847	0,3982	0,1757
A^1	0,5231	0,5003	0,7844	0,8732	0,3749	0,3911	0,19
A^2	0,4427	0,4123	0,5697	0,8466	0,3407	0,3501	0,1733
A^3	0,3821	0,3508	0,4458	0,8281	0,2577	0,2748	0,1604
A^4	0,3430	0,3098	0,3697	0,8161	0,2193	0,2399	0,1528

Π_{GR}	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A^{\frac{1}{2}}$	0,0146	0,0593	4,4133	0,9189	0,5322	0,8	0,8
A^1	0,0474	0,1403	4,2003	0,9184	0,5203	0,8	0,8
A^2	0,02178	0,2047	4,0138	0,9109	0,5045	0,8	0,8
A^3	0,0019	0,3595	3,9410	0,9061	0,4511	0,8	0,8
A^4	0,0176	0,4541	3,9004	0,9032	0,4271	0,8	0,8

Π_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A^{\frac{1}{2}}$	0,4690	0,4329	1,1199	0,8716	0,3954	0,4182	0,2352
A^1	0,4919	0,4854	1,0072	0,8747	0,3842	0,4088	0,24
A^2	0,4070	0,3942	0,8030	0,8472	0,3425	0,3552	0,224
A^3	0,3687	0,3571	0,5444	0,8287	0,2616	0,2829	0,1832
A^4	0,3383	0,3179	0,4138	0,8167	0,2221	0,2454	0,1635

Fig. 1. Relationship among A-IFE related to Π_{LK}, Π_{RB}, Π_{GR} and Π_0.

6 Conclusion

In this work, the concept of generalized Atanassov's intuitionistic fuzzy index was studied by dual and conjugate construction methods, in particular, by means of fuzzy (S, N)- and (T, N)-operators. We also extend the study of Atanassov's intuitionistic fuzzy entropy based on such two methodologies. Further work considers the extension of such study related to properties verified by the A-GIFIx(N_I) and A-IFE following the interval-valued intuitionistic fuzzy approach also including the study of correlation [23].

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References

1. E. Barrenechea Tartas, H. Bustince, M. Pagola, J. Fernandez, and J. Sanz. Generalized Atanassov's intuitionistic fuzzy index construction method. In *IFSA EUSFLAT Conference, Proceedings of the 8th Conf. of the European Society for Fuzzy Logic and Technology, Lisbon, Portugal, 2009*, pages 478–482, 2009.

2. H. Bustince, E. Barrenechea Tartas, M. Pagola, J. Fernandez, C. Guerra, P. Couto, and P. Melo-Pinto. Generalized Atanassov's intuitionistic fuzzy index: Construction of atanassov's fuzzy entropy from fuzzy implication operators. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 19(01):51–69, 2011.
3. L. Costa, M. Matzenauer, R. Zanotelli, A. Yamin R. Reiser, and M. Pilla. Atanassov's intuitionistic fuzzy entropy: Conjugation and duality. In M. Mesquita, G. Dimuro, R. Santiago, and E. Laureando, editors, *CBFS 2016 4th Brasilian Congress of Fuzzy Systems*, pages 31–42. SBMAC, 2016.
4. L. Lin and Z. Xia. Intuitionistic fuzzy implication operators: Expressions and properties. *Journal of Applied Mathematics and Computing*, 22(3):325–338, 2006.
5. E. Klement and M. Navara. A survey on different triangular norm-based fuzzy logics. *Fuzzy Sets and Systems*, 101(2):241–251, 1999.
6. H. Bustince, P. Burillo, and F. Soria. Automorphisms, negations and implication operators. *Fuzzy Sets and Systems*, 134(2):209– 229, 2003.
7. R. Reiser and B. Bedregal. K-operators: An approach to the generation of interval-valued fuzzy implications from fuzzy implications and vice versa. *Information Sciences*, 257(1):286 – 300, 2014.
8. V. Torra. Aggregation operators and models. *Fuzzy Sets and Systems*, 156(3):407–410, 2005.
9. A. Pradera G. Beliakov and T. Calvo. *Aggregation Functions – A Guide for Practitioners*. Springer Publishing Company, Incorporated, 2008.
10. M. Grabisch, J. Marichal, R. Mesiar, and E. Pap. *Aggregation Functions*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 2009.
11. J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publisher, Dordrecht, 1994.
12. M. Baczynski and B. Jayaram. On the characterization of (S,N)-implications. *Fuzzy Sets and Systems*, 158(15):1713–1727, 2007.
13. E. Trillas and L. Valverde. On implication and indistinguishability in the setting of fuzzy logic. In J. Kacprzyk and R. Yager, editors, *Management Decision Support Systems using Fuzzy Sets and Possibility Theory*, pages 198–212. Verlag TUV Rheinland, Cologne, 1985.
14. K. Atanassov and G. Gargov. Elements of intuitionistic fuzzy logic. *Fuzzy Sets and Systems*, 9(1):39–52, 1998.
15. C. Costa, B. Bedregal, and A. Dória Neto. Relating De Morgan triples with Atanassov's intuitionistic De Morgan triples via automorphisms. *International Journal of Approximate Reasoning*, 52:473–487, 2011.
16. M. Baczynski. Residual implications revisited. Notes on the Smets-Magrez. *Fuzzy Sets and Systems*, 145(2):267–277, 2004.
17. M. Liu and H. Ren. A new intuitionistic fuzzy entropy and application in multi-attribute decision making. *Information*, 5(4):587–601, 2014.
18. J. Ye. Two effective measures of intuitionistic fuzzy entropy. *Computing*, 87(1):55–62, 2010.
19. R. Verma and B. Sharma. Exponential entropy on intuitionistic fuzzy sets. *Kybernetika*, 49(1):114–127, 2013.
20. C. Wei, Z. Gao, and T. Guo. An intuitionistic fuzzy entropy measure based on trigonometric function. *Control and Decision*, 27(4):571–574, 2012.
21. Z. Yue, Y. Jia, and G. Ye. An approach for multiple attribute group decision making based on intuitionistic fuzzy information. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 17(03):317–332, 2009.
22. M. Baczyński and B. Jayaram. *Fuzzy implications*, volume 231 of *Studies in Fuzziness and Soft Computing*. Springer, Berlin, 2008.
23. R. Reiser and B. Bedregal. Correlation in interval-valued Atanassovs intuitionistic fuzzy sets conjugate and negation operators. *International Journal of Uncertainty, Fuzziness and Knowledge Based Systems*, 25(5):787–819, 2017.

Fuzzy versus probability: A discussion

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Abstract. There are in the literature various approaches to uncertainty and a vast debate about the advantages of one over another. In this work we are interested only in two of them namely fuzzy and probability theories because of the statement "anything that can be done with fuzzy logic ... or any other alternative to probability can better be done with probability" [1]. In this paper we argue that such an assertion is incomprehensible because each one of these theories arise from different approaches in the treatment of uncertainties and, consequently, their formal solutions for a given problem are not comparable.

Keywords: Probability measures, fuzzy sets, probability distribution, membership function.

1 Introduction

There is in the literature a debate about the necessity or not of fuzzy set theory in the mathematical treatment of uncertainty [1]. There are those who say that "anything that can be done with fuzzy logic ... or any other alternative to probability can better be done with probability" - this corresponds to a statement expressed by Lindley in [2] which was slightly adapted by Klir in his book (see page 315 of [1]) - other similar claims can be found, for instance, in [2–5]. Our goal is to provide some further insights that might add clarity to this debate.

The first theory designed to establish adequate mathematical tools to qualify and quantify uncertainties, initially with a "frequencist viewpoint", was the probability. Probability theory is an axiomatic theory based on measure theory. Fuzzy set theory is a much more recent theory. This theory was developed to deal with sets (or classes) with uncertainties (imprecision, subjectivity) at their borders. As the name itself suggests, it is a mathematical theory of sets designed to investigate the membership of elements of a universe in "subsets" [6, 7].

Fuzzy set theory is an extension of set theory and, therefore, it deals only with membership relation. In this respect, the theory of fuzzy sets is totally different from the axiomatic theory of probability which involves measure theory. Thus, a comparison of probability and fuzzy set theory necessarily involves a comparison

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of probabilities with crisp sets because they are particular cases of fuzzy sets. Here, we will treat both set theory and probability theory in an axiomatic way.

We do not intend to address general uncertainty theory. Here, we will deal with uncertainty given by means of fuzzy sets and random variables. In what follows, we will briefly present each one of these theories.

2 Basic Concepts of Probability Theory

The basic concepts of probability are well-known. They are presented here for the sake of clarity of our point of view.

Let Ω be a non-empty set and let $\mathcal{A}$ be a class of subsets of Ω. We say that $\mathcal{A}$ is a σ-algebra in Ω if it satisfies the following properties:

A1: $\Omega \in \mathcal{A}$ and $A^c \in \mathcal{A}$ whenever $A \in \mathcal{A}$, where $A^c := (\Omega \setminus A)$;
A2: if $A_i \in \mathcal{A}$, $i \in \mathbb{N}$, then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$.

The pair $(\Omega, \mathcal{A})$ is called $\mathcal{A}$-measurable space, or simply measurable space. In particular, the σ-algebra generated by all open subsets of $\mathbb{R} = \Omega$ is called *Borel* σ-algebra and is denoted by the symbol $\mathcal{B}$.

Definition 1 (Probability Measure). *Let $\mathcal{A}$ be a σ-algebra of a set Ω. A function P from $\mathcal{A}$ to $\mathbb{R}^+ := [0, +\infty)$ is said to be a* **probability measure** *if P satisfies the following properties:*

P1: $P(A) \leq 1$ for all $A \in \mathcal{A}$ and $P(\Omega) = 1$;
P2: If $A_1, \ldots, A_n, \ldots \in \mathcal{A}$, $A_i \bigcap A_j = \emptyset$, $i \neq j$, then $P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$.

The triple $(\Omega, \mathcal{A}, P)$ is called a probability space and $A \in \mathcal{A}$ is called an event.

One can show that a probability space $(\Omega, \mathcal{A}, P)$ also satisfies the following:

P4: $P(\emptyset) = 0$ and $P(A) \leq P(B)$ if $A \subseteq B$, $A, B \in \mathcal{A}$;
P5: $P(A) = 1 - P(A^c)$ for all $A \in \mathcal{A}$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $(E, \mathcal{E})$ be a measurable space where $E \subseteq \mathbb{R}$. A function $X : \Omega \to E$ is said to be a random variable if

$$X^{-1}(B) := \{\omega \in \Omega \,|\, X(\omega) \in B\} \in \mathcal{A}, \quad \forall B \in \mathcal{E}.$$

The composition of X with a continuous function $g : E \to E$, that is, $g \circ X : \Omega \to E$, produces another random variable that we denoted by the symbol $g(X)$. Given a random variable X, one can induce a probability measure $\bar{P}$ on the measurable space $(E, \mathcal{E})$ given by $\bar{P}(B) = P(X^{-1}(B))$ for all $B \in \mathcal{E}$. Thus, the range of the random variable X can be associated with the probability space $(E, \mathcal{E}, \bar{P})$. At this point, probability theory bifurcates in two cases: discrete and continuous. The first occurs when the range of X is an enumerable set. The second case occurs when the range of X is non-enumerable and there exists a function $f : \mathbb{R} \to \mathbb{R}^+$, called density of probability, such that $\bar{P}(B) = \int_B f(x)dx = P(X^{-1}(B))$ for all $B \in \mathcal{E}$. Conversely, we can obtain a probability measure from a given probability density function.

3 Fuzzy Set Theory

Fuzzy set theory only concerns itself with the membership relation within classical set theory [6], that is, its aim is to decide the membership degree of an element into a given set. Nevertheless, here, we will not present the axioms of classical set theory, which can be found in [6, 7].

Each subset A of Ω is uniquely associated with its indicator (or characteristic) function $\chi_A : \Omega \to \{0,1\}$ where $\chi_A(\omega) = 1$ if $\omega \in A$ and $\chi_A(\omega) = 0$ if $\omega \notin A$. The notion of fuzzy subset is based on the extension of the characteristic function as follows. A fuzzy (sub)set A of Ω is characterized by a unique function $\varphi_A : \Omega \to [0,1]$ called membership function where $\varphi_A(\omega)$ represents the membership degree of ω in A. The class of fuzzy sets of Ω is denoted by $F(\Omega) \equiv \{\varphi : \Omega \to [0,1]\}$.

Initially, let us remember that each fuzzy set A with membership function $\varphi_A : \Omega \to [0,1]$ can be associated with the following family of subsets of Ω

$$[A]^\alpha = \{\omega \in \Omega \,|\, \varphi_A(\omega) \geq \alpha\} \text{ for } 0 \leq \alpha \leq 1.$$

The sets $[A]^\alpha \subseteq \Omega$ are called α-levels (or α-cuts) of A.

The following theorem is of paramount importance in the study fuzzy set theory and indicates a necessary and sufficient condition for a family of classical subsets of Ω to be α-levels of a fuzzy subset of Ω [8].

Theorem 1 (The Representation Theorem of Negoita and Ralescu). *Let $\{A_\alpha\}_{\alpha\in[0,1]}$ be a family of classical subsets of Ω satisfying*

(i) $A_0 = \Omega$ *and* $A_\alpha \subset A_\beta$ *if* $\beta \leq \alpha$ *for* $\alpha, \beta \in [0,1]$*;*
(ii) $A_\alpha = \bigcap_{k\geq 0} A_{\alpha_k}$ *if* α_k *converge to* α *with* $\alpha_k \leq \alpha$.

Under these conditions there exists a unique fuzzy subset A of Ω such that $[A]^\alpha = A_\alpha$ for all $\alpha \in [0,1]$.

Given a family of subsets $\{A_\alpha\}_{\alpha\in[0,1]}$ satisfying (i)-(ii) of Theorem 1, the membership function of the fuzzy set A such that $A_\alpha = [A]^\alpha$, for all $\alpha \in [0,1]$, is given by [8]:

$$\varphi_A(\omega) = \sup\{\alpha \in (0,1] \,|\, \omega \in A_\alpha\}, \ \forall \omega \in \Omega, \tag{1}$$

where $\sup \emptyset = 0$. The next theorem is a consequence of (1) and Theorem 1.

Theorem 2. *Let $A, B \in F(\Omega)$. If $[A]^\alpha = [B]^\alpha$ for all $\alpha \in (0,1)$, then $A = B$.*

Proof. Let $\alpha_k = 1 - \frac{1}{1+k}$, for $k \geq 0$. From item (ii) of Theorem 1, we have that

$$[A]^1 = \bigcap_{k\geq 0} [A]^{\alpha_k} = \bigcap_{k\geq 0} [B]^{\alpha_k} = [B]^1.$$

Equation (1) implies that $A = B$.

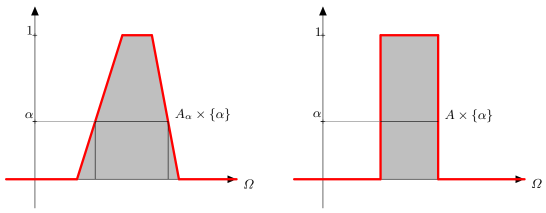

Fig. 1. From left to right, the gray area correspond respectively to the $\psi(\varphi_A)$ of a membership function of A and a characteristic function of A.

Next, let us provide some arguments that reinforces the use of the name "fuzzy set" for functions whose codomain is the unit interval $[0,1]$. Let

$$G = \Big\{ \bigcup_{\alpha \in (0,1)} (A_\alpha \times \{\alpha\}) \,|\, \{A_\alpha\} \text{ is a family satisfying (i)-(ii) of Theorem 1} \Big\}.$$

By Theorems 1 and 2, there is a bijection ψ between $F(\Omega)$ and G. This already allows us to identify each element A of $F(\Omega)$ as a subset $\psi(\varphi_A)$ of $\Omega \times (0,1)$ (see Figure 1).

On the other hand, ψ is also a bijection between $\chi(\Omega) = \{\chi : \Omega \to \{0,1\}\} \subset F(\Omega)$ and the subset $H \subset G$ given by

$$H = \Big\{ \bigcup_{\alpha \in (0,1)} (A_\alpha \times \{\alpha\}) \,|\, A_\alpha = A, \forall \alpha \in (0,1), A \subseteq \Omega \Big\}.$$

More precisely, for each $A \subseteq \Omega$ we have that $\psi(\chi_A) = A \times (0,1)$. Thus, we can interpret each element of H as a subset of Ω since there is a bijection between $\chi(\Omega)$ and $P(\Omega)$. By extension, we can also call the elements of $F(\Omega)$ as a "type of subsets" of Ω, more specifically, as fuzzy subsets of Ω. In short, we can identify $F(\Omega)$ with (classical) subsets of $\Omega \times (0,1)$ by means of the bijection ψ, that is, $\psi(F(\Omega)) \equiv G$ (see Figure 2).

One can verify that the set G is closed with respect to the operations of intersection and union of sets, that is, if $\mathcal{A}, \mathcal{B} \in G$ then $\mathcal{A} \cap \mathcal{B}, \mathcal{A} \cup \mathcal{B} \in G$. More than that, for $A, B \in F(\Omega)$ such that $\mathcal{A} = \psi(\varphi_A)$ and $\mathcal{B} = \psi(\varphi_B)$, we have that $\mathcal{A} \cap \mathcal{B} = \psi(\varphi_{A \cap B})$ and $\mathcal{A} \cup \mathcal{B} = \psi(\varphi_{A \cup B})$, where $A \cap B$ and $A \cup B$ correspond respectively to the usual intersection and union of fuzzy subsets, that is, $\varphi_{A \cap B}(\omega) = \min\{\varphi_A(\omega), \varphi_B(\omega)\}$ and $\varphi_{A \cup B}(\omega) = \max\{\varphi_A(\omega), \varphi_B(\omega)\}$ for all $\omega \in \Omega$. Moreover, the class H is closed with respect to the complement since the elements of H are of the form $A \times (0,1)$, $A \subseteq \Omega$, and, thereby, $(A \times (0,1))^c = A^c \times (0,1) \in H$. However, for any $\mathcal{A} \in G \setminus H$ we have that $\mathcal{A}^c = \Omega \times (0,1) \setminus \mathcal{A}$ does not belong to G. Hence, G is not closed with respect to the complement. This fact means that the law of excluded middle in the class G only holds if $\mathcal{A} \in H$, that is, if the corresponding fuzzy set A is crisp and so supports our claim that fuzzy set theory is an extension of classical set theory. This is a well-known fact in fuzzy set theory [9] that can be clarified as follows. Let $N : G \to G$ be the

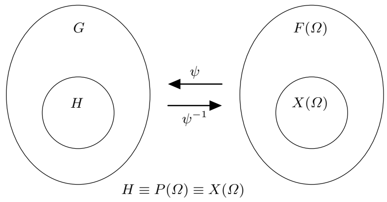

Fig. 2. One-to-one mapping between G and $F(\Omega)$ as well as between H and $\chi(\Omega)$.

function given by

$$N(\mathcal{A}) = \bigcup_{\alpha\in(0,1)} \left(([A]^{1-\alpha})^c \times \{\alpha\}\right), \quad \forall \mathcal{A} \in G,$$

where A is the fuzzy set of Ω such that $\mathcal{A} = \psi(\varphi_A)$, *i.e.*, $\mathcal{A} = \bigcup_{\alpha\in(0,1)} ([A]^\alpha \times \{\alpha\})$. The function N can be viewed as a generalization of the complement operator defined on H to the class of G since $N(A \times (0,1)) = A^c \times (0,1)$ for every $A \subseteq \Omega$. Note that the function N is related with the usual notion of complement of fuzzy set. In particular, we have that $\mathcal{A} = \psi(\varphi_A)$ if, and only if, $N(\mathcal{A}) = \psi(\varphi_{A^c})$, where A^c denotes the complement of the fuzzy set A given by $\varphi_{A^c}(\omega) = 1 - \varphi_A(\omega)$ for all $\omega \in \Omega$.

Recall that a fuzzy set A is contained in another fuzzy set B if $\varphi_A(\omega) \leq \varphi_B(\omega)$ for all $\omega \in \Omega$ and this is denoted by the symbol $A \subseteq B$. Equivalently, we have that $A \subseteq B$ if, and only if, $[A]^\alpha \subseteq [B]^\alpha$ for all $\alpha \in [0,1]$. This last observation implies that the inclusion relation on G also coincides to the notion of inclusion between fuzzy sets. Specifically, we have that $A \subseteq B$ if, and only if, $\psi(\varphi_A) \subseteq \psi(\varphi_B)$.

The above comments establish a connection between fuzzy set theory and the classical set theory on $G \subset \mathcal{P}(\Omega \times (0,1))$. This allows us to use typical terms of set theory to $F(\Omega)$ and to focus on the relation of interest from axiomatic theory of sets: the relation of membership. In the fuzzy case, the membership function φ_A plays the role of this relation in the same way as the characteristic function χ_A plays it in the classic case.

4 Fuzzy Set versus Probability

We next present an elementary situation that helps to illustrate the similarities and differences between the probability and fuzzy set theory. Both theories deal with uncertainties about "subsets" of a non-empty universal set Ω. Moreover, in general, in both cases these uncertainties about a "subset" A can be represented by (classical) sets of pairs $(a, f(a))$, where $f(a)$ is a real number that objectively represents the "evaluation" of the uncertainty associated with the element a. In probability theory, Ω is called a sample space and for some cases it is nothing more than a set (universe) equipped with a (probability density) function f and a measure obtained from f, such as was discussed at the end of Section 2. The subsets of Ω on which the uncertainties are evaluated are called events. In this case, the measure, called probability measure, is a function of sets, that is, its domain is the set of the events of Ω. We point out that the "uncertainty" of all events are evaluated by the same probability measure, which in turn is obtained from a unique function f. In contrast, in fuzzy set theory, the function f, which expresses the uncertainty (*i.e.* the membership degree of a to the fuzzy set A), varies for each fuzzy set. Such a function is used to characterize/describe the fuzzy set. We can say that these functions (called membership) are related to fuzzy subsets in the same way that the characteristic functions are related to the (classical) subsets of Ω.

The preceding comments demonstrate that, although both probability and fuzzy set theory can be formulated in terms of sets of pairs $(a, f(a))$, from a modeling point of view, these two mathematical theories are completely different and we do not see how to compare the quantitative results obtained by applying their methods of inferences. The examples below help us to elucidate the fundamentals of each theory and the impossibility of comparing the solutions produced by each one.

Example 1 *Are the "quantities"* $< 1, 2, 2, 3 >$ *and* $< 1, 2, 3 >$ *equal?*

From set theory, the answer is yes because $\{1, 2, 2, 3\} = \{1, 2, 3\}$*, that is, they have the same elements (note that we are dealing with membership relation here!). From the point of view of probability, if these quantities represent balls with same size in two baskets, then the probability that someone randomly removes the element "2" in the first basket is greater than the second basket. In this context, if we regard these quantities as events of probability spaces, then they are distinct because in the first event the element "2" has a "weight" greater than the other numbers. In the second event, all elements have the same "weight", that is, it is assumed that they are uniformly distributed. In short, these quantities are equal if we regard them as sets whereas they may be different if we regard them as events of distinct probability spaces.*

Example 2 *Consider the problem: John is fifty-ísh (A). How old is John?*

Let x be the age of John. According to the theoretical frameworks of each theory, the answers could be:

(a) Set theory: $x \in [40, 60] = A$*;*
(b) Probability theory: $x \in [40, 60] = A$ *with probability* $P([40, 60]) = 0.95$*;*
(c) Fuzzy set theory: x *is* A *with the following membership function:*

$$\varphi_A(x) = \begin{cases} 1 - \frac{|x-50|}{10} & \text{if } 40 \leq x \leq 60 \\ 0 & c.c. \end{cases}.$$

Note that (a) and (c) can be compared since one can say that (c) is more specific than (a) because $\varphi_A(x) \leq \chi_A(x)$, for all $x \in \mathbb{R}$. On the other hand, it is impossible to compare the items (a) and (c) with item (b), which presents the probability of A. Differently from (a) and (c), item (b) represents the term "fifty-ish" as an event and, based on some "extra" information, the probability of this event ($P([40, 60])$). This example is corroborated by Figure 19.1 of page 320 in [1] which indicates that there is some hierarchy between the theories in which the set theory precedes the others.

The discussions of these two previous examples suggest that the notion of events with uncertain boundaries should be studied, that is, fuzzy events. From the modeling point of view, while fuzzy set theory can be used to describe the boundary of an event, probability presents a "measure" of its occurrence. In the second example above, it would be interesting if one could obtain the probability of item (c) occurring. In our opinion, what is really meaningful and fruitful is the combination of fuzzy and probability.

5 Fuzzy Event

The notion of fuzzy events, introduced by Zadeh [10], illustrates the potential of the combination of these two theories: fuzzy, which deals with the identification of the event, and probability which deals with the occurrence of events. From both theoretical and practical point of view, Nguyen and Wu developed the foundations of statistics to deal with fuzzy data [11]. Massad *et al.* [12] applied the concept of fuzzy event to assess the risk of HIV contamination for sexually active individuals.

Definition 2. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. A* fuzzy event *on Ω is a fuzzy subset of Ω such that each α-level belongs to σ-algebra $\mathcal{A}$.*

The question that arises is how to calculate the probability of A? Obviously, a classic event A is in agreement to the definition above and A can thus be considered a fuzzy event. Note that the indicator (or characteristic) function of A is a discrete random variable since $\chi_A : \Omega \to \{0, 1\}$. On the other hand, if A is fuzzy, then its membership function $\varphi_A : \Omega \to [0, 1]$ is a random variable that may not be discrete.

Let A be a fuzzy event. Zadeh argues that the probability of A ($P(A)$) should be a real number whereas Buckley [13], Ralescu [14], and others argue that $P(A)$ should be a fuzzy number, that is, $P(A) \in \mathcal{F}(\mathbb{R})$. Here, we follow the Zadeh's idea, that is, we consider $P(A)$ as a real number.

5.1 Probability of Fuzzy Events

Given a classical event A of an arbitrary probability space $(\Omega, \mathcal{A}, P)$. The characteristic function $\chi_A : \Omega \to \{0, 1\}$ of A corresponds to a discrete random variable. In this case, we have that $P(A) = E(\chi_A) = 1.P(\chi_A = 1) + 0.P(\chi_A = 0)$, where $E(X)$ denotes the expected value of the random variable X. In view of the above comments, one may suggest the following definition [9, 15].

Definition 3. *Let A be a fuzzy event of a probability space $(\Omega, \mathcal{A}, P)$ with membership function $\varphi_A : \Omega \to [0, 1]$. The probability of A is given by $P(A) = E(\varphi_A)$.*

In order to explore the concepts presented so far, let us consider a probability space for which the underlying σ-algebra is contained in $\mathcal{P}(\mathbb{R})$ and the probability measure P is induced by the probability distribution of a random variable X. Here, we simply call this a fuzzy event of the probability space of a real event. Let A be a real event with membership function $\varphi_A : \mathbb{R} \to [0,1]$. By Definition 3, the probability of A is given by

$$P(A) = E(\varphi_A) = E(\varphi_A(X)) = \sum_{i=1}^{n} \varphi_A(x_i) P(X = x_i) \tag{2}$$

if X is discrete and by

$$P(A) = E(\varphi_A) = E(\varphi_A(X)) = \int_{\mathbb{R}} \varphi_A(x) f(x) dx = \int_{suppA} \varphi_A(x) f(x) dx \tag{3}$$

if X is continuous, where f denotes the probability density function of X. Note that, from above equations, it is not difficult to verify that $P(A)$ satisfies the axioms of probability.

5.2 Independence between fuzzy events

The concept of independence between fuzzy events is necessarily related to the concept of conditional probability. In order to extend the classical notion of conditional probability for the fuzzy case, we must revisit the mathematical model of simultaneous occurrence of two events A and B. In the classical case, the simultaneous occurrence is given by intersection $A \cap B$. However, in order to extend to the fuzzy case, we must assess the function indicator $\chi_{A \cap B}$. Any *t-norm* models the conjunction "and", in particular, the *product* or the *minimum* t-norm: $\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x)$ or $\chi_{A \cap B}(x) = \chi_A(x) \wedge \chi_B(x)$. Since in the classical case usually adopts the product ". " t-norm and we are interested in its extension, we opt to use the *product* t-norm to represent simultaneous occurrence. The notion of conditional probability can be extended as follows.

Definition 4. *Consider A and B, two fuzzy events of $\mathbb{R}$, with $P(B) > 0$ ($\Leftrightarrow$ $E(\varphi_B) > 0$). The conditional probability of A given B is defined by*

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{E(\varphi_A . \varphi_B)}{E(\varphi_B)}. \tag{4}$$

Similarly to the classic case, A is said to be independent of B if, and only if,

$$\frac{E(\varphi_A . \varphi_B)}{E(\varphi_B)} = E(\varphi_A) \Leftrightarrow E(\varphi_A . \varphi_B) = E(\varphi_A).E(\varphi_B). \tag{5}$$

Note that the independence of A and B does not mean that the random variables φ_A and φ_B are uncorrelated, since Equation (5) does not involve a bivariate join distribution.

According to the formula (4) we do not always have $P(B|B) = 1$ if B is not classical set (crisp set). This is the object of criticisms of some authors [16] who prefer to adopt the t-norm of the minimum instead of the product in (4). However, with the t-norm of the minimum, we lose the generalization - from the classic case to the fuzzy case - of the probabilistic independence interpretation via formula (5). Moreover, in our view, it is not clear why $P(B|B) = 1$ if B is a non-crisp fuzzy event, since it is not clear which are its elements.

Let us emphasize another difference between sets and events. From the point of view of (fuzzy) set theory, we have "$(A|B) = (B|A)$" since "$(X|Y)$" can be interpreted as set formed by the points of Y which also belong to X. However, $P(A|B) \neq P(B|A)$ if $P(A) \neq P(B)$.

This section developed the concept of events whose probability is a real number. However, there are at least two other ways to introduce the study of fuzzy random variables from a random experiment. The first one is only linked to uncertainty once the result of the experiment are revealed, *i.e.*, the revelation of the event. In this case, the probability of the fuzzy event is necessarily a real number. In the second case, beyond the uncertainties at the boundaries of the events, the mechanism of the experiment itself is also uncertain, for example, by the procedure of the draw. In such a case the probability may be uncertain. In the literature, the notion of fuzzy random variable deals with these both cases [14].

6 Linguistic Random Variable

Given a probability space $(\Omega, \mathcal{A}, P)$. Let $X : \Omega \to \mathbb{R}$ be a random variable and let $(Y, \bar{\mathcal{A}}, \bar{P})$ be the probability space induced by X, $Y \subseteq \mathbb{R}$. Consider the proposition $\bar{P}(X \text{ is } A)$, where A represents a linguistic term (*high*, *low*, *very high* ...) given by a real fuzzy event A, that is, $A \in \mathcal{F}(Y)$. Note that A is a fuzzy event and, thus, we have that $\bar{P}(A) = E(\varphi_A)$.

On the other hand, if A is a classical (crisp) event, then, from the point of view of classical theory, one can translate $\bar{P}(X \text{ is } A)$ as $\bar{P}(X \in A)$ and, depending if X is a discrete or a continuous variable, we have that $\bar{P}(X \in A) = \sum_i \chi_A(x_i)\bar{P}(X = x_i)$ or $\bar{P}(X \in A) = \int_{\mathbb{R}} \chi_A(x) f(x)dx$.

The previous subsections have shown that $\bar{P}(X \in A) = E(\chi_A) = \bar{P}(A)$. Now, for the case where A is a real fuzzy set that models a linguistic term such as "low", "medium", "high", etc. for the the variable X', it is natural, according our previous discussion, to define $\bar{P}(X \text{ is } A) = E(\varphi_A) = \bar{P}(A)$. This idea leads us to the concept of a linguistic random variable, that is, a random variable that can be associated with linguist terms given by fuzzy events.

Note that the notion of linguistic random variable can be used, for example, to assess the probability "that a person has a low salary". In this example, the sample space Ω and the random variable X can be given respectively by the set of all workers and a mapping that associates each worker with her/his salary. Moreover, the linguistic term "low salary" can be given in terms of a fuzzy set.

The term fuzzy random variables is used to represent certain functions from a probability space to the class of fuzzy sets. Thus, fuzzy random variables can be viewed as an extension of the notion of random variables and there are at least two approaches to this concept in the literature. It is worth noting that the concept of linguistic random variable that has been presented here may not coincide with these concepts of fuzzy random variable. The reader interested in the concept of fuzzy random variables can consult [14].

7 Final Comments

This paper presents some ideas and reflections about the subject "Fuzzy versus Probability". In particular, we argue that such a discussion, from the mathematical point of view, does not make sense since each one deals with distinct and incomparable mathematical frameworks. In order to support our conclusion, we have shown that the solutions from each theory are formally incomparable. Probability theory requires the notion of measure in order to produce an answer to a given problem. In (fuzzy) set theory such a requirement is not necessary. From a theoretical point of view, given a universal set Ω, probability theory requires a function P called measure whose domain is a class (σ-algebra) of subsets of Ω. On the other hand, in fuzzy set theory, a function φ called membership whose domain is the universal set Ω is required. It is worth noting that, according to the end of Section 3, fuzzy set theory is indeed an approach of set theory on the universal set $G \subseteq \mathcal{P}(\Omega \times (0, 1))$.

From the modeling point of view, if the domain Ω is a sample space, then the function φ is used to describe/characterize a fuzzy event whereas the function P evaluates the probability of the (fuzzy) event defined by φ. Note that in the theory of probability, uncertainty or doubt about an event only exists before its occurrence, for example, we can evaluate the probability of victory of whoever bets in the lottery, however, after the draw, we will actually know who won. On the other hand, even after the occurrence of a fuzzy event A, the uncertainty of which elements of the sample space Ω belong to A persists. For example, suppose we are interested in evaluating the probability of a young person winning the lottery, where the term "young" is given by a fuzzy event. In this case, even after knowing who won the lottery after the draw, the uncertainty or doubt about the winner being young may prevail.

Finally, we would like to emphasize that the question of interest is the combination of these two theories and not the competition between them. In this context, we devoted Section 5 to a review an approach to combine fuzzy and probability through the notion of fuzzy events.

References

1. Klir, G.: Fuzzy Sets : An Overview of Fundamentals, Applications and Personal Views. Beijing Normal University Press (2000)

2. Lindley, D.V.: The probability approach to the treatment of uncertainty in artificial intelligence and expert systems. Statistical Science **2**(1) (1987) 17–24
3. Cheeseman, P.: In defense of probability. In: Proceedings of the 9th International Joint Conference on Artificial Intelligence - Volume 2. IJCAI'85, San Francisco, CA, USA, Morgan Kaufmann Publishers Inc. (1985) 1002–1009
4. Cheeseman, P.: Probabilistic versus fuzzy reasoning. Machine Intelligence and Pattern Recognition **4** (1986) 85–102 Uncertainty in Artificial Intelligence.
5. Lindley, D.V.: Scoring rules and the inevitability of probability. International Statistical Review / Revue Internationale de Statistique **50**(1) (1982) 1–11
6. Di Prisco, C.A.: Una introducción a la teoría de conjuntos y los fundamentos de las matemáticas. Volume 20. UNICAMP, Centro de Lógica, Epistemología e Historia da Ciência, Coleção CLE (1997)
7. Halmos, P.R.: Naive set theory. Springer Science & Business Media (1998)
8. Negoita, C.V., Ralescu, D.A.: Applications of Fuzzy Sets to Systems Analysis. John Wiley & Sons, New York (1975)
9. Barros, L., Bassanezi, R., Lodwick, W.: A First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics: Theory and Applications. Volume 347 of Studies in Fuzziness and Soft Computing. Springer-Verlag Berlin Heidelberg, Campinas, SP, Brazil (2017)
10. Zadeh, L.A.: Probability measures of fuzzy events. J. Math. Analysis and Applications **23** (1968) 421–427
11. Nguyen, H.T., Berlin, W.: Fundamentals of statistics with fuzzy data. Springer-Verlag Berlin Heidelberg (2006)
12. Massad, E., Ortega, N.R.S., Barros, L.C., Struchiner, C.J.: Fuzzy Logic in Action: Applications in Epidemiology and Beyond. Springer, Berlin, Heidelberg (2008)
13. Buckley, J.J.: Fuzzy Probability and Statistics. Studies in Fuzziness and Soft Computing. Springer (2006)
14. Gil, M.A., López-Diaz, M., Ralescu, D.A.: Overview on the development of fuzzy random variable. Fuzzy Sets and Systems **157** (2006) 2546–2557
15. Zadeh, L.A.: Fuzzy sets as a basis for a theory of possibility. Fuzzy sets and systems **1**(1) (1978) 3–28
16. Sousa, J.M.C., Kaymak, U., Vieira, S.M.: Probabilistic fuzzy system tutorial: Part ii - probabilistic fuzzy models. (Jul 2013) IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2013), Hyderabad, India.

Constructing Generalized Mixture Functions from Bounded Generalized Mixture Functions

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Abstract. Some functions commonly found in the literature have the ability to aggregate a finite amount of information into a single data, e.g. OWA functions, mixture functions, generalized mixture functions and bounded generalized mixture functions. OWA functions can be characterized as aggregation functions, while the others cannot because of the lack of monotonicity. In this paper we will present a way of building generalized mixture functions starting from bounded generalized mixture functions, but we will not discuss about monotonicity, considering that monotonicity is not an essential factor in applications as image processing. Besides that, we show that this process preserves properties as idempotency and homogeneity.

Keywords: Aggregation functions; OWA functions, Mixture functions, Generalized Mixture functions, Bounded Generalized Mixture functions

1 Introduction

In several areas of applications, we need a mechanism that allows grouping finite collections of information (inputs) into a single representative data. One of these data grouping devices are called of aggregation functions, that can be used, for example: To identify tumors [1–3], in techniques which support dental treatments [4–6], in decision making [7–11], in image processing [12], to combine degrees of membership of fuzzy sets [13], etc.

Aggregation functions must satisfy the monotonicity property, but it has already been proven that this condition is not necessary for some applications [14]. It was in this sense that Farias *et al.* [15] investigated a class of non monotonic functions called **Generalized Mixture Functions** which generalize OWA functions [16] and mixture functions [17] in order to provide an alternative. Still in [15], Farias *et al.* introduced the **Bounded Generalized Mixture Functions**, which generalize all the other functions mentioned. In addition, other properties and applications of these functions have been considered in [18, 19].

Generalized mixture functions and *bounded generalized mixture functions*, like OWAs, are obtained from weights. In the case of OWAs a constant vector of weights (i.e. a $\mathbf{w} = (w_1, \cdots, w_n) \in [0,1]^n$, with $\sum_{i=1}^{n} w_i = 1$), is chosen and a new aggregation function $\mathsf{OWA}_{\mathbf{w}}$ is obtained. But in the case of *mixture functions*, the vector of weights is variable, more precisely, the weights come from functions with input $\mathbf{x} = (x_1, \cdots, x_n)$ which results in a more flexible solution.

In this paper we recall the notion of *generalized mixture functions* and a way to construct them from a broader class of functions: *The Bounded Generalized Mixture functions.* To achieve that, the text is structured as follows: Section 2 introduces some preliminary notions: Aggregation, OWAs and Mixture functions. In section 3 we expose *Generalized Mixture functions* and *Bounded Generalized Mixture function*; we also show how to build generalized mixture functions from bounded generalized mixture functions as well as some important aspects of this construction. Finally, section 4 states some final remarks.

2 Preliminaries

In this section we present some required concepts for this work.

2.1 Aggregation Functions

The class of **Aggregation functions** [20, 21] is an important class of functions used in several fields of knowledge, which ranges from Statistics to Computer Science. The basic intuition is that they collect a finite amount of data and produce a representative data of it.

Definition 1 *A* **n-dimensional aggregation function** *is an isotonic function* $A : [0,1]^n \longrightarrow [0,1]$ *which satisfies the boundary condition:* $A(0, \cdots, 0) = 0$ *and* $A(1, \cdots, 1) = 1$.

Applications of aggregation functions can be found, for example, in fuzzy logic [13, 22–24], in decision making problems [7, 9–11] and in image processing [1, 12, 25]. In addition, there are four types of aggregation functions: **Conjunctive**, **Disjunctive**, **Averaging** and **Mixed**.

Definition 2 *For any* $(x_1, \cdots, x_n) \in [0,1]^n$*, a function* $A : [0,1]^n \longrightarrow [0,1]$ *is:*

1. *Conjunctive, if* $\min(x_1, \cdots, x_n) \geq A(x_1, \cdots, x_n)$ *;*
2. *Disjunctive, if* $A(x_1, \cdots, x_n) \geq \max(x_1, \cdots, x_n)$*;*
3. *Averaging, if* $\min(x_1, \cdots, x_n) \leq A(x_1, \cdots, x_n) \leq \max(x_1, \cdots, x_n)$*;*
4. *Mixed, if it is neither conjunctive nor disjunctive nor averaging.*

Example 1 *The Minimum and Maximum functions, the Arithmetic Mean and the Weighted Average Mean are examples of Averaging aggregation functions.*

In this paper we investigate a subclass of averaging functions. We will not go into details on conjunctive, disjunctive and mixed functions. For further information we suggest the references: [20, 21].

Some aggregations satisfy important algebraic properties. Those properties are: **Idempotency (IP)**, **Homogeneity of order k ($\mathbf{HP_k}$)**; has **Neutral Element (NP)**; has **Annihilator element (AP)**; **Symmetry (SP)**; **Shift-invariance (SHP)**; has **Zero Divisor (ZP)** and has **One Divisor (OP)**. These properties are detailed below and **Table 1** shows some aggregations related with them.

Definition 3 *A function $A : [0, 1]^n \to [0, 1]$ satisfies:*

- **(IP)** *if, $A(x, \cdots, x) = x$, for all $x \in [0, 1]$;*
- **($\mathbf{HP_k}$)** *if, $A(\lambda x_1, \cdots, \lambda x_n) = \lambda^k \cdot A(x_1, \cdots, x_n)$, for any $\lambda \in [0, 1]$ and $(x_1, \cdots, x_n) \in [0, 1]^n$;*
- **(NP)** *if there is an element $e \in [0, 1]$ such that $A(e, \cdots, e, x_i, e, \cdots, e) = x_i$, for all $x_i \in [0, 1]$ and any coordinate $i \in \{1, \cdots, n\}$;*
- **(AP)** *if there is an element $a \in [0, 1]$ such that for all $i \in \{1, \cdots, n\}$ and $(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) \in [0, 1]^n$, we have $A(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) = a$;*
- **(SP)** *if $A(\sigma(\mathbf{x})) = A(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = A(x_1, \cdots, x_n)$, for any permutation $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$ and any $\mathbf{x} \in [0, 1]^n$;*
- **(SHP)** *if for all $\lambda \in [-1, 1]$, chosen properly, we have $A(x_1 + \lambda, \cdots, x_n + \lambda) = A(x_1, \cdots, x_n) + \lambda$ (or $A(\mathbf{x} + \lambda) = A(\mathbf{x}) + \lambda$);*
- **(ZP)** *if there is a element $a \in (0, 1)$ such that for any coordinate i and any $(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) \in (0, 1]^n$ we have $A(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) = 0$;*
- **(OP)** *if there is a element $a \in (0, 1)$ such that for any coordinate i and any $(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) \in [0, 1)^n$ we have $A(x_1, \cdots, x_{i-1}, a, x_{i+1}, \cdots, x_n) = 1$.*

Table 1. Aggregation functions and their properties

Function	Properties
max	**(IP)**, **($\mathbf{HP_1}$)**, **(NP)**, **(AP)**, **(SP)** and **(SHP)**
min	**(IP)**, **($\mathbf{HP_1}$)**, **(NP)**, **(AP)**,**(SP)** and **(SHP)**
Arith	**(IP)**, **($\mathbf{HP_1}$)**, **(SP)** and **(SHP)**
SBound	**(IP)**, **($\mathbf{HP_1}$)**, **(SP)** and **(SHP)**
WAvg	**(IP)**, **($\mathbf{HP_1}$)** and **(SHP)**
Prod	**(NP)**, **(AP)** and **(SP)**
GMean	**(AP)** and **(SP)**
$\max(x + y - 1.5, 0)$	**(SP)** and **(ZP)**
$\min(2.5 - x - y, 1)$	**(SP)** and **(OP)**

for $\mathbf{SBound}(x_1, \cdots, x_n) = min\left\{1, \sum_{i=1}^{n} x_n\right\}$, $\mathbf{Prod}(x_1, \cdots, x_n) = \prod_{i=1}^{n} x_i$ and $\mathbf{GMean}(x_1, \cdots, x_n) = \sqrt{x_1 x_2 \cdots x_n}$.

2.2 OWA Functions

The **Ordered Weighted Averaging (OWA) Functions** were introduced by Yager [16] and provide an important subclass of averaging aggregation functions. An OWA function is a parametric aggregation, it produces an aggregation, $\mathsf{OWA}_{\mathbf{w}}$, according to a fixed vector of weights, $\mathbf{w}$. Some of its applicability can be found in: [10, 12, 15, 22].

Definition 4 *An* **Ordered Weighted Averaging - OWA Function** *is:*

$$\mathit{OWA}_{\mathbf{w}}(x_1, \cdots, x_n) = \sum_{i=1}^{n} w_i x_{(i)},$$

where $\mathbf{w} = (w_1, \cdots, w_n) \in [0,1]^n$ *is a vector of weights, i.e.,* $\sum\limits_{i=1}^{n} w_i = 1$*, and* $(x_{(1)}, \cdots, x_{(n)}) = Sort(x_1, \cdots, x_n)$ *is the the input vector* $(x_1, \cdots, x_n)$ *decreasingly ordered.*

Example 2 *The functions* min*,* max*, Arith e WAvg are example of OWAs, with vector of weights* $\mathbf{w}_{\min} = (0, \cdots, 0, 1)$*,* $\mathbf{w}_{\max} = (1, 0, \cdots, 0)$*,* $\mathbf{w}_{Arith} = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$ *and* $\mathbf{w}_{WAvg} = \left(w_{(1)}, \cdots, w_{(n)}\right)$*. More examples of OWA can be found in [15, 19].*

Proposition 1 *If* $\mathbf{w} = (w_1, \cdots, w_n) \in [0,1]^n$ *is the vector of weights, then*

1. $\mathit{OWA}_{\mathbf{w}}$ *is a continuous averaging aggregation function that satisfies* **(IP)***,* **(HP)***,* **(SP)** *and* **(SHP)***;*
2. $\mathit{OWA}_{\mathbf{w}}$ *satisfy* **(NP)** *or* **(AP)** *if, and only if,* $\mathit{OWA}_{\mathbf{w}} = \max$ *or* $\mathit{OWA}_{\mathbf{w}} = \min$*.*

OWA functions are obtained from fixed vector of weights. It means that an application programmer must appropriately choose beforehand the vector of weights. In other words, this choice must be done externally to the algorithms and there is no place for algorithms to choose dynamically an appropriate vector of weights. However, there are other functions in which the vector of weights is not fixed, they are the mixture functions [17].

2.3 Mixture Functions and Generalized Mixture Functions

Beyond OWAs, there are other kinds of parametric functions which generalize OWAs. They are called **Mixture functions**, coming from the **Bajraktarevic mean** [17] and having the following form:

$$M(\mathbf{x}) = \frac{\sum\limits_{i=1}^{n} w_i(x_i)x_i}{\sum\limits_{i=1}^{n} w_i(x_i)},$$

with $w_i : [0,1] \longrightarrow [0, +\infty)$.

Since each w_i is a function, then the vectors of weights $(w_1(x_1), \ldots, w_n(x_n))$ are not fixed beforehand.

The Mixture Functions generalize OWAs, but are not always isotonic. They were also generalized, first by Pereira and Pasi [26] and subsequently by Farias *et al* [15, 19, 27, 28] to what is called **Generalized Mixture Functions**.

Definition 5 ([15, 19]) *A* ***Generalized Mixture function*** *(or simply GM function) is a function* $\mathsf{GM}_\Gamma : [0,1]^n \to [0,1]$ *given by:*

$$\mathsf{GM}_\Gamma(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) x_i,$$

where $\Gamma = \{f_i : [0,1]^n \to [0,1] | \ 1 \leq i \leq n\}$*, with* $\sum_{i=1}^{n} f_i(\mathbf{x}) = 1$*. The class of all Generalized Mixture functions is denoted by:* $\mathbb{GM}$*.*

Example 3
- *The functions:* min*,* max*, Arith, WAvg are examples of GM functions [15].*
- *Any OWA function is also a GM function, with* $f_i(\mathbf{x}) = w_{q(i)}$*, where* $q : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ *is the inverse permutation of* $p(i) = (i)$ *obtained in* $Sort(x_1, \cdots, x_n) = (x_{(1)}, \cdots, x_{(n)})$ [1]*.*

The class of GM functions extends that of OWAs, in fact it is a proper superclass, see the example below:

Example 4 *Given the functions:* $f_1, f_2 : [0,1]^2 \longrightarrow [0,1]$*, s.t.* $f_1(x,y) = \sin(x) \cdot y$ *and* $f_2(x,y) = 1 - \sin(x) \cdot y$*, then the function* $h(x,y) = (\sin(x) \cdot y) \cdot x + (1 - \sin(x) \cdot y) \cdot y$ *is a GM function which is not isotonic.*

2.4 Bounded Generalized Mixture Functions

Another generalized form of OWA was defined by Farias et. al. [15]. These new functions were called **Bounded Generalized Mixture Functions** or BGM functions.

Definition 6 *A* ***Bounded Generalized Mixture Function*** *(BGM) is a function BGM* $: [0,1]^n \longrightarrow [0,1]$ *defined by* $\mathsf{BGM}_\Gamma(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) x_i$*, with* $\Gamma = \{f_i : [0,1]^n \longrightarrow [0,1] | \ 1 \leq i \leq n\}$*,*

(I) $\sum_{i=1}^{n} f_i(\mathbf{x}) \leq 1$ *for any* $\mathbf{x} \in [0,1]^n$*, and*

[1] For example, for $n = 3$ and $(x_1, x_2, x_3) = (0.1, 0.5, 0.3)$, $Sort(x_1, x_2, x_3) = (0.5, 0.3, 0.1) = (x_2, x_3, x_1)$ and so $(1) = 2, (2) = 3, (3) = 1$. The inverse permutation of $p(i) = (i)$ is $q(1) = 3, q(2) = 1, q(3) = 2$, and so $GM(0.1, 0.5, 0.3) = w_3 \cdot 0.1 + w_1 \cdot 0.5 + w_2 \cdot 0.3 = w_1 \cdot 0.5 + w_2 \cdot 0.3 + w_3 \cdot 0.1 = OWA_{\mathbf{w}}(0.1, 0.5, 0.3)$.

(II) $\sum_{i=1}^{n} f_i(1, \cdots, 1) = 1$ *for all* $i \in \{1, 2, \cdots, n\}$.

The class of all Bounded Generalized Mixture functions is denoted by: $\mathbb{BGM}$.

Remark 1 *Any* GM *function is also a* BGM *function such that:*

$$\sum_{i=1}^{n} f_i(\mathbf{x}) = 1, \text{ for all } \mathbf{x} \in [0,1]^n.$$

Remark 2 *There are* BGM *functions which are not* GM *functions, e.g.* $\mathsf{BGM}(\mathbf{x}) = \sum_{i=1}^{n} \frac{x_i^2}{n}$ *— obtained from weights:* $f_i(\mathbf{x}) = \frac{x_i}{n}$.

For further information and application see: [15, 18, 19].

It is possible to build a GM function from a BGM function:

Proposition 2 *Given a* BGM *function, with weights* $\Gamma = \{f_i : [0,1]^n \longrightarrow [0,1] \mid 1 \leq i \leq n\}$, *the function below is a generalized mixture function:*

$$\mathsf{GM}(\mathbf{x}) = \begin{cases} \sum_{i=1}^{n} \frac{f_i(\mathbf{x})x_i}{\sum_{j=1}^{n} f_j(\mathbf{x})}, & \text{if } \sum_{j=1}^{n} f_i(\mathbf{x}) \neq 0 \\ \sum_{i=1}^{n} \frac{x_i}{n}, & \text{otherwise.} \end{cases}.$$

In other words, there is a function $\pi : \mathbb{BGM} \to \mathbb{GM}$ *such that:*

$$\pi(\mathsf{BGM}_{\Gamma}) = \mathsf{GM}_{\Gamma'},$$

where $\Gamma' = \{f_i'\}$ *with* $f_i' : [0,1]^n \longrightarrow [0,1]$ *given by:*

$$f_i'(\mathbf{x}) = \begin{cases} \sum_{i=1}^{n} \frac{f_i(\mathbf{x})}{\sum_{j=1}^{n} f_j(\mathbf{x})}, & \text{if } \sum_{j=1}^{n} f_i(\mathbf{x}) \neq 0 \\ \frac{1}{n}, & \text{otherwise.} \end{cases}.$$

Proof. Just see that $\sum_{i=1}^{n} f_i'(\mathbf{x}) = 1$.

Example 5 *Given* $f_i(\mathbf{x}) = \frac{x_i}{n}$ *and* $\mathsf{BGM}(\mathbf{x}) = \sum_{i=1}^{n} \frac{x_i^2}{n}$, *we have:*

$$\pi(\mathsf{BGM})(\mathbf{x}) = \begin{cases} \sum_{i=1}^{n} \frac{x_i^2}{\sum_{j=1}^{n} x_j}, & \text{if } (x_1, \cdots, x_n) \neq (0, \cdots, 0) \\ \sum_{i=1}^{n} \frac{x_i}{n}, & \text{otherwise.} \end{cases}.$$

Proposition 3 π *is surjective and* $\pi \circ \pi = \pi$.

Proof. In order to verify that π is surjective, it is sufficient to note that if $\mathsf{GM}_\Gamma \in \mathbb{GM}$, then $\pi(\mathsf{GM}_\Gamma) = \mathsf{GM}_\Gamma$. In other words , just shows that $\pi|_{\mathbb{GM}} = Id_{\mathbb{GM}}$, and for this

$$\sum_{i=1}^{n} f_i(\mathbf{x}) = 1 \Longrightarrow \sum_{i=1}^{n} f_i'(\mathbf{x}) = \sum_{i=1}^{n} \frac{f_i(\mathbf{x})}{\sum\limits_{j=1}^{n} f_j(\mathbf{x})} = 1.$$

Now, to prove that $\pi \circ \pi = \pi$, see that $\pi(\mathsf{BGM}_\Gamma) \in \mathbb{GM}$. Therefore, as $\pi|_{\mathbb{GM}} = Id_{\mathbb{GM}}$, we have to

$$\pi(\pi(\mathsf{BGM}_\Gamma)) = \pi(\mathsf{BGM}_\Gamma),\ \text{para todo } \mathsf{BGM}_\Gamma \in \mathbb{BGM}.$$

In fields as image processing, some properties are necessary and can not be discarded. This properties are **idenpotency** and **homoneity**. In the propositions 4 and 5 show that π conserve these important properties.

Proposition 4 *For any $BGM_\Gamma \in \mathbb{BGM}$, $\pi(BGM_\Gamma)$ satisfies* **(IP)**.

Proof. Note that $\pi(\mathsf{BGM}_\Gamma)$ satisfies **(IP)** if, and only if, $\sum\limits_{i=1}^{n} f_i'(x, \cdots, x) = 1$ for all $x \in [0, 1]$, and also

$$\sum_{i=1}^{n} f_i'(x, \cdots, x) = \begin{cases} \sum\limits_{i=1}^{n} \frac{f_i(x,\cdots,x)}{\sum\limits_{j=1}^{n} f_j(x,\cdots,x)} = 1, & \text{if } \sum\limits_{i=1}^{n} f_i(x, \cdots, x) \neq 0 \\ \sum\limits_{i=1}^{n} \frac{1}{n} = 1, & \text{otherwise.} \end{cases}$$

Proposition 5 *If $BGM_\Gamma \in \mathbb{BGM}$ is such that $f_i \in \Gamma$ satisfies* $(\mathbf{HP_k})$, *with $k \geq 1$, then $\pi(BGM_\Gamma)$ satisfies* $(\mathbf{HP_1})$.

Proof. If $\sum\limits_{i=1}^{n} f_i(x_1, \cdots, x_n) = 0$, then $\sum\limits_{i=1}^{n} f_i(\lambda x_1, \cdots, \lambda x_n) = \lambda^k \sum\limits_{i=1}^{n} f_i(x_1, \cdots, x_n) = 0$, like this $f_i'(x_1, \cdots, x_n) = f_i'(\lambda x_1, \cdots, \lambda x_n) = \frac{1}{n}$. Meanwhile, if $\sum\limits_{i=1}^{n} f_i(x_1, \cdots, x_n) \neq 0$ and $\lambda \neq 0$, then

$$\begin{aligned} f_i'(\lambda x_1, \cdots, \lambda x_n) &= \frac{f_i(\lambda x_1, \cdots, \lambda x_n)}{\sum\limits_{j=1}^{n} f_j(\lambda x_1, \cdots, \lambda x_n)} = \frac{\lambda^k f_i(x_1, \cdots, x_n)}{\sum\limits_{j=1}^{n} \lambda^k f_j(x_1, \cdots, x_n)} \\ &= \frac{f_i(x_1, \cdots, x_n)}{\sum\limits_{j=1}^{n} f_j(x_1, \cdots, x_n)} = f'(x_1, \cdots, x_n) \end{aligned}$$

Therefore, $\pi(\mathsf{BGM}_\Gamma)(\lambda x_1, \cdots, \lambda x_n) = \lambda \pi(\mathsf{BGM}_\Gamma)(x_1, \cdots, x_n)$, if $\lambda \neq 0$. In the case that $\lambda = 0$, we have

$$\pi(\mathsf{BGM}_\Gamma)(0x_1, \cdots, 0x_n) = \sum_{i=1}^{n} \frac{0 \cdot x_i}{n} = 0 = 0 \cdot \pi(\mathsf{BGM}_\Gamma)(x_1, \cdots, x_n).$$

Thus, $\pi(\mathsf{BGM}_\Gamma)$ satisfies $(\mathbf{HP_1})$.

Another interesting property is the **Shift-invariance**, which is observed in functions as *means*, *mode*, *minimun*, *maximum* and *median*. In the next proposition we show that shift-invariance also is preserved by π.

Proposition 6 *If $BGM_\Gamma \in \mathbb{BGM}$ is such that $f_i(x_1+\lambda, \cdots, x_n+\lambda) = f_i(x_1, \cdots, x_n)$ for all $i \in \{1, \cdots, n\}$ and $x_1, \cdots, x_n \in [0,1]$, then $\pi(BGM_\Gamma)$ satisfies* **(SHP)**.

Proof. It is sufficient to prove that $f'_i(x_1 + \lambda, \cdots, x_n + \lambda) = f'_i(x_1, \cdots, x_n)$ for all $i \in \{1, \cdots, n\}$ and $x_1, \cdots, x_n \in [0,1]$. For this, note that: If $0 = \sum_{i=1}^{n} f_i(x_i + \lambda, \cdots, x_n+\lambda) = \sum_{i=1}^{n} f_i(x_i, \cdots, x_n)$, then $f'_i(x_i+\lambda, \cdots, x_n+\lambda) = \frac{1}{n} = f'(x_i, \cdots, x_n)$. In the other case:

$$
\begin{aligned}
f'_i(x_i + \lambda, \cdots, x_n + \lambda) &= \frac{f_i(x_i + \lambda, \cdots, x_n + \lambda)}{\sum_{j=1}^{n} f_j(x_i + \lambda, \cdots, x_n + \lambda)} \\
&= \frac{f_i(x_i, \cdots, x_n)}{\sum_{j=1}^{n} f_j(x_i, \cdots, x_n)} \\
&= f'_i(x_i, \cdots, x_n)
\end{aligned}
$$

3 Final Remarks

In this paper we investigate two classes of functions: The Generalized Mixture functions and the Bounded Generalized Mixture Functions. We prove that there is a relation between them. We also show that it is possible to obtain Generalized Mixture functions from Bounded Generalized Mixture functions and guarantee important properties such as: idempotency and homogeneity, which are important properties, for example, in image processing [12]. Applications of Generalized Mixture functions can be found in [15, 18].

The current work will allow to establish a series of generalized mixture functions whose applicability will be investigated in future contributions. Currently, we are investigating the application on Decision Making problems (work in progress).

References

1. R. P. Joseph, C. S. Singh, and M. Manikandan. Brain tumor mri image segmentation and detection in image processing. *International Journal of Research and Tecnology*, 3, 2014.
2. J. Mihailović, A. Savić, J. Bogdanović-Pristov, and K. Radotić. Mri brain tumors images by using independent component analysis. In *Intelligent Systems and Informatics (SISY), 2011 IEEE 9th International Symposium on*, pages 433–435, Sept 2011.

3. S. K. Woo, K. M. Kim, T. S. Lee, J. H. Jung, J. G. Kim, J. S. Kim, T. H. Choi, G. I. An, and G. J. Cheon. Registration method for the detection of tumors in lung and liver using multimodal small animal imaging. *IEEE Transactions on Nuclear Science*, 56(3):1454–1458, June 2009.
4. A. J. Solanki, K. R. Jain, and N. P. Desai. Isef based identification of rct/filling in dental caries of decayed tooth. *International Journal of Image Processing*, 7(2):149 – 162, 2013.
5. S. C. Dighe and R. Shriram. Dental biometrics for human identification based on dental work and image properties in periapical radiographs. In *TENCON 2012 - 2012 IEEE Region 10 Conference*, pages 1–6, Nov 2012.
6. Suprijanto, Gianto, E. Juliastuti, Azhari, and L. Epsilawati. Image contrast enhancement for film-based dental panoramic radiography. In *System Engineering and Technology (ICSET), 2012 International Conference on*, pages 1–5, Sept 2012.
7. R. R. Yager, G. Gumrah, and M. Z. Reformat. Using a web personal evaluation tool – pet for lexicographic multi-criteria service selection. *Knowledge-Based Systems*, 24(7):929 – 942, 2011.
8. S.-M. Zhou, F. Chiclana, R. I. John, and J. M. Garibaldi. Type-1 owa operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers. *Fuzzy Sets and Systems*, 159(24):3281 – 3296, 2008. Theme: Fuzzy Intervals and Optimisation.
9. S. J. Chen and C. L Hwang. *Fuzzy Multiple Attribute Decision Making: Methods and Applications*, volume 375 of *Lecture Notes in Economics and Mathematical Systems*. Springer, Berlin, 1992.
10. H. Bustince, M. Galar, B. Bedregal, A. Kolesarova, and R. Mesiar. A new approach to interval-valued choquet integrals and the problem of ordering in interval-valued fuzzy set applications. *IEEE Transactions on Fuzzy Systems*, 21(6):1150 – 1162, Dec 2013.
11. D. Paternain, A. Jurio, E. Barrenechea, H. Bustince, B. Bedregal, and E. Szmidt. An alternative to fuzzy methods in decision-making problems. *Expert Systems with Applications*, 39(9):7729 – 7735, 2012.
12. D. Paternain, J. Fernandez, H. Bustince, R. Mesiar, and G. Beliakov. Construction of image reduction operators using averaging aggregation functions. *Fuzzy Sets and Systems*, 261:87 – 111, 2015. Theme: Aggregation operators.
13. A. D. S. Farias, L. R. A. Lopes, B. C. Bedregal, and R. H. N. Santiago. Closure properties for fuzzy recursively enumerable languages and fuzzy recursive languages. *Journal of Intelligent & Fuzzy Systems*, 31:1795–1806, 2016.
14. G. Lucca, J. A. Sanz, G. P. Dimuro, B. Bedregal, R. Mesiar, A. Kolesárová, and H. Bustince. Preaggregation functions: Construction and an application. *IEEE Transactions on Fuzzy Systems*, 24(2):260–272, April 2016.
15. A. D. S. Farias, V. S. Costa, L. R. A. Lopes, B. C. Bebregal, and R. H. N. Santiago. A method for image reduction based on a generalization of ordered weighted averaging functions. *arXiv:1601.03785*, 2016.
16. R. R. Yager. Ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Transactions on Systems, Man, and Cybernetics*, 18:183 – 190, 1988.
17. G. Beliakov, H. Bustince, and T. Calvo. *A Practical Guide to Averaging Functions*, volume 329 of *Studies in Fuzziness and Soft Computing*. Springer, 2016.
18. A. D. S. Farias, V. S. Costa, R. H. N. Santiago, and B. Bedregal. The image reduction process based on generalized mixture functions. In *2016 Annual Conference of the North American Fuzzy Information Processing Society (NAFIPS)*, pages 1–6, Oct 2016.

19. A. D. S. Farias, R. H. N. Santiago, and B. Bedregal. Some properties of generalized mixture functions. In *2016 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 288–293, July 2016.
20. G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners*, volume 221 of *Studies in Fuzziness and Soft Computing*. Springer, Berlin, 2007.
21. G. Beliakov, H. Bunstince, and T. Calvo. *A Practical Guide to Averaging Functions*, volume 329 of *Studies in Fuzziness and Soft Computing*. Springer, Switzeland, 1 edition, 2016.
22. D. Dubois and H. Prade. On the use of aggregation operations in information fusion processes. *Fuzzy Sets and Systems*, 142(1):143 – 161, 2004. Aggregation Techniques.
23. E. Hancer, B. Xue, M. Zhang, D. Karaboga, and B. Akay. A multi-objective artificial bee colony approach to feature selection using fuzzy mutual information. In *Evolutionary Computation (CEC), 2015 IEEE Congress on*, pages 2420–2427, May 2015.
24. L. Lingling, Z. Xian, H. Pengju, and L. Zhigang. The research on the method of fuzzy information processing. In *System Science, Engineering Design and Manufacturing Informatization (ICSEM), 2012 3rd International Conference on*, volume 2, pages 47–50, Oct 2012.
25. G. Beliakov, H. Bustince, and D. Paternain. Image reduction using means on discrete product lattices. *IEEE Transactions on Image Processing*, 21(3):1070 – 1083, March 2012.
26. R. A. M. Pereira and G. Pasi. On non-monotonic aggregation: mixture operators. In *Proc. 4th Meeting of the EURO Working Group on Fuzzy Sets (EUROFUSE'99) and 2nd Internat. Conf. on Soft and Intelingent Computing (SIC'99)*, Budapest, Hungary,, 1999.
27. R. A. M. Pereira. The orness of mixture operators: the exponential case. In *Proc. 8th Internat. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'200)*, Madrid, Spain, 2000.
28. R. A. M. Pereira and R. A. Ribeiro. Aggregation with generalized mixture operators using weighting functions. *Fuzzy Sets and Systems*, 137(1):43 – 58, 2003. Preference Modelling and Applications.

Image Filters as Reference Functions for Morphological Associative Memories in Complete Inf-Semilattices

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Abstract. Mathematical morphology (MM) is concerned with the analysis and processing of images and signals by means of algebra and geometry. The algebraic framework of MM is usually given by complete lattices. In this paper, we employ the more recent theory of MM on inf-semilattices (cisls) based on reference elements and we describe a semi-lattice based auto-associative morphological memory (SL-AMM) model in this framework. The choice of appropriate reference elements or, more generally, reference functions in applications of SL-AMMs is an open research issue. Here, we employ different image filters as reference functions in cisl-based auto-associative memories and we perform some experiments concerning the recall of gray-scale images that are highly corrupted by salt and pepper or random-valued impulsive noise. We compare the experimental results produced by our approach with the ones achieved by the image filters as well as number of associative memory models.

Keywords: Mathematical morphology, complete inf-semilattice, reference element, auto-associative memory, image filter, image restoration.

1 Introduction

Mathematical morphology was first initiated in 1960s by G. Matheron and J. Serra as the part of binary image processing that is concerned with image filtering and geometric analysis by means of structuring elements [16]. Afterwards, MM was extended to gray-scale functions (images) using the umbra approach [18]. More recent approaches to MM include fuzzy MM [3, 17, 2] and $\mathbb{L}$-fuzzy MM [19]. Note that, in all approaches towards MM, the class of images represents a partially ordered set and in most cases, including the ones mentioned before, a complete lattice [1] on which morphological operators are defined [1, 6, 5, 20]. In this framework, MM has many applications such as image restoration, segmentation, edge detection, classification and analysis.

In the complete lattice setting, a morphological operator is determined by the specific partial ordering on the underlying image space and the choice what is foreground and what is background. This choice, which is never made explicit and for that reason usually goes unnoticed, causes morphological operators to usually come in dual pairs such as dilation/erosion and opening/closing [6]. In contrast, an operator ψ is called self-dual if $\psi(f^*) = (\psi(f))^*$ for any input image f. Here, f^* denotes the dual image of the image f. In many applications such as image filtering or image denoising, self-duality is a desirable property.

Keshet and Heijmans employed complete inf-semilattices in order to develop an algebraic approach towards self-dual mathematical morphology [7, 9]. Specifically, Keshet and Heijmans equipped the image space with a self-dual partial ordering so as to generate a complete inf-semilattice on which self-dual erosion operators can be defined. The resulting theory of MM on cisls is not only interesting from the mathematical point of view but can also be applied to problems in nonlinear image analysis such as fast motion detection, image restoration, innovation extraction, compression of segmented sequences [9].

A complete inf-semilattice is a set in which every arbitrary subset has an infimum (but not necessarily a supremum). A complete inf-semilattice can be derived from a conditionally complete lattice-ordered group by using a so called reference element r. The resulting cisl has r as its least element. Given an arbitrary element x of a complete lattice-ordered group, a reference element r is given by the value $\rho(x)$ of a so called reference function ρ.

Previously, Sussner and Medeiros introduced a cisl-based auto-associative morphological memories [21] that we shall call semi-lattice associative memories (SLAMs) in this paper. The definition of a SLAM model uses a reference function whose choice represents an open research problem. Since this paper includes simulations regarding the restoration of images that are corrupted by high levels of salt and pepper as well as random-valued impulsive noise (RVIN), some competitive impulse noise filter, namely adaptive median filter (AMF) [8] and adaptive fuzzy transform based image filter (AFT-IF) [15], were used as reference functions. In our experiments, the SLAM model that employs the AFT-IF as a reference function exhibited a better image restoration performance in terms of three different measures (NMSE, PSNR, SSIM) than the respective filters alone. In addition, the AFT-IF based SLAM also outperformed the some neural associative memory models, namely the optimal linear associative memory and the complex-valued Hopfield net [10, 22].

2 Some Mathematical Background

A non-empty set $\mathcal{P}$ on which a reflexive, antisymmetric and transitive binary relation " $\leq$ " is defined is called a partially ordered set (poset). Additionally, if we have either $x \leq y$ or $y \leq x$ in a poset $\mathcal{P}$, then $\mathcal{P}$ is said to be totally ordered and is called a chain. An operator $\psi : \mathcal{P} \rightarrow \mathcal{P}$ is said to be increasing or isotone if $x \leq y$ implies that $\psi(x) \leq \psi(y)$.

A poset $\mathbb{L}$ is called a *lattice* if every non-empty finite subset of $\mathbb{L}$ has an infimum and a supremum in $\mathbb{L}$ [1]. In particular, every totally ordered set or *chain* such as $\mathbb{R}$ and $\mathbb{Z}$ is a lattice. For any $X \subseteq \mathbb{L}$, we denote the infimum of X using the symbol $\bigwedge X$ and the supremum of X using the symbol $\bigvee X$. If

$X = \{x_j \in \mathbb{L} : j \in J\}$ for some index set J, then we write $\bigwedge_{j\in J} x_j$ and $\bigvee_{j\in J} x_j$ instead of $\bigwedge X$ and $\bigvee X$, respectively. If $X = \{x, y\} \subseteq \mathbb{L}$, then we write $x \wedge y$ and $x \vee y$ instead of $\bigwedge X$ and $\bigvee X$, respectively.

A lattice $\mathbb{L}$ is *complete* if every subset of $\mathbb{L}$ has an infimum and a supremum in $\mathbb{L}$. A lattice $\mathbb{L}$ is called *conditionally complete* if every bounded subset of $\mathbb{L}$ has an infimum and a supremum in $\mathbb{L}$. In particular, the *set of finite elements* of a complete lattice $\mathbb{L}$, i.e., $\mathbb{L} \setminus \{\bigwedge \mathbb{L}, \bigvee \mathbb{L}\}$, is conditionally complete. If every non-empty finite subset of a poset $\mathbb{L}$ has an infimum in $\mathbb{L}$, then $\mathbb{L}$ constitutes an *inf-semilattice*. If every subset of $\mathbb{L}$ has an infimum in $\mathbb{L}$ then $\mathbb{L}$ is called a *complete inf-semilattice* or *cisl*. If $\mathbb{L}$ is a (complete) lattice or an inf-semilattice, then the direct product $\mathbb{L}^n$ is also a (complete) lattice or an inf-semilattice with partial order induced by $\mathbb{L}$. Similarly $\mathbb{L}^X$, the class of functions from a set $X \neq \emptyset$ to $\mathbb{L}$, gives rise to a complete lattice when considering the following partial order on $\mathbb{L}^X$:

$$f \leq g \Leftrightarrow f(x) \leq g(x) \quad \forall x \in X. \tag{1}$$

A function $\phi : \mathbb{L} \to \mathbb{M}$, where $\mathbb{L}$ and $\mathbb{M}$ are cisls, is called a *cisl homomorphism* or (algebraic) *erosion* if we have the following equation for all index sets J and all $x_j \in \mathbb{L}$:

$$\phi(\bigwedge_{j\in\mathbb{L}} x_j) = \bigwedge_{j\in\mathbb{L}} \phi(x_j)\,. \tag{2}$$

A lattice that also forms a group in which every group translation $x \mapsto a+x+b$ is isotone, is called *lattice-ordered group*, for short l-group [1]. If $\mathbb{G}$ is a complete lattice whose set of finite elements forms a group with isotone group translation then we refer to $\mathbb{G}$ as a *complete l-group extension* [20]. Of course, a *conditionally complete lattice* $\mathbb{F}$ can form a group at the same time and, in this case $\mathbb{F}$ is simply called a conditionally complete l-group [1]. For example, the l-groups $\mathbb{R}$ and $\mathbb{Z}$ are conditionally complete.

From now on, let $\mathbb{F}$ stand for an arbitrary conditionally complete l-group. Note that $\mathbb{F}$ induces conditionally complete l-groups $\mathbb{F}^n$, $\mathbb{F}^{m\times n}$, and $\mathbb{F}^X$. Given a matrix $A \in \mathbb{F}^{m\times p}$ and a matrix $B \in \mathbb{F}^{p\times n}$, the matrix $C = A \boxvee B$ called the max-product of A and B and the matrix $D = A \boxwedge B$, called the min-product of A and B are respectively defined by the following equations for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$:

$$c_{ij} = \bigvee_{\xi=1}^{k} (a_{i\xi} + b_{\xi j}), \quad d_{ij} = \bigwedge_{\xi=1}^{k} (a_{i\xi} + b_{\xi j}). \tag{3}$$

If $x \in \mathbb{F} \subseteq \mathbb{G}$, then its conjugate x^* is simply given by $-x$ and X^* is given by $(X^*)_{ij} = -x_{ij}$ for all $X \in \mathbb{F}^{n\times m} \subseteq \mathbb{G}^{n\times m}$. Note that $\mathbb{F} = \mathbb{G} \setminus \{\pm\infty\}$.

The cone $\mathbb{F}^+$ is defined as $\{x \in \mathbb{F} : 0 \leq x\}$, where 0 denotes the neutral element with respect to the group operation of addition. Note that $(\mathbb{F}^+, \leq)$ represents a cisl. The positive part x^+ and the negative part x^- of an element x of $\mathbb{F}$ are respectively given by $x^+ = x \vee 0$ and $x^- = -x \vee 0$, where 0 denotes the neutral element of the group $\mathbb{F}$. Every $x \in \mathbb{F}$ can be written as $x = x^+ - x^-$. The element x^+ and x^- of the cone $\mathbb{F}^+$ are said to be disjoint because $x^+ \wedge x^- = 0$. Defining the following partial order $\preceq$ on $\mathbb{F}$ turns $\mathbb{F}$ into a cisl [7].

Proposition 1. *Consider the binary relation $\preceq_0$ on $\mathbb{F}$ that is defined as follows:*

$$x \preceq_0 y \Leftrightarrow x^+ \leq y^+ \, and \, x^- \leq y^- \tag{4}$$

We have that $(\mathbb{F}, \preceq_0)$ is a cisl whose least element is 0. The infimum of an arbitrary subset $\{x_i : i \in I\}$ of $\mathbb{F}$ is given by

$$\curlywedge_{i \in I} x_i = \bigwedge_{i \in I} (x_i)^+ - \bigwedge_{i \in I} (x_i)^-. \tag{5}$$

In particular the infimum operation in the cisl $(\mathbb{F}, \preceq_0)$ satisfies

$$(\curlywedge_{i \in I} x_i)^+ = \bigwedge_{i \in I} (x_i)^+ \, and \, (\curlywedge_{i \in I} x_i)^- = \bigwedge_{i \in I} (x_i)^- \tag{6}$$

The cisl $(\mathbb{F}, \preceq_0)$ is also denoted using the symbol $\mathbb{F}_0$. The neutral element of addition 0 plays an important role in $\mathbb{F}_0$ whose construction is based on the fact that 0 represents a reference element of the lattice $(\mathbb{F}, \leq)$. Recall that an arbitrary element r of a lattice $\mathbb{L}$ is called a reference element if the following statement is satisfied for all $x, y \in \mathbb{L}$:

$$x \wedge r = y \wedge r \, and \, x \vee r = y \vee r \Leftrightarrow x = y. \tag{7}$$

If $\mathbb{F}$ is conditionally complete l-group then every $r \in \mathbb{F}$ is reference element of the lattice $(\mathbb{F}, \leq)$ and a cisl arises via the following definition of "$\preceq_r$" which constitutes a partial order on $\mathbb{F}$. The resulting cisl $(\mathbb{F}, \preceq_r)$ can be denoted using the symbol $\mathbb{F}_r$.

$$x \preceq_r y \Leftrightarrow x \vee r \leq y \vee r \text{ and } y \wedge r \leq x \wedge r \tag{8}$$

For an arbitrary $X \subseteq \mathbb{F}$, the infimum of X in the cisl $\mathbb{F}_r$ is denoted using the symbol $\curlywedge_r X$. In the special case where $X = \{x_j \in \mathbb{L} : j \in J\}$ for some index set J, $\curlywedge_r X$ is also denoted

$$\curlywedge_{j \in J}{}_{\mathrm{r}} \, x_j. \tag{9}$$

Moreover, (z_r) denotes $z - r$ for all $z, r \in \mathbb{F}$. Note that $x \preceq_r y$ is equivalent to having both $(x_r)^+ \leq (y_r)^+$ and $(x_r)^- \leq (y_r)^-$. This observation leads to the following expression:

$$\curlywedge_{j \in J}{}_{\mathrm{r}} \, x_j = \bigwedge_{j \in J} (x_j)_r^+ - \bigwedge_{j \in J} (x_j)_r^- + r. \tag{10}$$

3 Semilattice Associative Memories

Associative memories (AMs) are designed to store a finite set of pattern associations $(\mathbf{x}^\xi, \mathbf{y}^\xi)$, where $\xi = 1, \ldots, k$, called set of fundamental memories [4]. Moreover, an AM should permit the retrieval of a desired output upon presentation of a possibly noisy or incomplete version of a input pattern.

In this paper, the focus is on auto-associative memories, i.e., the case where $\mathbf{y}^\xi = \mathbf{x}^\xi$ for all $\xi = 1, \ldots, k$. Furthermore, the patterns $\mathbf{x}^\xi$ are assumed to be in $\mathbb{F}^n$, where $\mathbb{F}$ is a conditionally complete l-group. Hence, the auto-associative memory described in this paper corresponds to a mapping $\mathcal{M} : \mathbb{F}^n \to \mathbb{F}^n$. Ideally, $\mathcal{M}$ exhibits perfect recall of the original pattern, that is, $\mathcal{M}(\mathbf{x}^\xi) = \mathbf{x}^\xi$ for all $\xi \in \mathcal{K}$ and some tolerance with respect to noise, that is, $\mathcal{M}\left(\widetilde{\mathbf{x}}^\xi\right) = \mathbf{x}^\xi$ for noisy or incomplete versions $\widetilde{\mathbf{x}}^\xi$ of $\mathbf{x}^\xi$.

The classical auto-associative morphological memories (AMMs) [13], [14], also referred to as lattice auto-associative memories, are defined in terms of the min- and max-products. Originally, defined as mappings $\mathbb{F}^n \to \mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{Z}$, AMMs can also be viewed as mappings $\mathbb{G}^n \to \mathbb{G}^n$, where $\mathbb{G} = \mathbb{R} \cup \{-\infty, +\infty\}$ or $\mathbb{G} = \mathbb{Z} \cup \{-\infty, +\infty\}$ [20]. As observed in [21], AMMs can also be defined as follows for an arbitrary complete lattice-ordered group $\mathbb{F}$.

Let $X \in \mathbb{F}^{n \times k}$ be the matrix whose ξ-th column is $\mathbf{x}^\xi$ for $\xi = 1, \ldots, k$. The AMM $\mathcal{M}_{XX}$ is the mapping $\mathbb{F}^n \to \mathbb{F}^n$ determined by the equation

$$\mathcal{M}_{XX}(\mathbf{x}) = M_{XX} \mathbin{\boxed{\wedge}} \mathbf{x} \;\forall \mathbf{x} \in \mathbb{F}^n, \tag{11}$$

where the synaptic weight matrix M_{XX} is given by $M_{XX} = X \mathbin{\boxed{\vee}} X^*$. The dual AMM model $\mathcal{W}_{XX} : \mathbb{F}^n \to \mathbb{F}^n$ is determined by the equation

$$\mathcal{W}_{XX}(\mathbf{x}) = W_{XX} \mathbin{\boxed{\vee}} \mathbf{x} \;\forall \mathbf{x} \in \mathbb{F}^n, \tag{12}$$

where the synaptic weight matrix W_{XX} is given by $W_{XX} = X \mathbin{\boxed{\wedge}} X^*$. If $\mathbb{G}$ denotes the completion of $\mathbb{F}$, then $\mathbb{G}$ represents a complete l-group extension and Equations (11) and (12) can also be applied to $\mathbf{x} \in \mathbb{G}^n$. As shown in [20], the respective extended mappings $\mathbb{G}^n \to \mathbb{G}^n$ represent elementary operations of MM in complete lattices - in this case from the complete lattice $\mathbb{G}^n$ to the complete lattice $\mathbb{G}^n$ - and this is the reason why the AMs $\mathcal{M}_{XX}$ and $\mathcal{W}_{XX}$ are called "morphological".

Let us briefly review the SLAM model of Sussner and Medeiros [21] As before, we consider patterns $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{F}^n$. Let $\rho : \mathbb{F}^n \to \mathbb{F}^n$ be an arbitrary function. Given an arbitrary element $\mathbf{x}$ of $\mathbb{F}^n$, $\rho(\mathbf{x})$ will play the role of a reference element and therefore we may refer to ρ as a "reference function". Let $X_\rho^\pm \in \mathbb{F}^{n \times 2k}$ be the matrix whose ξth column is $(\mathbf{x}^\xi - \rho(\mathbf{x}^\xi))^+$ and $(\xi + k)$th column is given by $(\mathbf{x}^\xi - \rho(\mathbf{x}^\xi))^-$ for all $\xi = 1, \ldots, k$. In addition, let M_{XX}^ρ denotes the matrix $M_{X_\rho^\pm X_\rho^\pm} \in \mathbb{F}^{n \times n}$. The following equation yields an auto-associative memory $\mathcal{M}_\rho : \mathbb{F}^n \to \mathbb{F}^n$ [21]:

$$\mathcal{M}_\rho(\mathbf{x}) = M_{XX}^\rho \mathbin{\boxed{\wedge}} (\mathbf{x} - \rho(\mathbf{x}))^+ - M_{XX}^\rho \mathbin{\boxed{\wedge}} (\mathbf{x} - \rho(\mathbf{x}))^- + \rho(\mathbf{x}) \,\forall \,\mathbf{x} \in \mathbb{F}^n. \tag{13}$$

Theorem 1. *Let $\mathcal{M}_\rho : \mathbb{F}^n \to \mathbb{F}^n$ be defined as in Equation 13. The function $\mathcal{M}_\rho$ represents an associative memory model, that is guaranteed to yield perfect recall for an arbitrary set of patterns $\{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subset \mathbb{F}^n$ for an arbitrary number of patterns $k \in \mathbb{N}$. Formally we have:*

$$\mathcal{M}_\rho(\mathbf{x}^\xi) = \mathbf{x}^\xi \, \forall \, \xi = 1, \ldots, k \,, \tag{14}$$

For an arbitrary input pattern $\mathbf{x} \in \mathbb{F}^n$, the output pattern $\mathcal{M}_\rho(\mathbf{x}) \in \mathbb{F}^n$ satisfies

$$\rho(\mathbf{x}) \preceq_{\rho(\mathbf{x})} \mathcal{M}_\rho(\mathbf{x}) \preceq_{\rho(\mathbf{x})} \mathbf{x}. \tag{15}$$

If $\rho(\mathbf{x}) = \mathbf{r} \in \mathbb{F}^n$ for all $\mathbf{x} \in \mathbb{F}^n$, then $\mathcal{M}_\rho$ represents an erosion from the cisl $\mathbb{F}_0^n$ to the cisl $\mathbb{F}_\rho^n$.

4 Simulations in Gray-Scale Image Reconstruction

In this section we perform some experiments using the ten images (i.e., Lena, Airplane, Watch, Cameraman, House, Vehicle, Tree, Papav, Boat, Church and Clock) that are available at the internet site of the Mathematical Imaging and Computational Intelligence Laboratory, University of Campinas [11]. These images have size 128 $\times$ 128 and 256 gray levels. For each of these images, we generated a vector $\mathbf{x}^\xi$ of length n = 16384.

Recall that SLAM depends on the choice of reference vectors which can be accomplished by means of a reference function. In this papers, we chosen the image filters i.e., adaptive median filter (AMF) with maximum window size [8] and adaptive fuzzy transform based image filter (AFT-IF) [15] as reference functions. Since the AFT-IF is based on the fuzzy transform [12], we converted the images into ten fuzzy images by normalizing the respective pixel values within the range $[0, 1]$. In order to verify the tolerance of SLAM models using aforementioned image filters, we corrupted the original images by introducing the following types of noise:

1. Salt and pepper (S & P) noise with different density levels;
2. Random-valued impulse noise (RVIN) with various probability;

We conducted this experiment 100 times for each type of noise and each original pattern $\mathbf{x}^\xi$, where $\xi = 1, \ldots, 10$ and we used several measures to evaluate the performance of the SLAM model.

Let $\mathbf{x}$ be the original image and let $\mathbf{y}$ be another image such as the image that was restored from a corrupted version of $\mathbf{x}$. Recall that the normalized mean squared error (NMSE) is given by

$$\text{NMSE}(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{x} - \mathbf{y}\|_2^2}{\|\mathbf{x}\|_2^2} = \frac{\sum_{i=1}^{n}(x_i - y_i)^2}{\sum_{i=1}^{n}(x_i)^2} \tag{16}$$

where $\|.\|_2$ denotes the usual Euclidean norm. The peak signal to noise ratio (PSNR) (a bigger PSNR value represents a better visual quality of the restored image) and the mean structural similarity index (SSIM) (a number from 0 to 1, where 1 corresponds to two identical images [23]) are given as follows:

$$\mathrm{PSNR}(\mathbf{x},\mathbf{y}) = 10\log_{10}\left(\frac{n}{\sum_{j=1}^{n}(x_j - y_j)^2}\right) \quad \text{and} \tag{17}$$

$$\mathrm{SSIM}(\mathbf{x},\mathbf{y}) = \frac{(2\eta_{\mathbf{x}}\eta_{\mathbf{y}} + c_1)(2\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} + c_2)}{(\eta_{\mathbf{x}}^2 + \eta_{\mathbf{y}}^2 + c_1)(\sigma_{\mathbf{x}}^2 + \sigma_{\mathbf{y}}^2 + c_2)} \tag{18}$$

where $\eta_{\mathbf{x}}$ and $\sigma_{\mathbf{x}}$ are respectively the mean and the variance of the image $\mathbf{x}$, and c_1 and c_2 are two small constants added to provide stability. In this paper, we used $c_1 = (k_1 \cdot L)^2$ and $c_2 = (k_2 \cdot L)^2$ as parameters, where L is the dynamic range of the pixel-values (typically this is $2^{\#\text{ bits per pixel}} - 1$, i.e., $L = 255$) and $k_1 = 0.01$ and $k_2 = 0.03$ by default.

Table 1 lists the NMSE, the PSNR, and the SSIM values produced by AMF and AFT-IF (shortly AFTIF), as reference filters as well as the SLAM models $\mathcal{M}_{AMF}$ and $\mathcal{M}_{AFTIF}$ for salt and pepper noise with density levels d = 0.4, 0.5, 0.6 and random-valued impulse noise with probability p = 0.4, 0.5 and 0.6. Note that, for aforementioned noise, the SLAM models yield slightly lower NMSEs than the image filters alone. In these experiments, we employed the same fixed AFT-IF parameters for both salt and pepper noise and RVIN.

Furthermore, we also performed some experiments using the optimal linear associative memory (OLAM) proposed by Kohonen and Ruohonen [10] as well as the complex-valued Hopfield net of Tanaka et al. [22]. The synaptic weight matrix M_O of the OLAM model is given by $M_O = XX^\dagger$, where $X^\dagger$ denotes the pseudo-inverse of the matrix $X = [\mathbf{x}^1 \ldots \mathbf{x}^k] \in \mathbb{F}^{n\times k}$. Table 1 reveals that the SLAM model $\mathcal{M}_{AFTIF}$ that employs the AFTIF as a reference function exhibits the highest tolerance with respect to salt and pepper noise and RVIN for noise levels of 0.4, 0.5, and 0.6.

Fig. 1 depicts detailed views of the outputs of the aforementioned image filters and the corresponding SLAM models as well as the complex-valued Hopfield net and the OLAM upon presentation of a corrupted Lena image. Note that the output scene produced by $\mathcal{M}_{AFTIF}$ is the most similar to the corresponding scene of the original image.

In Fig. 2 we used a wide range of impulse noise with probability from 0.30 to 0.75 with step size 0.05 and measured the quality of the reconstructions produced by the AMF, the AFTIF and several associative memories in terms of the PSNR and the SSIM. Fig. 2 shows that the $\mathcal{M}_{AFTIF}$ exhibited the best error correction capability in this experiment.

5 Concluding Remarks

This paper discusses a semilattice-based autoassociative memory. We conducted some experiments concerning the restoration of gray-scale images that were corrupted by salt and pepper and random-valued impulse noise. In these experiments, the SLAM model $\mathcal{M}_{AFTIF}$ outperformed not only the AMF and the

Table 1. NMSEs, PSNRs, and SSIMs produced by the AMF, AFTIF and the corresponding SLAMs as well by the ccmplex Hopfield net and OLAM.

Type of noise	AMF	AFTIF	$\mathcal{M}_{AMF}$	$\mathcal{M}_{AFTIF}$	C. Hopfield	OLAM
NMSE						
S & P: d = 0.4	0.0991	0.0840	0.0937	**0.0801**	0.1086	0.1821
d = 0.5	0.1146	0.1060	0.1136	**0.1035**	0.3252	0.2253
d = 0.6	0.1301	0.1163	0.1298	**0.1102**	0.7903	0.2736
RVIN: p = 0.4	0.1901	0.0843	0.1866	**0.0805**	0.0896	0.0987
p = 0.5	0.2198	0.1088	0.2177	**0.1046**	0.2445	0.1112
p = 0.6	0.2711	0.1179	0.2710	**0.1176**	0.2760	0.1379
PSNR						
S & P: d = 0.4	25.98	29.11	26.13	**30.12**	29.61	28.41
d = 0.5	24.13	27.80	25.27	**29.55**	25.31	26.06
d = 0.6	22.89	26.98	23.85	**28.23**	23.32	25.34
RVIN: p = 0.4	24.10	28.21	25.34	**29.35**	28.18	27.78
p = 0.5	22.74	27.11	23.36	**28.93**	24.12	26.67
p = 0.6	21.11	26.12	21.89	**27.91**	22.12	25.79
SSIM						
S & P: d = 0.4	0.5010	0.7844	0.5545	**0.8314**	0.8211	0.8011
d = 0.5	0.4813	0.7455	0.5234	**0.8244**	0.7098	0.7825
d = 0.6	0.4698	0.7295	0.4811	**0.8011**	0.5712	0.7633
RVIN: p = 0.4	0.4909	0.7835	0.5495	**0.8299**	0.8133	0.7913
p = 0.5	0.4712	0.7423	0.4912	**0.8111**	0.6812	0.7712
p = 0.6	0.4614	0.7234	0.4612	**0.7913**	0.5812	0.7591

AFT-IF image filters but also other associative memory models. In future research, we intend to make further steps towards solving the problem of choosing a reference function for a SLAM model.

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References

1. Birkhoff, G. *Lattice Theory*, 3rd ed. American Mathematical Society, Providence, 1993.
2. Bloch, I., and Maitre, H. Fuzzy mathematical morphologies: a comparative study. *Pattern Recognition 28*, 9 (1995), 1341–1387.

Fig. 1. The first row depicts a detailed view of the original image, a noisy image (corrupted by impulse noise with p = 0.5), the outputs produced by AMF and AFTIF. The second row depicts the $\mathcal{M}_{AMF}$, $\mathcal{M}_{AFTIF}$, and the outputs produced by the complex Hopfield net and the OLAM, respectively.

3. De Baets, B. Fuzzy morphology: A logical approach. In *Uncertainty Analysis in Engineering and Science: Fuzzy Logic, Statistics, and Neural Network Approach*, B. M. Ayyub and M. M. Gupta, Eds. Kluwer Academic Publishers, Norwell, 1997, pp. 53–67.
4. Hassoun, M. H. Dynamic associative neural memories. In *Associative Neural Memories: Theory and Implementation*, M. H. Hassoun, Ed. Oxford University Press, Oxford, U. K., 1993.
5. Heijmans, H. J. A. M. *Morphological Image Operators*. Academic Press, New York, NY, 1994.
6. Heijmans, H. J. A. M. Self-dual morphology operators and filters. *Journal of Mathematical Imaging and Vision 6(1)* (1996), 15–36.
7. Heijmans, H. J. A. M., and Keshet, R. Inf-semilattice approach to self-dual morphology. *Journal of Mathematical Imaging and Vision 17* (2002), 55–80.
8. Hwang, H., and Haddad, R. A. Adaptive median filters: New algorithms and results. *IEEE Trans. Image Process 4* (1995), 499502.
9. Keshet, R. Mathematical morphology on complete semilattices and its applications to image processing. *Fundamenta Informatica 41* (2000), 33–456.
10. Kohonen, T. *Self-Organization and Associative Memory*. Springer Verlag, 1984.
11. Internet site on the mathematical imaging and computational intelligence lab (http://www.milab.ime.unicamp.br).
12. Perfilieva, I. Fuzzy transforms: Theory and applications. *Fuzzy sets and Systems 157* (2006), 993–1023.
13. Ritter, G. X., and Sussner, P. An introduction to morphological neural networks. In *Proceedings of the 13th International Conference on Pattern Recognition* (Vienna, Austria, 1996), pp. 709–717.
14. Ritter, G. X., Sussner, P., and de Leon, J. L. D. Morphological associative memories. *IEEE Transactions on Neural Networks 9*, 2 (1998), 281–293.

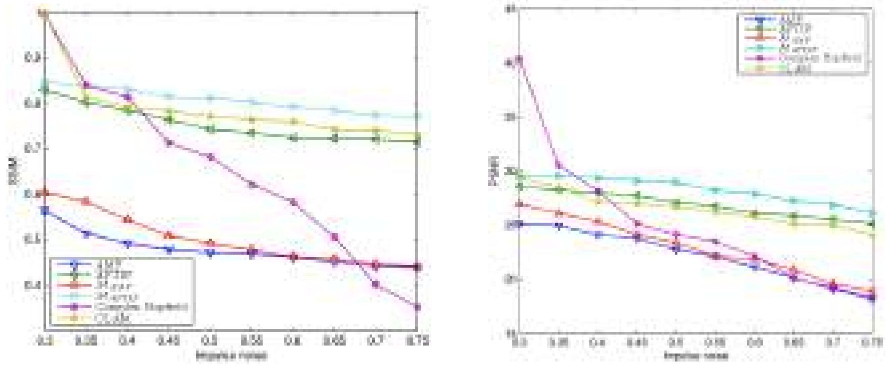

Fig. 2. The left side shows the average SSIMs and the right side shows the average PSNRs produced by the image filters and the AMs versus the probability of impulse noise.

15. Schuster, T., and Sussner, P. An adaptive image filter based on the fuzzy transform for impulse noise reduction. *Submitted for Soft Computing journal* (2017).
16. Serra, J. *Image Analysis and Mathematical Morphology.* Academic Press, London, 1982.
17. Sinha, D., and Dougherty, E. R. Fuzzy mathematical morphology. *J. Vis. Commun. Image Represent 3*, 3 (September 1995), 286–302.
18. Sternberg, S. R. Grayscale morphology. *Computer Vision, Graphics and Image Processing 35* (1986), 333–355.
19. Sussner, P. Lattice fuzzy transforms from the perspective of mathematical morphology. *Fuzzy sets and Systems 288* (2016), 135–149.
20. Sussner, P., and Esmi, E. Morphological perceptrons with competitive learning: Lattice-theoretical framework and constructive learning algorithm. *Information Sciences 181*, 10 (2011), 1929–1950.
21. Sussner, P., and Medeiros, C. R. An introduction to morphological associative memories in complete lattices and inf-semilattices. *World Congress on Computational intelligence* (2012), 1 – 8.
22. Tanaka, G., and Aihara, K. Complex-valued multistate associative memory with nonlinear multilevel functions for gray-level image reconstruction. in. *IEEE International Joint Conference on Neural Networks 4* (2008), 3086–3092.
23. Wang, Z., Bovik, A. C., Sheikh, H. R., and Simoncelli, E. P. Image quality assessment: From error measurement to structural similarity. *IEEE Transactions on Image Processingl 13(1)* (2004), 1–14.

The Class of Max-C Projection Autoassociative Fuzzy Memories

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Abstract. A fuzzy associative memory (FAM) is an input-output system that allows for the storage and recall of fuzzy sets. In this paper, we present and discuss a new class of autoassociative FAM models, called max-C projection autoassociative fuzzy memories (max-C PAFMs), which have been derived from autoassociative fuzzy morphological memories (AFMMs). In few words, a max-C PAFM projects the input fuzzy set into the set of the max-C combinations of the stored items. We illustrate and compare the performance of some max-C PAFM models with other AFMMs for the storage and recall of corrupted scale-gray images.

Keywords: associative memory, mathematical morphology, adjunction, fuzzy relational inequalities, image reconstruction.

1 Introduction

Associative memories (AMs) are mathematical models inspired by the human brain ability to store and retrieve a determined information by association [1]. An AM is categorized as either heteroassociative or autoassociative. An autoassociative memory is designed for the storage of a finite set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\}$. The famous Hopfield neural network is an example of an autoassociative memory [2].

We speak of a fuzzy associative memory (FAM) if the AM is designed for the storage and recall of fuzzy sets, that is, $\mathcal{A} \subseteq [0,1]^n$ [3, 4]. Applications of fuzzy associative memories include control [3], classification [5], time series prediction [6, 7, 8], and restoration of corrupted images [7, 9]. The reader interested on a comprehensive review on fuzzy associative memories is invited to consult [4]. In this paper, we only address autoassociative fuzzy memories. Precisely, we focus on the class of autoassociative fuzzy morphological memories (AFMMs) proposed by Valle and Sussner [7, 10].

The AFMMs can be seen as fuzzy versions of the autoassociative morphological memories (AMMs) introduced by Ritter and Sussner [11]. Like the AMM models, AFMMs exhibit unlimited absolute storage capacity and an excellent tolerance to either erosive or dilative noise. Despite the successful applications of morphological and fuzzy morphological autoassociative memories [6, 7, 8, 12], these memories suffer from a large number of spurious memories.

Recently, Valle introduced the max-plus projection autoassociative morphological memory (max-plus PAMM), which have less spurious memories than the original AMM [13]. Inspired by the max-plus PAMM, we proposed the new class of max-C projection autoassociative fuzzy memories (max-C PAFMs) [14]. In general terms, a max-C PAFM projects the input fuzzy set into the set whose elements are all max-C combinations of the stored items. Such as the max-plus PAMMs, the max-C PAFMs exhibit unlimited absolute storage capacity and an excellent tolerance to dilative noise. On the downside, they are very sensitive to either erosive or mixed noise.

The paper is organized as follows. The next section briefly reviews the min-D AFMMs. The max-C PAFMs are introduced in Section 3. Computational experiments comparing the performance of the new models and min-D AFMMs for the storage and recall of scale-gray images are given in Section 4. The paper finishes with the concluding remarks in Section 5. We would like to point out that this paper corresponds to an improved version of the conference paper [14]. Namely, in this paper we provide some new theoretical results and we provide further computational experiments.

2 Autoassociative Fuzzy Morphological Memories

In this section, we briefly review the autoassociative fuzzy morphological memories (AFMM). The reader interested on a detailed account on this subject is invited to consult [7, 10]. Let us begin by recalling some basic concepts from fuzzy set theory and fuzzy logic.

Throughout the paper, we only consider fuzzy sets on a finite universe of discourse. A fuzzy set $\mathbf{x}$ on a finite universe $U = \{u_1, u_2, \ldots, u_n\}$ corresponds to a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in [0,1]^n$, where the component $x_j = \mathbf{x}(u_j)$ denotes the degree of membership of u_j on the fuzzy set $\mathbf{x}$.

A fuzzy disjunction D is an increasing mapping $D : [0,1] \times [0,1] \to [0,1]$ such that $D(0,0) = 0$ and $D(1,0) = D(0,1) = 1$. An hybrid monotonous (decreasing in the first argument and increasing in the second argument) operator $J : [0,1] \times [0,1] \to [0,1]$ is a fuzzy co-implication if it satisfies $J(0,0) = J(1,1) = 0$ and $J(0,1) = 1$ [15]. We say that a commutative fuzzy disjunction D and a fuzzy co-implication J form an adjunction if and only if the following equivalence holds for $x, y, z \in [0,1]$:

$$D(x,y) \geq z \quad \Longleftrightarrow \quad x \geq J(y,z). \tag{1}$$

From now on, we assume that a fuzzy disjunction D is commutative and forms an adjunction with a certain fuzzy co-implication J.

An autoassociative fuzzy memory is designed for the storage of a family of fuzzy sets $\mathcal{A} = \left\{\mathbf{a}^1, \ldots, \mathbf{a}^k\right\} \subseteq [0,1]^n$, called fundamental memories set. In mathematical terms, an autoassociative fuzzy memory is an application $\mathcal{M} : [0,1]^n \to [0,1]^n$ such that the identity $\mathcal{M}(\mathbf{a}^\xi) = \mathbf{a}^\xi$ holds as far as possible for all $\xi \in \mathcal{K} = \{1, 2, \ldots, k\}$. Moreover, $\mathcal{M}$ should exhibit some noise tolerance, i.e., $\mathcal{M}(\widetilde{\mathbf{a}}^\xi) = \mathbf{a}^\xi$ is expected to hold true for noisy or incomplete versions $\widetilde{\mathbf{a}}^\xi$ of $\mathbf{a}^\xi$.

Motivated by concepts from mathematical morphology, Valle and Sussner introduced the autoassociative fuzzy morphological memories (AFMMs) [4, 10]. Due to page limit, in this paper we focus only on the min-D AFMM.

Let D be a commutative fuzzy disjunction and J a fuzzy co-implication such that D and J forms an adjunction. Given a fundamental memory set $\mathcal{A} = \left\{\mathbf{a}^1, \ldots, \mathbf{a}^k\right\} \subseteq [0,1]^n$, a min-$D$ AFMM $\mathcal{M} : [0,1]^n \to [0,1]^n$ is defined by

$$\mathcal{M}(\mathbf{x}) = M \bullet \mathbf{x}, \tag{2}$$

where the symbol "$\bullet$" denotes a min-D product based on the fuzzy disjunction D. The matrix $M \in [0,1]^{n\times n}$ is called the synaptic weight matrix of the AFMM. We would like to point out that $\mathcal{M}$ given by (2) corresponds to an erosion of fuzzy mathematical morphology [16, 17]. Hence the name fuzzy morphological memory. A new theoretical justification for a subclass of AFMMs has been proposed recently by Perfilieva et al. using fuzzy preorder [18].

The synaptic weight matrix M can be determined using the fuzzy learning by adjunction [10]. Let $A = [\mathbf{a}^1, \ldots, \mathbf{a}^k]$ be the matrix whose columns correspond to the fundamental memories. The fuzzy learning by adjunction establishes that the matrix M is the best approximation from above of the matrix A in terms of the min-D product. Formally, the fuzzy learning by adjunction defines

$$M = \bigwedge \left\{V \in [0,1]^{n\times n} : V \bullet A \geq A\right\}. \tag{3}$$

Alternatively, the solution of (3) can be expressed using the equation

$$M = A \blacktriangleright A^T, \tag{4}$$

where the symbol "$\blacktriangleright$" denotes the max-J product based on the fuzzy co-implication J which forms an adjunction with the fuzzy disjunction D [10].

The following proposition reveals that a min-D AFMM $\mathcal{M}$ projects the input $\mathbf{x}$ into the set of all fixed points of the memory [7]. Alternatively, the output $\mathcal{M}(\mathbf{x})$ is the largest fixed point less than or equal to the input $\mathbf{x}$.

Proposition 1 *If D is a commutative and associative fuzzy disjunction with an identity then, for any input $\mathbf{x} \in [0,1]^n$, the output of a min-D AFMM $\mathcal{M}$ satisfies*

$$\mathcal{M}(\mathbf{x}) = \bigvee \left\{\mathbf{z} \in \mathcal{I}(\mathcal{A}) : \mathbf{z} \leq \mathbf{x}\right\}, \tag{5}$$

where $\mathcal{I}(\mathcal{A}) = \{\mathbf{z} \in [0,1]^n : \mathcal{M}(\mathbf{z}) = \mathbf{z}\}$ denotes the set of all fixed points of $\mathcal{M}$, which depends on the fundamental memory set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\}$.

In the light of Proposition 1, from now on we assume that the fuzzy disjunction D is commutative, associative, and has an identity. In this case, it is not hard to show that $\mathcal{A} \subseteq \mathcal{I}(\mathcal{A})$, i.e., all fundamental memories are fixed points of the min-D AFMM. In other words, the identity $\mathcal{M}(\mathbf{a}^\xi) = \mathbf{a}^\xi$ holds true for any

$\xi \in \{1, \ldots, k\}$. Also, we can show that several transformations of the fundamental memories as well as any minimax combinations of these transformations are fixed points of $\mathcal{M}$ [7]. Now, since an element on the set difference $\mathcal{I}(\mathcal{A}) \setminus \mathcal{A}$ is a spurious memory of $\mathcal{M}$, we deduce that a min-D AFMM has a large number of spurious memories. Moreover, we conclude from (5) that the identity $\mathcal{M}(\mathbf{x}) = \mathbf{a}^\xi$ is satisfied only if $\mathbf{x} \geq \mathbf{a}^\xi$. In other words, a min-$D$ AFMM exhibits tolerance with respect to dilative noise, but it is extremely sensitive to either erosive or mixed (= dilative + erosive) noise. We would like to recall that $\mathbf{x}$ corresponds to a dilated version of a fundamental memory $\mathbf{a}^\xi$ if $\mathbf{x} \geq \mathbf{a}^\xi$. Dually, $\mathbf{x}$ is an eroded version of the fundamental memory $\mathbf{a}^\xi$ if $\mathbf{x} \leq \mathbf{a}^\xi$ [11].

Example 1 *Consider the fundamental memory set*

$$\mathcal{A} = \left\{ \mathbf{a}^1 = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.1 \\ 0.8 \end{bmatrix}, \mathbf{a}^2 = \begin{bmatrix} 0.9 \\ 0.3 \\ 0.5 \\ 0.2 \end{bmatrix}, \mathbf{a}^3 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.6 \\ 0.8 \end{bmatrix} \right\} \tag{6}$$

and the fuzzy disjunction $D_M(x, y) = x \vee y$. Note that D_M is commutative, associative, and has an identity. Furthermore, $J_M(x, y) = \begin{cases} 0, & x \geq y, \\ y, & x < y, \end{cases}$ is the fuzzy co-implication that forms an adjunction with D_M. From (4)*, we obtain the matrix*

$$M_M = A \blacktriangleright_M A^T = \begin{bmatrix} 0 & 0.9 & 0.9 & 0.9 \\ 0.5 & 0 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0 & 0.5 \\ 0.8 & 0.8 & 0.8 & 0 \end{bmatrix}, \tag{7}$$

where "$\blacktriangleright_M$" is the max-J product based on the fuzzy co-implication J_M. Now, given the input vector

$$\mathbf{x} = \begin{bmatrix} 0.7 & 0.6 & 0.1 & 0.9 \end{bmatrix}^T, \tag{8}$$

the min-D_M AFMM yields

$$\mathcal{M}_M(\mathbf{x}) = M \bullet_M \mathbf{x} = \begin{bmatrix} 0.7 & 0.5 & 0.1 & 0.8 \end{bmatrix}^T \neq \mathbf{a}^1, \tag{9}$$

where "$\bullet_M$" is the min-D product based on the fuzzy disjunction D_M. Note that the input $\mathbf{x}$ corresponds to a dilated version of the fundamental memory $\mathbf{a}^1$. In fact, we have $\mathbf{x} = \mathbf{a}^1 + \begin{bmatrix} 0.1 & 0.1 & 0 & 0.1 \end{bmatrix}^T \geq \mathbf{a}^1$. In this example, however, the min-D_M AFMM failed to produce the desired output $\mathbf{a}^1$. Also, since $\mathcal{M}(\mathbf{x})$ does not belong to the fundamental memory set $\mathcal{A}$, the fuzzy set $[0.7, 0.5, 0.1, 0.8]^T$ is a spurious memory.

3 Max-C Projection Autoassociative Fuzzy Memories

First of all, an increasing mapping $C : [0, 1] \times [0, 1] \to [0, 1]$ is a fuzzy conjunction if $C(1, 0) = C(0, 1) = 0$ and $C(1, 1) = 1$. Given a fuzzy conjunction C, the set of all max-C combinations of vectors from $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\} \subseteq [0,1]^n$ is

$$\mathcal{C}(\mathcal{A}) = \left\{ \bigvee_{\xi=1}^{k} C(\lambda_\xi, \mathbf{x}^\xi) : \lambda_\xi \in [0,1] \right\}. \tag{10}$$

Note that $\mathbf{z} \in \mathcal{C}(\mathcal{A})$ if and only if $z_i = \bigvee_{\xi=1}^{k} C(\lambda_\xi, a_i^\xi)$ for all $i = 1, \ldots, n$.

A fuzzy implication $I : [0,1]\times[0,1] \to [0,1]$ is defined as an hybrid monotonous mapping such that $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$ [15]. A fuzzy implication I and a fuzzy conjunction C form an adjunction if and only if

$$C(x,y) \leq z \quad \Longleftrightarrow \quad x \leq I(y,z), \quad \forall x, y, z \in [0,1]. \tag{11}$$

Similarly, in this paper we assume that C is a fuzzy conjunction that form an adjunction with a certain fuzzy implication I.

A max-C projection autoassociative fuzzy memory (max-C PAFM), introduced recently by us in [14], is obtained by replacing $\mathcal{I}(\mathcal{A})$ by the set of all max-C combinations of the fundamental memories in (5). Formally, given a fundamental memory set $\mathcal{A}$, a max-C PAFM $\mathcal{V} : [0,1]^n \to [0,1]^n$ is defined by

$$\mathcal{V}(\mathbf{x}) = \bigvee \left\{ \mathbf{z} \in \mathcal{C}(\mathcal{A}) : \mathbf{z} \leq \mathbf{x} \right\}, \quad \forall \mathbf{x} \in [0,1]^n. \tag{12}$$

In words, a max-C PAFM projects the input $\mathbf{x}$ into the set of all max-C combination of the fundamental memories. Therefore, like a min-D AFMM, a max-C PAFM exhibits a certain tolerance to dilative noise but it is extremely sensitive to either erosive or mixed noise. Furthermore, the following theorem provides an effective formula for the implementation of a max-C PAFM [14]:

Theorem 1 *Let C be a fuzzy conjunction that forms an adjunction with a fuzzy implication I. Given a fundamental memory set $\mathcal{A} = \left\{\mathbf{a}^1, \ldots, \mathbf{a}^k\right\}$, a max-$C$ PAFM satisfies the following for any input $\mathbf{x} = [x_1, \ldots, x_n]^T \in [0,1]^n$:*

$$\mathcal{V}(\mathbf{x}) = \bigvee_{\xi=1}^{k} C(\lambda_\xi, \mathbf{a}^\xi), \text{ where } \lambda_\xi = \bigwedge_{j=1}^{n} I(a_j^\xi, x_j), \ \forall \xi \in \{1, \ldots, k\}. \tag{13}$$

Moreover, $\mathcal{V}(\mathbf{x}) \leq \mathbf{x}$ and $\mathcal{V}(\mathcal{V}(\mathbf{x})) = \mathcal{V}(\mathbf{x})$ holds for all $\mathbf{x} \in [0,1]^n$.

Remark 1 *The parameter λ_ξ in* (13) *measures, in the sense of Bandler-Kohout, the degree of inclusion of the fundamental memory $\mathbf{a}^\xi$ in the input $\mathbf{x}$ [17].*

The following theorem says that, like the min-D AFMM, a max-C PAFM exhibit optimal absolute storage capacity if the fuzzy conjunction C has a left identity.

Theorem 2 *If C is a fuzzy conjunction with an identity, the max-C PAFMs satisfies $\mathcal{V}(\mathbf{a}^\xi) = \mathbf{a}^\xi$ for any fundamental memory set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\}$.*

Example 2 *Consider the fundamental memory set $\mathcal{A}$ given by* (6)*. Let C_M and I_M be the minimum fuzzy conjunction and Gödel fuzzy implication defined respectively by $C_M(x,y) = x \wedge y$ and $I_M(x,y) = \begin{cases} 1, & x \leq y, \\ y, & x > y. \end{cases}$ Note that C_M is a fuzzy conjunction with 1 as identity. Moreover, C_M and I_M form an adjunction. We synthesized the max-C_M PAFM designed for the storage of $\mathcal{A}$ using the adjunction pair (I_M, C_M). We first confirmed that the equation $\mathcal{V}_M(\mathbf{a}^\xi) = \mathbf{a}^\xi$ holds for $\xi = 1,2,3$. Then, we presented the vector $\mathbf{x}$ defined by* (8) *as input to the max-C_M PAFM. We obtained from* (13) *the following coefficients:*

$$\lambda_1 = 1.0, \quad \lambda_2 = 0.1, \quad \textit{and} \quad \lambda_3 = 0.1. \tag{14}$$

Hence, the output of the max-C_M PAFM is

$$\begin{aligned} \mathcal{V}_M(\mathbf{x}) &= C_M(\lambda_1, \mathbf{a}^1) \vee C_M(\lambda_2, \mathbf{a}^2) \vee C_M(\lambda_3, \mathbf{a}^3) \\ &= \begin{bmatrix} 0.6 \\ 0.5 \\ 0.1 \\ 0.8 \end{bmatrix} \vee \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} \vee \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.5 \\ 0.1 \\ 0.8 \end{bmatrix} = \mathbf{a}^1. \end{aligned}$$

Different from the min-D_M AFMM, the max-C_M PAFM retrieved the fundamental memory $\mathbf{a}^1$. Moreover, this example shows that max-C_M PAFMs can be more robust to dilative noise than its corresponding min-D_M AFMM.

Let us conclude the section by emphasizing that we cannot ensure optimal absolute storage capacity if C does not have an identity.

Example 3 *Consider the "compensatory and" fuzzy conjunction defined by*

$$C_Z(x,y) = \sqrt{(xy)(x+y-xy)}. \tag{15}$$

Note that C_z does not have a left identity. Moreover, the fuzzy implication that forms an adjunction with C_z is

$$I_Z(x,y) = \begin{cases} 1, & x = 0, \\ 1 \wedge \left[\dfrac{-x^2 + \sqrt{x^2 + 4x(1-x)y^2}}{2x(1-x)} \right], & 0 < x < 1, \\ y^2, & x = 1. \end{cases} \tag{16}$$

Now, let $\mathcal{V}_Z : [0,1]^4 \to [0,1]^4$ be the max-C_Z PAFM designed for the storage of the fundamental memory set $\mathcal{A}$ given by (6)*. Upon presentation of the fundamental memory $\mathbf{a}^1$ as input we have from* (13) *the following coefficients:*

$$\lambda_1 = 0.28, \quad \lambda_2 = 0.04, \quad \textit{and} \quad \lambda_3 = 0.03. \tag{17}$$

Thus, the output of the max-C_Z PAFM $\mathcal{V}_Z$ is

$$\mathcal{V}_Z(\mathbf{a}^1) = C_Z(\lambda_1, \mathbf{a}^1) \vee C_Z(\lambda_2, \mathbf{a}^2) \vee C_Z(\lambda_3, \mathbf{a}^3) = \begin{bmatrix} 0.35 \\ 0.30 \\ 0.10 \\ 0.43 \end{bmatrix} \neq \mathbf{a}^1. \tag{18}$$

Fig. 1. Gray-scale images that correspond to the fundamental memories $\mathbf{a}^1, \ldots, \mathbf{a}^{12}$.

In a similar fashion, we obtain from the fundamental memories $\mathbf{a}^2$*, and* $\mathbf{a}^3$*, the fuzzy sets*

$$\mathcal{V}_Z(\mathbf{a}^2) = \begin{bmatrix} 0.57 \\ 0.26 \\ 0.37 \\ 0.2 \end{bmatrix} \neq \mathbf{a}^2 \quad \textit{and} \quad \mathcal{V}_Z(\mathbf{a}^3) = \begin{bmatrix} 0.20 \\ 0.37 \\ 0.42 \\ 0.52 \end{bmatrix} \neq \mathbf{a}^3. \tag{19}$$

Note that, in accordance with Theorem 1, the inequality $\mathcal{V}_Z(\mathbf{a}^\xi) \leq \mathbf{a}^\xi$ *holds for* $\xi = 1, 2, 3$*. On the downside, the max-*C_Z *PAFM failed to yield the desired responses* $\mathbf{a}^1$*,* $\mathbf{a}^2$*, and* $\mathbf{a}^3$*.*

4 Computational Experiments

Let us compare the performance of max-C PAFMs and min-D AFMMs for the storage and recall of scale-gray images. Precisely, consider the twelve gray-scale images of size 64×64 shown in Fig. 1. These images have been identified with fuzzy sets $\mathbf{a}^\xi \in [0,1]^{4096}$, for all $\xi = 1, 2, \ldots, 12$. The fundamental memory set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^{12}\} \subseteq [0,1]^{4096}$ was stored in the min-D AFMMs $\mathcal{M}_M$, $\mathcal{M}_P$, and $\mathcal{M}_L$, which are obtained using respectively the maximum, the probabilistic sum, and the Łukasiewicz fuzzy disjunction [7]. We also used the fundamental memory set $\mathcal{A}$ to synthesize the max-C PAFMs $\mathcal{V}_M, \mathcal{V}_P$, and $\mathcal{V}_L$, obtained using respectively the minimum, the product, and the Łukasiewicz fuzzy conjunction[3]. For comparison purposes, we also stored the same fundamental memory set into the optimal linear associative memory (OLAM) [19], the kernel associative memory (KAM) [20], and the recurrent exponential fuzzy associative memory (RE-FAM) [9].

We first confirmed that the nine AMs exhibit optimal absolute storage capacity. Then, we fed them with dilated versions of the fundamental memories. Precisely, in the first scenario, the original images have been corrupted by adding

[3] The experiments were conducted on `MATLAB` in a computer with processor Intel Core i7-5500U, 2.50GHZ, and 8GB RAM.

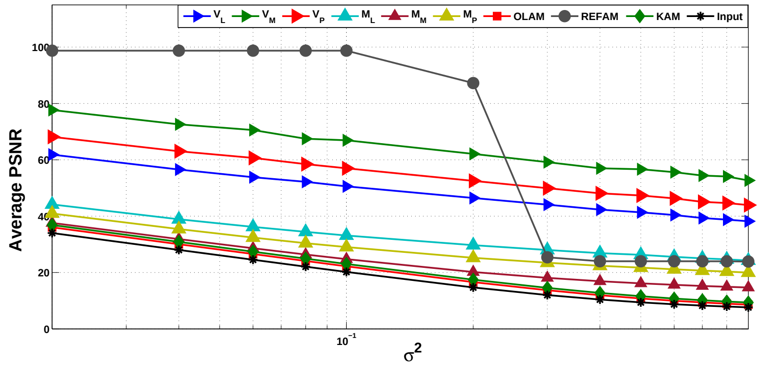

Fig. 2. Average PSNR by the variance σ^2 of the dilative Gaussian noise.

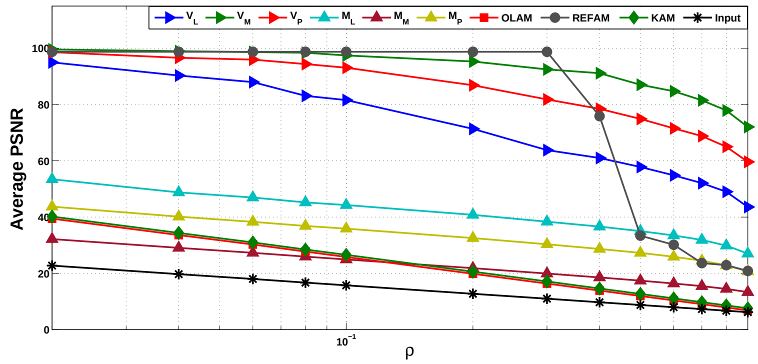

Fig. 3. Average PSNR by the probability ρ of (dilative) salt noise.

the absolute value of a Gaussian noise with zero mean and variance σ^2 between 0.02 and 0.9. In the second scenario, we introduced salt noise with probability ρ ranging from 0.02 to 0.9. The average peak signal-to-noise ratio (PSNR) obtained from 360 trials, i.e., each original image was corrupted 30 times for a certain noise intensity, is shown in Figures 2 and 3. We would like to point out that, in order to avoid infinities, we bounded the PSNR rates by 100.

Note that the new memories $\mathcal{V}_M, \mathcal{V}_P$ and $\mathcal{V}_L$ outperformed the models $\mathcal{M}_L$, $\mathcal{M}_P$, $\mathcal{M}_M$, OLAM, and KAM. For large intensities of noise, the new models yielded PSNR rates larger than the RE-FAM. Furthermore, the min-D_M AFMM $\mathcal{M}_M$ is the worst min-D AFMM. In contrast, the max-C_M PAFM $\mathcal{V}_M$ yielded excellent PSNR rates in both scenarios.

Finally, Fig. 4 provides a visual interpretation of the error correction capability of the min-D_L AFMM and the max-C_M PAFM – the best min-D AFMM

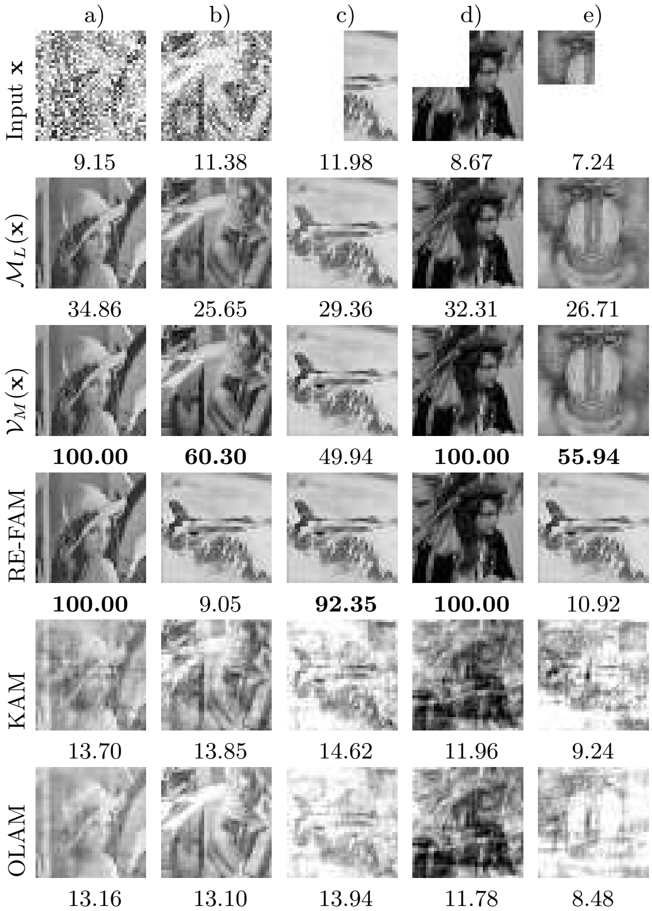

Fig. 4. Error correction capability of the autoassociative memories.

and the best max-C PAFM. This figure also shows the error correction capability of the RE-FAM, KAM, and OLAM models. Specifically, the first row exhibits dilated versions of the fundamental memories obtained by: a) introducing salt noise with probability $\rho = 0.4$. b) adding the absolute value of a Gaussian noise with mean and variance $\sigma^2 = 0.3$. c), d), and e) deleting significant parts of a fundamental memory. The following rows present the corresponding images retrieved by the AMs models. Furthermore, we included below each image in Fig. 4 the corresponding PSNR rate.

5 Concluding Remarks

In this paper, we introduced the class of max-C projection autoassociative fuzzy memories (PAFMs) for the storage and retrieval of fuzzy sets or vectors on the hypercube $[0,1]^n$. Precisely, a max-C PAFM $\mathcal{V}$ is designed for the storage of a fundamental memory set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\}$. Afterward, given an input vector $\mathbf{x} \in [0,1]^n$, which can be a dilated version of fundamental memory $\mathbf{a}^\xi$, the max-C PAFM produces as output $\mathcal{V}(\mathbf{x})$ the projection of $\mathbf{x}$ into the set of all max-C combinations of $\mathbf{a}^1, \ldots, \mathbf{a}^k$. Besides the formal definition, Theorem 1 provides an effective formula for computing the coefficients of the max-C combination and, thus, for the implementation of a max-C PAFM. Furthermore, we showed that a max-C PAFM exhibits optimal absolute storage capacity if the fuzzy conjunction C has a left identity.

In this paper, we also presented some computational experiments concerning the reconstruction of corrupted gray-scale images. The max-C PAFMs always outperformed the min-D AFMMs, KAM, and OLAM for the reconstruction of gray-scale images corrupted by dilative noise. For large intensities of noise, the new memories produced the largest PSNR rates. In particular, the max-C_M PAFM $\mathcal{V}_M$, which is based on the minimum fuzzy conjunction, is very robust in the presence of dilative noise.

In the future, we plan to investigate further the properties of the max-C PAFMs and perform more computational experiments. Furthermore, we intent to establish relations between the new models and others associative memories models from the literature.

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References

[1] Hassoun, M.H., Watta, P.B.: Associative Memory Networks. In Fiesler, E., Beale, R., eds.: Handbook of Neural Computation. Oxford University Press (1997) C1.3:1–C1.3:14

[2] Hopfield, J.J.: Neural Networks and Physical Systems with Emergent Collective Computational Abilities. Proceedings of the National Academy of Sciences **79** (April 1982) 2554–2558

[3] Kosko, B.: Neural Networks and Fuzzy Systems: A Dynamical Systems Approach to Machine Intelligence. Prentice Hall, Englewood Cliffs, NJ (1992)

[4] Sussner, P., Valle, M.E.: Fuzzy Associative Memories and Their Relationship to Mathematical Morphology. In Pedrycz, W., Skowron, A., Kreinovich, V., eds.: Handbook of Granular Computing. John Wiley and Sons, Inc., New York (2008) 733–754

[5] Esmi, E., Sussner, P., Bustince, H., Fernandez, J.: Theta-fuzzy associative memories (theta-fams). IEEE Transactions on Fuzzy Systems **23**(2) (April 2015) 313–326

[6] Sussner, P., Miyasaki, R., Valle, M.E.: An Introduction to Parameterized IFAM Models with Applications in Prediction. In: Proceedings of the 2009 IFSA World Congress and 2009 EUSFLAT Conference, Lisbon, Portugal (July 2009) 247–252
[7] Valle, M.E., Sussner, P.: Storage and Recall Capabilities of Fuzzy Morphological Associative Memories with Adjunction-Based Learning. Neural Networks **24**(1) (January 2011) 75–90
[8] Sussner, P., Schuster, T.: Interval-valued fuzzy associative memories based on representable conjunctions with applications in prediction. In: IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), 2013 Joint. (2013) 344–349
[9] Valle, M.E., de Souza, A.C.: On the recall capability of recurrent exponential fuzzy associative memories based on similarity measures. Mathware and Soft Computing Magazine **22** (2015) 33–39
[10] Valle, M.E., Sussner, P.: A General Framework for Fuzzy Morphological Associative Memories. Fuzzy Sets and Systems **159**(7) (2008) 747–768
[11] Ritter, G.X., Sussner, P., de Leon, J.L.D.: Morphological Associative Memories. IEEE Transactions on Neural Networks **9**(2) (1998) 281–293
[12] Sussner, P., Valle, M.E.: Recall of Patterns Using Morphological and Certain Fuzzy Morphological Associative Memories. In: Proceedings of the IEEE World Conference on Computational Intelligence 2006, Vancouver, Canada (2006) 209–216
[13] Valle, M.E.: An Introduction to Max-plus Projection Autoassociative Morphological Memory and Some of Its Variations. In: Proceedings of the IEEE International Conference on Fuzzy Systems 2014 (FUZZ-IEEE 2014), Beijing, China (July 2014) 53–60
[14] Santos, A.S., Valle, M.E.: Uma introdução às memórias autoassociativas fuzzy de projeções max-C. In: Recentes Avanços em Sistemas Fuzzy. Sociedade Brasileira de Matemática Aplicada e Computacional, Volume 1 , São Carlos, Brasil (Nov. 2016) ISBN: 78-85-8215-079-5.
[15] De Baets, B.: Coimplicators, the forgotten connectives. Trata Mountains Mathematical Publications **12** (1997) 229–240
[16] Deng, T., Heijmans, H.: Grey-scale morphology based on fuzzy logic. Journal of Mathematical Imaging and Vision **16**(2) (2002) 155–171
[17] Sussner, P., Valle, M.E.: Classification of Fuzzy Mathematical Morphologies Based on Concepts of Inclusion Measure and Duality. Journal of Mathematical Imaging and Vision **32**(2) (October 2008) 139–159
[18] Perfilieva, I., Vajgl, M.: Autoassociative fuzzy implicative memory on the platform of fuzzy preorder. In: Proceedings of the 16th World Congress of the International Fuzzy Systems Association (IFSA) and the 9th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), Gijón, Spain, Atlantis Press (June 2015) 1598–1603
[19] Kohonen, T.: Self-organization and associative memory. 3rd edition edn. Springer-Verlag New York, Inc., New York, NY, USA (1989)
[20] Zhang, B.L., Zhang, H., Ge, S.S.: Face Recognition by Applying Wavelet Subband Representation and Kernel Associative Memory. IEEE Transactions on Neural Networks **15**(1) (January 2004) 166–177

Three Interval-valued Associative Memory Versions for Predicting a Certain Socioeconomic Index

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Abstract. The field of type-2 fuzzy systems and in particular interval type-2 fuzzy systems have been under active development in the last decades. Heartened by this progress, we present in this paper some theoretical foundations and applications of three different models of interval-valued fuzzy morphological associative memories (IV-FMAMs) as a rule-based system, two in conjunction with interval-valued fuzzy clustering techniques and one that handles the crisp data directly. We perform simulations regarding the application of IV-FMAMs to the prediction of the monthly rate of participation of certain age groups in the work force of the metropolitan area of São Paulo. The proposed IV-FMAM approaches outperformed a conventional Mamdani-Assilian type-2 fuzzy inference system in terms of mean absolute error.

1 Introduction

Type-2 fuzzy sets, and in particular interval-valued fuzzy sets have found a wide variety of applications in engineering, control, computing with words, and approximate reasoning [9]. Like general type-2 fuzzy sets, interval-valued fuzzy sets can be employed to model the inherent uncertainties regarding fuzzy set membership functions. An approach for dealing with interval-valued fuzzy data is given by IV-FMAMs [15], a recent extension of fuzzy morphological associative memories (FMAMs). Both FMAMs and IV-FMAMs represent lattice computing (LC) approaches toward computational intelligence. LC is a recent and increasingly popular paradigm for processing lattice-order data such as numbers, graphs and fuzzy sets. Examples of LC based fuzzy systems include lattice extensions of fuzzy inference systems [7] and the interactive fuzzy lattice reasoning [8].

Previously, IV-FMAMs were employed to implement interval-valued fuzzy systems in time series prediction problems [15]. In these applications, the crisp input was fuzzified as interval-valued Gaussian fuzzy sets by means of the *interval-valued fuzzy subtractive clustering* (IV-SC) method [11]. Note that some previous knowledge about the problem is required in order to define the parameters of the IV-SC. In this paper we propose an alternate cluster-free IV-FMAM methodology for the cases where such knowledge is not available or not easily obtained.

We also propose to use IV-FMAMs in conjunction with the *interval-valued fuzzy c-means* (IV-FCM) [6] to fuzzify the crisp data. A performance comparison between the three designs will be made by applying the proposed IV-FMAM models to the problem of predicting the monthly percentages of participation of certain age groups in the work force of the metropolitan area of São Paulo [1].

The paper is organized as follows. Section 2 provides a brief review of pertinent notions on lattice theory, $\mathbb{L}$-fuzzy logical operators and $\mathbb{L}$-fuzzy mathematical morphology. In Section 3, we present the IV-FMAMs. Section 4 describes three experimental setups of the IV-FMAMs as rule-based systems and an application to a socioeconomic index time series prediction. The results are compared with the ones obtained using a Mamdani-Assilian type-2 fuzzy system [9], followed by some concluding remarks.

2 Theoretical background

2.1 Some Relevant Concepts of Lattice Theory and Mathematical Morphology

A complete lattice $\mathbb{L}$ is a partially ordered set such that every subset $X \in \mathbb{L}$ has an infimum $\bigwedge X$ and a supremum $\bigvee X$ in $\mathbb{L}$. Recall that a *partial order* is a reflexive, antisymmetric, and transitive binary relation "$\leq$" [4]. The unit interval $[0,1]$ with the usual (total) ordering yields an example of a complete lattice. Another example is given by $\mathbb{I} = \{u = [\underline{u}, \overline{u}] \subseteq [0,1]\}$ if we consider the partial order $u \leq v \Leftrightarrow \underline{u} \leq \underline{v}$ and $\overline{u} \leq \overline{v}$. Note the conceptual difference between $[0,1]$ in which every pair of elements x, y is comparable, i.e., $x \leq y$ or $y \leq x$, and $\mathbb{I}$ in which there are elements u, v that are incomparable. A component-wise partial order can also be defined on $\mathbb{L}^n$ by setting

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \Leftrightarrow a_i \leq b_i,\ i = 1, \ldots, n. \tag{1}$$

If $\mathbb{L}$ is a (complete) lattice, then $\mathbb{L}^n$ is also a (complete) lattice. Similarly, we have that if $\mathbb{L}$ is a (complete) lattice then the class of functions $X \to \mathbb{L}$, $\mathbb{L}^X$, is also a (complete) lattice. Here, the partial order on $\mathbb{L}^X$ is given as follows for all $f, g \in \mathbb{L}^X$:

$$f \leq g \Leftrightarrow f(x) \leq g(x)\ \forall x \in X. \tag{2}$$

Let $\mathbb{L}$ be a complete lattice and $X \neq \emptyset$. The partial order on $\mathbb{L}^X$ of Equation (2) induces a partial order on $\mathcal{F}_{\mathbb{L}}(X)$, the class of $\mathbb{L}$-fuzzy sets over the universe X. Recall that an $\mathbb{L}$-fuzzy set A consists of a universe X together with a membership function $\mu_A : X \to \mathbb{L}$ [3]. For $\mathbb{L}$-fuzzy sets A and B, we have $A \leq B$ if and only if $\mu_A \leq \mu_B$. The class of fuzzy sets over the universe X, denoted $\mathcal{F}(X)$, and the class of interval-valued fuzzy sets over the universe X, denoted $\mathcal{F}_{\mathbb{I}}(X)$, represent particular classes of $\mathbb{L}$-fuzzy sets for particular choices of $\mathbb{L}$.

It is well established that complete lattices form an appropriate framework for *mathematical morphology* (MM) [5]. In this framework, the basic operators of MM are the *algebraic erosion*, $\varepsilon : \mathbb{L} \to \mathbb{M}$ and the *algebraic dilation*, $\delta : \mathbb{L} \to \mathbb{M}$, that satisfies, respectively, $\varepsilon(\bigwedge M) = \bigwedge \varepsilon(M)$ and $\delta(\bigvee M) = \bigvee \delta(M)$, for all $M \subseteq \mathbb{L}$. Moreover, the concept of an *adjunction* plays a prominent role in MM.

Definition 1. *A pair* (ε, δ) *consisting of mappings* $\varepsilon : \mathbb{M} \to \mathbb{L}$ *and* $\delta : \mathbb{L} \to \mathbb{M}$ *is called an* adjunction *between* $\mathbb{M}$ *and* $\mathbb{L}$ *if and only if for all* $x \in \mathbb{L}$ *and all* $y \in \mathbb{M}$*:*

$$\delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y). \tag{3}$$

If the pair (ε, δ) is an adjunction then ε is an erosion and δ a dilation [5].

2.2 $\mathbb{L}$-Fuzzy Operators

The purpose of this section is to recall the definitions of some morphological operators and matrix products on the complete lattice $\mathcal{F}_{\mathbb{L}}(X)$ [15]. Let us begin by the definition of $\mathbb{L}$-fuzzy conjunctions and implications [2]:

Definition 2. *Let* $\mathbb{L}$ *be a complete lattice.*

- *A* conjunction *on* $\mathbb{L}$ *or* $\mathbb{L}$-fuzzy conjunction *is defined as an increasing mapping* $\mathcal{C} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ *that satisfies* $C(0_{\mathbb{L}}, 0_{\mathbb{L}}) = \mathcal{C}(0_{\mathbb{L}}, 1_{\mathbb{L}}) = \mathcal{C}(1_{\mathbb{L}}, 0_{\mathbb{L}}) = 0_{\mathbb{L}}$ *and* $\mathcal{C}(1_{\mathbb{L}}, 1_{\mathbb{L}}) = 1_{\mathbb{L}}$*.*
- *An operator* $\mathcal{I} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ *that is decreasing in the first argument and that is increasing in the second argument is called an* implication *on* $\mathbb{L}$ *or* $\mathbb{L}$-fuzzy implication *if the equations* $\mathcal{I}(0_{\mathbb{L}}, 0_{\mathbb{L}}) = \mathcal{I}(0_{\mathbb{L}}, 1_{\mathbb{L}}) = \mathcal{I}(1_{\mathbb{L}}, 1_{\mathbb{L}}) = 1_{\mathbb{L}}$ *and* $\mathcal{I}(1_{\mathbb{L}}, 0_{\mathbb{L}}) = 0_{\mathbb{L}}$ *are satisfied.*
- *If* $\mathbb{L}$ *equals the complete chain* $[0, 1]$*, then one obtains the usual fuzzy logical operators. In the special case where* $\mathbb{L} = \mathbb{I}$*, we speak of* interval-valued fuzzy (IV fuzzy) operators*, in particular of* IV fuzzy conjunctions and implications*.*

Consider the following matrix product derived from $\mathbb{L}$-fuzzy operators [15]:

Definition 3. *Given an* $\mathbb{L}$*-fuzzy conjunction* $\mathcal{C}$ *and an* $\mathbb{L}$*-fuzzy implication* $\mathcal{I}$*, the* sup-$\mathcal{C}$ product *of* $A \in \mathbb{L}^{m \times k}$ *and* $B \in \mathbb{L}^{k \times n}$*, denoted by* $E = A \circ_{\mathcal{C}} B$*, and the* inf-$\mathcal{I}$ product*, denoted by* $G = A \circledast B$ *are defined, respectively, for all* $i = 1, \ldots, m$ *and* $j = 1, \ldots, n$*, as follows:*

$$e_{ij} = \bigvee_{\xi=1}^{k} \mathcal{C}(a_{i\xi}, b_{\xi j}), \text{ and } g_{ij} = \bigwedge_{\xi=1}^{k} \mathcal{I}(b_{\xi j}, a_{i\xi}) \ . \tag{4}$$

The following proposition holds for every complete lattice $\mathbb{L}$:

Proposition 1 *The operator* $\delta_W : \mathbb{L}^n \to \mathbb{L}^m$ *that are given by*

$$\delta_W(\mathbf{x}) = W \circ \mathbf{x}, \ \forall \mathbf{x} \in \mathbb{L}^n. \tag{5}$$

represents a dilation for every $W \in \mathbb{L}^{m \times n}$ *from the complete lattice* $\mathbb{L}^n$ *to the complete lattice* $\mathbb{L}^m$ *if and only if* $\mathcal{C}(w, \cdot) : \mathbb{L} \to \mathbb{L}$ *represents a dilation for every* $w \in \mathbb{L}$*.*

As we shall point out in the next section for the interval-valued case, the output of an interval-valued fuzzy associative memory can be modeled by means of Equation (5). Furthermore, the model given by Equation (5) will be called *morphological*. The next section focuses on IV-FMAMs.

3 Interval-Valued FMAMs

In this section we will present an associative memory model for interval-valued fuzzy sets. Consider initially X and Y arbitrary universes and a set of pairs or associations $\left\{\left(\mathbf{p}^\xi, \mathbf{q}^\xi\right) : \xi \in \mathcal{K}\right\} \subseteq \mathcal{F}_{\mathbb{I}}(X) \times \mathcal{F}_{\mathbb{I}}(Y)$, called the *fundamental memory set* [13]. An *IV fuzzy associative memory* (*IV*-FAM) is an input-output system given by a mapping $\mathcal{W} : \mathcal{F}_{\mathbb{I}}(X) \to \mathcal{F}_{\mathbb{I}}(Y)$ that should ideally satisfy $\mathcal{W}(\mathbf{p}^\xi) = \mathbf{q}^\xi$ for all $\xi \in \mathcal{K}$ and $\mathcal{W}(\tilde{\mathbf{p}}^\xi) = \mathbf{q}^\xi$ if $\tilde{\mathbf{p}}^\xi$ is approximately equal to $\mathbf{p}^\xi$, in symbols $\tilde{\mathbf{p}}^\xi \approx \mathbf{p}^\xi$. The meaning of the symbol $\approx$ depends on the application and on the practitioner's requirements. Since our objective is to employ IV-FAMs in order to implement rule-based systems, it would be more adequate to consider the condition $\tilde{\mathbf{p}}^\xi \approx \mathbf{p}^\xi \Rightarrow \mathcal{W}(\tilde{\mathbf{p}}^\xi) \approx \mathbf{q}^\xi$ instead.

For simplicity, we concentrate on the case where X, Y, and $\mathcal{K}$ are finite. Let $|X| = n$, $|Y| = m$, and $|\mathcal{K}| = k$. Thus, we view an IV-FAM as a mapping $\mathcal{W} : \mathbb{I}^n \to \mathbb{I}^m$. We say that $\mathcal{W}$ represents a *sup-$\mathcal{C}$ IV-FAM* if $\mathcal{W}(\mathbf{x}) = \delta_W(\mathbf{x}) = W \circ_{\mathcal{C}} \mathbf{x}$, $\forall \mathbf{x} \in \mathbb{I}^n$, for some $W \in \mathbb{I}^{m \times n}$. In this paper we focus on the construction of IV-FMAMs based on representable IV fuzzy conjunctions and their adjoint implications [2,14]. We will now present a recipe for constructing IV fuzzy conjunctions and their adjoint implications, beginning with the definition of a representable IV fuzzy conjunction [2]:

Proposition 2 *If C is a fuzzy conjunction, then the operator $\mathcal{C}^r_C$, that is defined as follows for all $u = [\underline{u}, \overline{u}], v = [\underline{v}, \overline{v}] \in \mathbb{I}$, yields an IV fuzzy conjunction.*

$$\mathcal{C}^r_C(u, v) = [C(\underline{u}, \underline{v}), C(\overline{u}, \overline{v})]. \tag{6}$$

Definition 4. *The IV fuzzy conjunction $\mathcal{C}^r_C$ of Equation* (6) *is referred to as the* representable conjunction with representative *C.*

Let $\mathcal{C}^r_C$ be a representable interval-valued fuzzy conjunction with representative C and I the adjoint fuzzy implication of C. The adjoint IV fuzzy implication of $\mathcal{C}^r_C$ can be obtained by means of Equation (7), as follows [14]:

$$\mathcal{I}^n_I(u, v) = [I(\underline{u}, \underline{v}) \wedge I(\overline{u}, \overline{v}), I(\overline{u}, \overline{v})]. \tag{7}$$

An example of dilative fuzzy conjunction is the cross-ratio uninorm [17], denoted using the symbol C_F. The formulas for C_F and its adjoint implication I_F can be found below:

$$C_F(x,y) = \begin{cases} 1, & \text{if } (x,y) = (0,1) \text{ or } (1,0) \\ \frac{xy}{(1-y)(1-x)+xy}, & \text{otherwise.} \end{cases}, \quad I_F(x,y) = \begin{cases} 1, & \text{if } (x,y) = (0,0) \text{ or } (1,1) \\ \frac{(1-x)y}{y(1-x)+x(1-y)}, & \text{otherwise.} \end{cases}$$

Observe that the representable IV fuzzy conjunction $\mathcal{C}^r_{C_F}$ and its adjoint implication $\mathcal{I}^n_{I_F}$, given by means of Equation (7), represents a pair consisting of an erosion and a dilation. Hence, an IV-FAM based on $\mathcal{C}^r_{C_F}$, that will be denoted here by $\mathcal{W}^r_F$, will be morphological. In this research paper, we will employ sup-$\mathcal{C}^r_C$ matrix products in the recall phases of IV-FAMs, in particular sup-$\mathcal{C}^r_C$ matrix products that give rise to dilations and, therefore, IV-FMAMs.

Let $P = [\mathbf{p}^1, ..., \mathbf{p}^k] \in \mathbb{I}^{n \times k}$, $Q = [\mathbf{q}^1, ..., \mathbf{q}^k] \in \mathbb{I}^{m \times k}$ be matrices which columns are formed by the pairs $(\mathbf{p}^j, \mathbf{q}^j)$ of fundamental memories and let $\mathcal{C}_C^r$ be an IV fuzzy conjunction. Consider the problem of determining the weight matrix W of a sup-$\mathcal{C}_C^r$ IV-FMAM given by the dilation δ_W. As an extension of the *fuzzy learning by adjunction* [13] for sup-C FMAMs to *IV fuzzy learning by adjunction (IV-FLA)* for sup-$\mathcal{C}_C^r$ FMAMs, we propose to construct its weight matrix $W \in \mathbb{I}^{n \times k}$ as follows:

$$W = Q \circledast_{n,I} P^t, \tag{8}$$

where the IV fuzzy implication used in the Inf-$\mathcal{I}$ product is the adjoint pair of $\mathcal{C}_C^r$. Finally, upon presentation of an input pattern $\mathbf{x} \in \mathbb{I}^n$, the output pattern $\mathbf{y} \in \mathbb{I}^m$ of the corresponding IV-FMAM can be calculated by the sup-$\mathcal{C}_C^r$ product of W and $\mathbf{x}$. In the following section, we present three experimental setups of the IV-FMAM $\mathcal{W}_F^r$ in order to implement a IV fuzzy inference system for a time series prediction problem.

4 Some Experimental Setups of IV-FMAMs Towards Time Series Prediction

In this section, we present three versions of the experimental setup regarding the use of sup-$\mathcal{C}_C^r$ IV-FMAMs for building interval-valued fuzzy rule-based systems. The first two approaches rely on IV-fuzzy clustering techniques, namely, the *IV fuzzy c-means clustering technique* [6] and the *IV fuzzy subtractive clustering technique* [11], both extensions of the widely known fuzzy c-means and the fuzzy subtractive clustering [12], respectively. These approaches are appropriate when expert knowledge is available, since clustering methods require parameters such as the number of clusters of data and cluster radius. Our third approach introduced in Section (4.2) is preferable when there is little knowledge about the problem, since it employs only the IV-FMAM and crisp data. In the Section (4.3), we present an application of the IV-FMAMs in a time series forecasting.

4.1 Cluster-based IV-FMAM Approach

The problem of time series forecasting consists of predicting b future values using the past a values. In order to tackle this problem, we employ a sup-$\mathcal{C}_F^r$ IV-FMAM to model a rule-based system consisting of k rules, each one having a IV fuzzy antecedents and b IV fuzzy consequents, each rule being build by means of a fuzzy clustering. Specifically, we employed the IV-FCM and the IV-SC on the crisp training data, that are contained along with the testing data in a finite universe $\mathcal{U} \times \mathcal{V} \subseteq \mathbb{R}^a \times \mathbb{R}^b$.

In the case of the IV-FCM, given a number k of clusters and a "fuzzifier" parameter $\mathbf{m} = [m_1, m_2]$, it produces the cluster centers $c_{\mathbf{p}}^{\xi} \in \mathbb{R}^a$ and $c_{\mathbf{q}}^{\xi} \in \mathbb{R}^b$ with respective component-wise standard deviations $\sigma_{\mathbf{p}} \in \mathbb{R}^a$ and $\sigma_{\mathbf{q}} \in \mathbb{R}^b$ of IV fuzzy Gaussian membership functions $\mathbf{p}^{\xi}$ and $\mathbf{q}^{\xi}$ for $\xi = 1, \ldots, k$. Similarly, given the cluster radius $r \in \mathbb{R}^a$ and the "fuzzifier" parameter $\mathbf{m}$, the IV-SC produces k cluster centers $c^\xi_{\mathbf{p}} \in \mathbb{R}^a$ and $c^\xi_{\mathbf{q}} \in \mathbb{R}^b$ with respective component-wise standard deviations $\sigma_{\mathbf{p}} \in \mathbb{R}^a$ and $\sigma_{\mathbf{q}} \in \mathbb{R}^b$ of $\mathbf{p}^\xi$ and $\mathbf{q}^\xi$.

The antecedents $\mathbf{p}^\xi$ and the consequents $\mathbf{q}^\xi$ produced by either IV-FCM or IV-SC are respectively contained in $\mathcal{F}_{\mathbb{I}}(\mathcal{U})$ and $\mathcal{F}_{\mathbb{I}}(\mathcal{V})$, where $\mathcal{U} = \{\mathbf{u}^1, \ldots, \mathbf{u}^m\}$ and $\mathcal{V} = \{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$. Their components can be calculated as follows:

$$p^\xi_j = exp\left[-\frac{1}{2}\sum_{l=1}^a \left|\frac{(\mathbf{u}^j)_l - (\mathbf{c}^\xi_{\mathbf{p}})_l}{(\sigma_{\mathbf{p}})_l}\right|^{\frac{2}{m_2-1}}, -\frac{1}{2}\sum_{l=1}^a \left|\frac{(\mathbf{u}^j)_l - (\mathbf{c}^\xi_{\mathbf{p}})_l}{(\sigma_{\mathbf{p}})_l}\right|^{\frac{2}{m_1-1}}\right], \tag{9}$$

$$q^\xi_i = exp\left[-\frac{1}{2}\sum_{l=1}^b \left|\frac{(\mathbf{v}^i)_l - (\mathbf{c}^\xi_{\mathbf{q}})_l}{(\sigma_{\mathbf{q}})_l}\right|^{\frac{2}{m_2-1}}, -\frac{1}{2}\sum_{l=1}^b \left|\frac{(\mathbf{v}^i)_l - (\mathbf{c}^\xi_{\mathbf{q}})_l}{(\sigma_{\mathbf{q}})_l}\right|^{\frac{2}{m_1-1}}\right]. \tag{10}$$

In principle, the pairs $(\mathbf{p}^\xi, \mathbf{q}^\xi)$ corresponding to the training data can be used to compute the weight matrix $W \in \mathbb{I}^{m\times n}$ by means of Equation (8). Briefly, training is performed as follows:

1. Apply the IV-FCM clustering technique with k centers or the IV-SC with radius center r, and fuzzifier parameter $\mathbf{m}$ to the training data in $\mathcal{U} \times \mathcal{V} \subseteq \mathbb{R}^a \times \mathbb{R}^b$ in order to obtain k centers $c^\gamma_{\mathbf{p}} \in \mathbb{R}^a$ and $c^\xi_{\mathbf{q}} \in \mathbb{R}^b$ with respective component-wise standard deviations $\sigma_{\mathbf{p}} \in \mathbb{R}^a$ and $\sigma_{\mathbf{q}} \in \mathbb{R}^b$;
2. Create the discrete intervalar gaussian antecedents $\mathbf{p}^\xi$ in $\mathcal{F}_{\mathbb{I}}(\mathcal{U})$ and consequents $\mathbf{q}^\xi$ in $\mathcal{F}_{\mathbb{I}}(\mathcal{V})$ using Equations (9) and (10);
3. Compute the weight matrix $W \in \mathbb{I}^{m\times n}$ by means of $W = (\mathbf{q}^\xi)^t \circledast \mathbf{p}^\xi$.

Given the weight matrix W, for each $\mathbf{p}$ in the test data set, testing is performed as follows:

1. Compute $\mathbf{q} = W \circ \mathbf{p}$ to obtain an interval-valued fuzzy vector;
2. Average $\mathbf{q}$ component-wise to type-reduce it;
3. Defuzzify the type-reduced vector using the *centroid* defuzzification method to obtain a real valued output (Nie-Tan pointwise average type-reduction and defuzzification method [10]).

However, an explicit construction of W is not required since the representation as an IV fuzzy set of the antecedent part of a crisp test datum $\mathbf{u}^d$ in $\mathcal{U}$ yields an input pattern $\mathbf{p} \in \mathcal{F}_{\mathbb{I}}(\mathcal{U})$ of the form $\mathbf{p} = [0_{\mathbb{I}}, \ldots, 0_{\mathbb{I}}, e_{\mathbb{I}}, 0_{\mathbb{I}}, \ldots, 0_{\mathbb{I}}]^T$. More precisely, we have $p_j = \begin{cases} e_{\mathbb{I}}, & \text{if } j = d \\ 0_{\mathbb{I}}, & \text{otherwise.}\end{cases}$. Here, the symbol $e_{\mathbb{I}}$ denotes the identity element $[0.5, 0.5]$ of the IV fuzzy conjunction $\mathcal{C}^r_F$. For $\mathbf{p} \in \mathcal{F}_{\mathcal{I}}(\mathcal{U}) \simeq \mathbb{I}^m$ of this form, $\mathbf{q} = W \circ_{r,F} \mathbf{p}$ has the following components for $i = 1, \ldots, m$:

$$q_i = W \circ_{r,F} p_i = \bigvee_{j=1}^{n} \mathcal{C}^r_F(w_{ij}, p_j) = w_{id} = \bigwedge_{\xi=1}^{k} \mathcal{I}^n_F(p^\xi_j, q^\xi_i)\,. \tag{11}$$

We obtain a final, crisp prediction value from $\mathbf{q}$ after an application of type-reduction and defuzzification using the *Nie-Tan* method that requires a very low computational effort [16].

4.2 Cluster-free IV-FMAM Approach

Given crisp training data $(\mathbf{x}^\xi, \mathbf{y}^\xi) \in \mathbb{R}^a \times \mathbb{R}^b$, where $\xi = 1, \ldots, t$, we propose to build a rule-based system based on the sup-$\mathcal{C}_F^r$ IV-FMAM such that each training point $(\mathbf{x}^\xi, \mathbf{y}^\xi)$ yields a rule with a IV-fuzzy antecedents and b IV-fuzzy consequents. To this end, we considered a discrete universe of the form $\mathcal{U} \times \mathcal{V} \subseteq \mathbb{R}^a \times \mathbb{R}^b$, where $\mathcal{U} = \{\mathbf{u}^1, \ldots, \mathbf{u}^m\}$ and $\mathcal{V} = \{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ and we computed antecedents $\mathbf{p}^\xi$ and consequents $\mathbf{q}^\xi$, that are respectively contained in $\mathbb{I}^m \simeq \mathcal{F}_\mathbb{I}(\mathcal{U})$ and $\mathbb{I}^n \simeq \mathcal{F}_\mathbb{I}(\mathcal{V})$, as described below.

Let $\sigma_\mathbf{x} \in \mathbb{R}^a$ and $\sigma_\mathbf{y} \in \mathbb{R}^b$ denote respectively the standard deviations of $\mathbf{x}^\xi$ and $\mathbf{y}^\xi$. Given $\mathbf{m} = [m_1, m_2]$, the components $p_j^\xi = [\underline{p_j^\xi}, \overline{p_j^\xi}]$ and $q_i^\xi = [\underline{q_i^\xi}, \overline{q_i^\xi}]$ of $\mathbf{p}^\xi$ and $\mathbf{q}^\xi$ can be calculated as follows:

$$p_j^\xi = exp\left[-\frac{1}{2}\sum_{l=1}^{a}\left|\frac{(\mathbf{u}^j)_l - x_l^\xi}{(\sigma_\mathbf{x})_l/k}\right|^{\frac{2}{m_2-1}}, -\frac{1}{2}\sum_{l=1}^{a}\left|\frac{(\mathbf{u}^j)_l - x_l^\xi}{(\sigma_\mathbf{x})_l/k}\right|^{\frac{2}{m_1-1}}\right], \tag{12}$$

$$q_i^\xi = exp\left[-\frac{1}{2}\sum_{l=1}^{a}\left|\frac{(\mathbf{v}^i)_l - y_l^\xi}{(\sigma_\mathbf{y})_l/k}\right|^{\frac{2}{m_2-1}}, -\frac{1}{2}\sum_{l=1}^{a}\left|\frac{(\mathbf{v}^i)_l - y_l^\xi}{(\sigma_\mathbf{y})_l/k}\right|^{\frac{2}{m_1-1}}\right]. \tag{13}$$

The pairs $(\mathbf{p}^\xi, \mathbf{q}^\xi)$ corresponding to the training data can be used to compute the weight matrix $W \in \mathbb{I}^{m\times n}$ by means of Equation (8). Training is performed as follows:

1. Create the discrete IV-fuzzy Gaussian antecedents $\mathbf{p}^\xi$ in $\mathbb{I}^m \simeq \mathcal{F}_\mathbb{I}(\mathcal{U})$ and consequents $\mathbf{q}^\xi$ in $\mathcal{F}_\mathbb{I}(\mathcal{V})$ using Equations (12) and (13);
2. Compute the weight matrix $W \in \mathbb{I}^{m\times n}$ using $W = \mathbf{q}^\xi \circledast_{n,F} (\mathbf{p}^\xi)^t$.

Testing is performed exactly as the aforementioned cluster-based methodology.

4.3 Prediction of a Brazilian Socioeconomic Index

The Economically Active Population Index (PEA) refers to the percentage of economically active persons, i.e., people who are currently employed or actively looking for a job, within certain age groups. The data for the computation of the PEA index are collected on a monthly basis by DIEESE, the “Inter-Union Department of Statistics and Socioeconomic Studies” [1], and provides valuable information concerning the current state of the workforce.

Here, we employed the proposed IV-FMAM methodologies in order to forecast the PEA index of the metropolitan area of São Paulo from January 1985 to December 2012. The population under consideration is divided into the following age brackets: 10-15, 16-24, 25-39, 40-49, 50-59 and 60+. In view of the seasonal differences in the PEA values and the differences in the time series generated by the age groups, we chose to employ one predictor model for each month and age group. Let $s_\gamma \in \mathbb{R}$ be standardized samples of the time series. The goal is to estimate the s_q from a subset of the past values $\{s_1, s_2, ..., s_{q-1}\}$. In particular, we simply used the last three monthly PEA indices $\{s_{\gamma-3}, s_{\gamma-2}, s_{\gamma-1}\}$ to predict the next index s_γ. We considered finite universes of discourse $\mathcal{U} = \{\mathbf{u}^1, \ldots, \mathbf{u}^m\}$ and $\mathcal{V} = \{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ comprising $m = 50^3$ and $n = 50$ equally spaced points in $[-5, 5]^3$ and $[-5, 5]$, respectively. The centers $c^\xi_{\mathbf{p}}, c^\xi_{\mathbf{q}}$ and standard deviations $\sigma_{\mathbf{p}}, \sigma_{\mathbf{q}}$ produced by the clustering methods can be used to compute the entries $p^\xi_j = [\underline{p^\xi_j}, \overline{p^\xi_j}]$ and $q^\xi_i = [\underline{q^\xi_i}, \overline{q^\xi_i}]$ of the finite Gaussian IV fuzzy sets $\mathbf{p}^\xi \in \mathcal{F}_{\mathbb{I}}(\mathcal{U}) \simeq \mathbb{I}^n$ and $\mathbf{q}^\xi \in \mathcal{F}_{\mathbb{I}}(\mathcal{V}) \simeq \mathbb{I}^m$ via Equations (9) to (10). In the case of the cluster-free approach, the antecedents and consequents of the rules are derived directly from the training data via Equations (12) and (13).

The performance of the IV-FMAM approaches suggested in this paper depends on the choices of the number of clusters k of the IV-FCM, a cluster radius r of the IV-SC, and a fuzzifier parameter $\mathbf{m} = [m_1, m_2]$. In our experiments, we employed $k = 10$, $r = 0.1$, and $\mathbf{m} = [1.9, 2.1]$ since they displayed the best training results for all proposed models. Here, the performance was measured in terms of the mean absolute error (MAE).

We compared the prediction results produced by the sup-$\mathcal{C}^r_F$ IV-FMAM models with the ones produced by an interval type-2 fuzzy inference system (IT2-FIS) with the same fuzzifier parameter using the test data from January 2003 to December 2012. Table (1) displays the testing errors produced by the $\mathcal{W}^r_F$ models and the interval type-2 fuzzy inference system. Figures (1) and (2) illustrate the prediction results obtained by all $\mathcal{W}^r_F$ models and a conventional interval type-2 fuzzy system, respectively, in comparison with the real data.

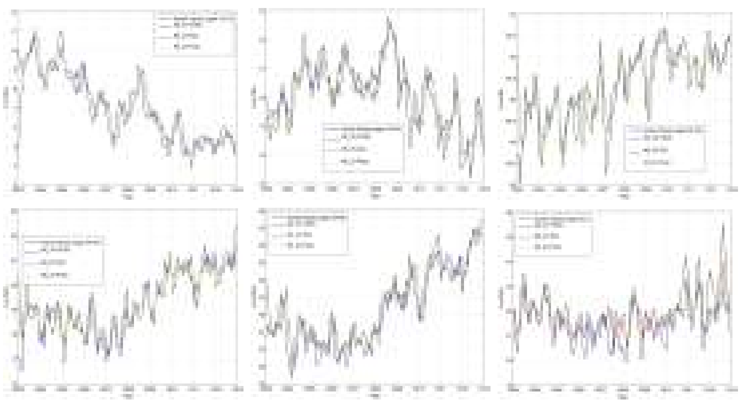


Fig. 1. Predictions obtained by the three versions of $\mathcal{W}^r_F$: with IV-FCM, IV-SC, and the cluster-free approach (C-Free), for the age groups: 10-15, 16-24 , 25-39, 40-49, 50-59 and 60+ from left to right, top to bottom.

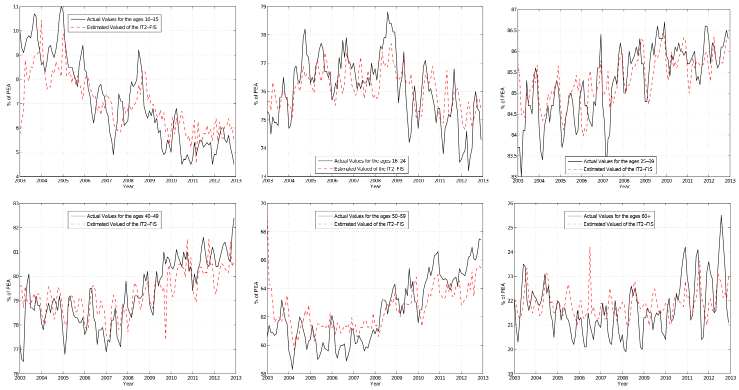


Fig. 2. Predictions obtained by the IT2-FIS for the age groups: 10-15, 16-24 , 25-39, 40-49, 50-59 and 60+ from left to right, top to bottom.

Table 1. MAEs produced by the three versions of $\mathcal{W}_F^r$ IV-FMAM models and the IT2-FIS on the test data for the each age group of the PEA index.

Method / Age Bracket	10-15	16-24	25-39	40-49	50-59	60+
$\mathcal{W}_F^r$	0.60	0.49	0.35	0.46	0.72	0.71
$\mathcal{W}_F^r$ (IV-FCM)	0.88	0.62	0.51	0.62	0.95	0.87
$\mathcal{W}_F^r$ (IV-SC)	0.60	0.50	0.35	0.46	0.71	0.66
IT2-FIS	1.15	0.78	0.60	0.82	1.40	0.97

5 Concluding Remarks

Following the recent advances in type-2 fuzzy systems and, in particular, interval type-2 fuzzy systems, we presented in this paper three versions of our alternative approach towards an IV fuzzy system based on the sup-$\mathcal{C}_C^r$ IV-FMAM, two in conjunction with clustering techniques and one employing raw data directly. We applied the proposed systems to the problem of forecasting the monthly rates of participation of given age groups in the work force of the metropolitan area os São Paulo. The performance in terms of mean absolute error were compared with the ones of a Mamdani-Assilian IT2-FIS. In our simulations all the sup-$\mathcal{C}_C^r$ IV-FMAM approaches exhibited significantly better results than the conventional IT2-FIS.

Note that the IV-FMAM approach presented in this paper depends on the complete lattice structure of $\mathbb{I}$ and in particular on the fact that elements of $\mathbb{I}$ were partially ordered in terms of Equation (1). In the future, we intend to investigate the suitability of other partial orders in conjunction with IV-FAMs. Furthermore, we plan to develop full type-2 fuzzy morphological associative memories (T2-FMAMs) and employ them in type-2 fuzzy inference systems.

References

1. "Departamento Intersindical de Estatística e Estudos Socioeconômicos." [Online]. Available: www.dieese.org.br
2. G. Deschrijver and C. Cornelis, "Representability in interval-valued fuzzy set theory," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 15, no. 3, pp. 345–361, 2007.
3. J. Goguen, "L-fuzzy sets," *Journal of Mathematical Analysis and Applications*, vol. 18, no. 1, pp. 145–174, Apr. 1967.
4. G. A. Grätzer, *Lattice Theory: First Concepts and Distributive Lattices*. San Francisco, CA: W. H. Freeman, 1971.
5. H. Heijmans and C. Ronse, "The algebraic basis of mathematical morphology I. dilations and erosions," *Computer Vision, Graphics, and Image Processing*, vol. 50, no. 3, pp. 245 – 295, 1990.
6. C. Hwang and F. C.-H. Rhee, "Uncertain Fuzzy Clustering : Interval Type-2 Fuzzy Approach to C-Means," *IEEE Transaction on Fuzzy Systems*, vol. 15, no. 1, pp. 107–120, 2007.
7. V. G. Kaburlasos and A. Kehagias, "Fuzzy inference system (FIS) extensions based on the lattice theory," *IEEE Trans. Fuzzy Systems*, vol. 22, no. 3, pp. 531–546, 2014.
8. V. G. Kaburlasos and G. A. Papakostas, "Learning Distributions of Image Features by Interactive Fuzzy Lattice Reasoning in Pattern Recognition Applications," *IEEE Computational Intelligence Magazine*, vol. 10, no. 3, pp. 42–51, 2015.
9. J. M. Mendel, "Type-2 fuzzy sets and systems: An overview," *IEEE Computational Intelligence Magazine*, vol. 2, no. 2, pp. 20–29, 2007.
10. M. Nie and W.W. Tan, "Towards an efficient type-reduction method for interval type-2 fuzzy logic systems," in *IEEE International Conference on Fuzzy Systems*, 2008, pp. 1425-1432.
11. L. T. Ngo and B. H. Pham, "A Type-2 Fuzzy Subtractive Clustering," *Adv. Intell. Soft Comput.*, vol. 125, pp. 395–402, 2012.
12. W. Pedrycz and F. Gomide, *Fuzzy systems engineering: toward human-centric computing.*, Wiley-IEEE Press, 1 ed., 2007.
13. P. Sussner and M. E. Valle, "Fuzzy associative memories and their relationship to mathematical morphology," in *Handbook of Granular Computing*, W. Pedrycz, A. Skowron, and V. Kreinovich, Eds. New York: John Wiley and Sons, Inc., 2008, ch. 33.
14. P. Sussner, M. Nachtegael, T. Mélange, G. Deschrijver, E. Esmi, and E. Kerre, "Interval-valued and intuitionistic fuzzy mathematical morphologies as special cases of L-fuzzy mathematical morphology," *Journal of Mathematical Imaging and Vision*, vol. 43, no. 1, pp. 50–71, Apr. 2012.
15. P. Sussner and T. Schuster, "Interval-valued fuzzy morphological associative memories based on representable conjunctions," in *Proceedings of the IFSA-NAFIPS Joint Congress 2013*, Edmonton, CA, Jul. 2013, pp. 344–349.
16. D. Wu, "An overview of alternative type-reduction approaches for reducing the computational cost of interval type-2 fuzzy logic controllers," in *2012 IEEE International Conference on Fuzzy Systems*, Jun. 2012, pp. 1–8.
17. R. Yager and A. Rybalov, "Uninorm aggregation operators," *Fuzzy Sets and Systems*, vol. 80, no. 1, pp. 111–120, 1996.

CONFERENCE REPORT

Revisiting the SFLA2017 and EVIA2017 summer schools

The 3rd EUSFLAT Summer School on Fuzzy Logic and Applications (SFLA2017[77]) and the 3rd Spanish Summer School on Artificial Intelligence (EVIA2017[78]) were held as collocated events, 17-21 July 2017, in Santiago de Compostela, Spain.

SFLA is the summer school promoted by EUSFLAT. SFLA was collocated with EVIA which is the summer school promoted by the Spanish Association for Artificial Intelligence (AEPIA[79]). Both summer schools were organized by the Intelligent Systems Research Group of the Research Center in Information Technologies ("Centro Singular de Investigación en Tecnoloxías da Información", CiTIUS[80]) of the University of Santiago de Compostela (USC).

Eyke Hüllermeier giving a joint SFLA-EVIA plenary talk. Title: "Fuzzy Logic in Machine Learning". Venue: CiTIUS Assembly Hall.

SFLA took place in the facilities of CiTIUS-USC while EVIA took place in the nearby USC School of Engineering. In addition, it is worthy to note that four plenary talks (Eyke Hüllermeier, Óscar Cordón, Francisco Herrera and Edy Portmann) were common to both schools; thus, yielding about 70 attendants to each talk. Also, other social activities, such as coffee breaks, lunches, tourist visit, and Schools dinner were also jointly organized.

SFLA 2017

The EUSFLAT summer school included 17 lecturers given by world-leading experts in the field; thus, providing attendants with a broad view of fuzzy theory and applications. Lectures were scheduled in five days (from Monday 17th to Friday 21st, July 2017). From Monday to Thursday, each lecture took 1h30 except for two lectures which took 2h30 because they included practical session in the lab. Notice this was the first time SFLA included practical sessions as a complementary activity to the usual master classes.

In addition, Friday program was very different from what was offered the rest of the week as well as previous editions of SFLA. The last day of the school was devoted to think about alternatives to students after finishing their PhDs. It included two lectures and two round tables, 1h each. The first round table focused on spin-offs and how to go from research to market. It was moderated by Senén Barro and we counted with three invited speakers: Victor Álvarez (*Situm Technologies*[81], USC spin-off), David Llorente (*Narrativa*[82]) and Luis Espinosa (*Savana*[83]).

Practical session in the CiTIUS robotics lab.

The second round table was entitled "The Role of Artificial Intelligence for Big Companies". It was moderated by Óscar Cordón and we had four speakers: Pedro Rey (*Accenture-Spain*), Nuria Oliver (*Vodafone*), Gabriella Pasi (*Universitá di Milano-Bicocca*) and Alberto Bugarín (CiTIUS-USC). We had two speakers (Gabriella and Alberto) who explained their experience on technology transfer from public universities and R&D centers to the industry. Also, two speakers (Pedro and Nuria) talking about their own experience in the industry side.

[77] http://citius.usc.es/SFLA2017/
[78] http://citius.usc.es/EVIA2017/
[79] http://www.aepia.org/
[80] http://citius.usc.es/
[81] https://situm.es/en
[82] http://www.narrativa.com/
[83] https://www.savanamed.com/

Round table on "The Role of Artificial Intelligence for Big Companies". From left to right in the table: Gabriella Pasi, Óscar Cordón, Pedro Rey and Alberto Bugarín. On the TV screen, Nuria Oliver who was connected remotely.

In addition, for the first time, SFLA2017 included the "Your Thesis in a Dram (YTD)" contest. YTD participants were 8 of the EUSFLAT grant holders who presented their ongoing research work in three sessions, 1h each (from Tuesday to Thursday). They were asked to briefly describe the key points of their work in a short time (3 minutes) and use a language style not only oriented to specialists in their respective topics, but for non-specialized attendants. Notice that the evaluation and improvement of their communicative and dissemination skills were one of the main objectives in the YTD contest. All presentations were judged by three members of the SFLA Scientific Program Committee (Professors Gabriella Pasi, Didier Dubois and Enric Trillas). The jury had 12 minutes to make comments, questions and/or suggestions after each presentation.

Taniya Seth receives the YTD award.

Together with the closing session we celebrated an awards ceremony. All SFLA attendants received their attendance certificates. Moreover, two students were recognized with two *ex-aequo* YTD awards: Taniya Seth, from the South Asian University, India and Sarah Uttendorf, from the University of Hanover - Institut für Integrierte Produktion Hannover, Germany.

Sarah Uttendorf receives the YTD award.

To conclude, let us highlight that SFLA2017 scientific program included 15 lectures, 2 sessions in the lab, 2 round tables, and 3 YTD sessions. The social program included a welcome reception, a guided visit to the USC patrimony, and a school dinner. The family picture includes the 23 attendants (being 9 EUSFLAT grant holders), some of the 17 lecturers, and part of the local organization committee. It is worthy to note that attendants were not coming only from Europe but all around the world (including India, Africa, and Asia). As a result, the EUSFLAT summer school become more a global international event than just an European school.

The family picture after the closing session.

EVIA 2017

The 2017 AEPIA summer school included 11 lectures and 2 practical sessions (held in parallel to the SFLA practical sessions). All in all, 14 speakers and 28 students. This was the third edition of EVIA, following the two previous ones held in A Coruña (2014) and Seville (2016).

For the first time the school was a collocated event with SFLA, with all lectures given in 3 days (from Tuesday 18th July to Thursday 20th July). As usual, lecturers were among the leading researchers in Spain in different areas of AI, such as Semantic Web and Open Data, Natural Language Processing, Time-series Prediction, Planning or Recommendation Systems. Topics related to Fuzzy Logic, Soft Computing, and applications (Big Data, Data Mining, Social Networks, and Cognitive Cities), were held in common with SFLA. Globally, a broad review of many research general topics in AI were presented to the students. Following a recent tradition, due to the close cooperation between the Spanish and the Portuguese AI Associations, one of the lecturers, dealing with Decision Support Systems, was delivered by Dr. Goreti Marreiros, from the Polytechnic Institute of Porto.

Lecture by Goreti Marreiros, from the Portuguese AI Society. Venue: USC School of Engineering.

Furthermore, the program included two labs, which were organized by two Spanish Research Networks: "Big Data and Scalable Data Analysis" (BigDADE[84]) and "Fuzzy Logic and Soft Computing" (LODISCO[85]). Students had a hands-on practical interactive exposure to "Big Data Analysis" and "Digital Image Processing" in these labs.

Finally, the opening and closing sessions of EVIA included the addresses by, respectively, AEPIA President, Prof. Amparo Alonso-Betanzos, and the President of the Spanish Computer Science Scientific Society (SCIE) -and former AEPIA president- Prof. Antonio Bahamonde. During the closing session the attendance certificates to all participants were delivered.

Acknowledgments

We would like to remark that this was the first time SFLA was collocated with another event and the experience was successful. However, it is worthy to note this was only possible thanks to the great support provided by several institutions. Firstly, the two scientific associations (EUSFLAT and AEPIA) which actively promote these schools among their members with the aim of become high quality and world-leading events. In addition, EUSFLAT provided 9 student grants. Secondly, the two sponsors (Vodafone[86] and Accenture[87]) which generously contribute not only with economical support but also taking part in one of the round tables in SFLA as well as in the YTD awards ceremony. Thirdly, the two Spanish thematic networks (LODISCO and BigDADE) which contributed decisively to attract Spanish attendants for both schools. Both networks offered a great economical support. They provided many students with travel grants, but they also covered the travel expenses for some lecturers. We would like to warmly thank Professors Humberto Bustince (head of LODISCO) and Francisco Herrera (head of BIGDADE) for their great support. Finally, we want to explicitly recognize the support given by the host institution (CiTIUS-USC) as well as the valuable work made by all members of the local organization committees. Moreover, all student volunteers are gratefully acknowledged because of their outstanding work.

Dr. Jose M. Alonso (SFLA2017 General Chair)
Dr. Alberto Bugarín Diz (EVIA2017 General Chair)

EVIA closing session with (from left to right) Prof. Alberto Bugarín, Prof. Antonio Bahamonde (SCIE President) and Dr. Juan Carlos Vidal Aguiar (EVIA co-chair) delivering the attendance certificates to all participants.

[84] http://bahia.ugr.es/bigdade/
[85] http://lodisco.net/
[86] https://www.vodafone.es/
[87] https://www.accenture.com/

CONFERENCE REPORT

FuzzyMAD 2017

FuzzyMAD 2017 was held at Complutense University of Madrid, December 19, 2017. Being the 10th edition of this series of conferences, we prepared a number of surprises to our 70 attendants. The structure of the conference was in principle similar to past editions:

- The first part of the conference was devoted to three conferences on hot topics (this time we addressed several aspects related to intelligent information processing in the framework of big data, particularly focused on data analytics in telecommunication and energy companies).
- The second part of the conferences was devoted to allow Ph.D. student to present their recent advances.
- And the third part was devoted to discuss around the posters each research group prepared to show their research, so new joint projects can be explored while having a buffet, this time ending with a FuzzyMAD cake.

Since Madrid is a busy city, FuzzyMAD is designed to offer an intense time to share research and help students in the region of Madrid. This time we had a wonderful concert of celtic music (a group were one of the main organizers of FuzzyMAD plays, J. Tinguaro Rodríguez). We also gave some FuzzyMAD pin awards to those that have attended to most editions of FuzzyMAD, to acknowledge our gratitude for supporting FuzzyMAD these years (during these 10 years more than 170 people have attended at least once FuzzyMAD). And a special gift was given to those who have been working harder in the organization of past FuzzyMAD editions. We had a wonderful time!

FuzzyMAD 2017 was again organized by the FORaid group at Complutense University, in collaboration with the CASI-CAM-CM network, participated by another three Universities of Madrid: Autonomous University of Madrid, Carlos III University of Madrid and Technical University of Madrid.

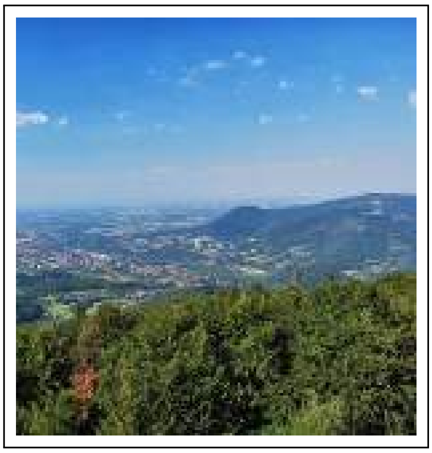
FuzzyMAD 2017 was possible thanks to the support of the Government of Spain (grant TIN2015-66471-P), the Community of Madrid (grant S2013/ICCE-2845), the Institute for Interdisciplinary Mathematics (IMI) and the Faculty of Mathematics at Complutense University, and thanks to the dedication of some of my colleagues: Daniel Gómez, Begoña Vitoriano, Javier Yáñez, Fabián A. Castiblanco, Inmaculada. Flores, Pablo A. Flores-Vidal, Carely Guada, Federico Liberatore, Pablo Olaso, J. Tinguaro Rodríguez, Adán Rodríguez, Karina Rojas and Guillermo Villarino, all of them under the coordination of Javier Martín.

Javier Montero, Complutense University of Madrid, Spain

CALLS

ISCAMI 2018. 19th International Student Conference on Applied Mathematics and Informatics

Malenovice (Czech Republic), 10 - 13 May 2018



It is already the 19th International Student Conference on Applied Mathematics and Informatics (ISCAMI) organized jointly by the Centre of Excellence IT4Innovations - Division of the University of Ostrava - Institute for Research and Applications of Fuzzy Modeling (IRAFM), by the Department of Mathematics of Faculty of Civil Engineering, Slovak University of Technology in Bratislava and by the Czech Technical University in Prague.

Based on the successful experience from the previous six years, ISCAMI 2018 will be organized jointly with the 7th Summer School on Applied Mathematics and Informatics. This means that the programme will alter between sections with student contributions and blocks of tutorials given by invited leading researchers.

The conference will be organized again in Malenovice, a beautiful village situated on the root of the highest peak in Beskydy mountains near Ostrava on May 10 - 13, 2018, `http://www.malenovice.com/`

We are happy and proud that the conference, as the only student conference, is marked as EUSFLAT endorsed event, for which we are grateful to EUSFLAT.

The main purpose of ISCAMI is to bring together young researchers and students and to give them an opportunity to present their achievements and ideas in the area of applied mathematics, informatics and various applications. Authors are invited to prepare short abstracts that will be published in Book of Abstracts equipped with ISBN.

The conference is considered to be a low cost conference with the registration fee 150 EUR. The registration fee includes:

- accommodation,
- full board,
- coffee breaks,
- social programme.

For further details, please visit: `http://irafm.osu.cz/iscami/` or contact the Organizing Committee: iscami@osu.cz .

The 11th Conference of the
European Society for Fuzzy Logic and Technology

EUSFLAT2019

Prague, Czech Republic
September 9-13, 2019

The aim of the conference is to bring together theoreticians and practitioners working on fuzzy logic, fuzzy systems, soft computing and related areas. It will provide a platform for the exchange of ideas among scientists, engineers and students.

The topics addressed by the Conference cover all aspects of fuzzy logic and soft computing, namely (but not limited to):

Approximate reasoning
Clustering and classification
Cognitive modeling
Intelligent data analysis and data-mining
Data aggregation and fusion
Database management and querying
Theory and applications of decision-making
Forecasting and time series modeling
Fuzzy control
Theoretical foundations of fuzzy logic and fuzzy set theory
Imprecise probabilities and fuzzy methods in statistics
Image processing and computer vision
Information retrieval
Knowledge representation and knowledge engineering
Linguistic modelling
Machine learning
Natural language processing
Neuro-fuzzy systems
Stochastic and fuzzy optimization
Possibility theory and applications
Rough sets theory
Semantic web
Uncertainty modeling

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