Mathware & Soft Computing

The magazine of the European Society for Fuzzy Logic and Technology

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LETTER FROM THE EDITOR-IN-CHIEF(January 2023)

HUMBERTO BUSTINCE

Dear colleagues, dear friends:

Once more, with the first days of the New Year, your Mathware&Soft Computing online magazine arrives at your e-mail, bringing you news from our community. As usual, we send it to you in the hope that it will help you to know better what is happening around our EUSFLAT society, as well as beyond. And also expecting that this knowledge may help to improve the collaboration among us. Because collaboration is, no doubt, one of the main stones upon which science is built.

In this new issue, let me make a different letter, rather than reviewing the contents of this issue. Not because the latter are lacking interest or value. On the contrary, the topics in this issue range from an excellent talk between Vladik Kreinovich and László Kóczy, to a set of selected works from our Brazilian friends, as well as news, summaries of conferences... Thanks a lot, once again, to all of you for making this magazine possible.

In this case, I want to make a different letter because I want to make reflexion about the past and the future. The past, looking how our magazine has grown along the years, and has become an important element in our community. And this has been possible thanks to your efforts, your interest and your hard work. I will say once and again, this magazine is your magazine, our magazine, and only you have made it possible to arrive here. So thank you very much, specially for your collaboration along these difficult last years.

But I want to share with you also my thoughts about the future. Because we are living hard and strange times, and I think that science is a key to help our society and to clarify the situation. We, as scientists, have the moral obligation of helping our societies to improve and to overcome the problems that are appearing along the way. And I think that our fuzzy community has a lot of say and to do in this sense. First of all, because we come from all around the world, and we can display a model of successful collaboration. And secondly, because from its very origin fuzzy theory has been devoted to help society by improving communication with machines and even by understanding how communication between human beings work.

So in this New Year I want to think on the possibilities of our work, of our research, an on how it can help all the other ones. I am convinced we can feel proud of our community, and it can become a basis for the improvement of our societies.

And of course, this magazine will be there to help in such a relevant task. But, of course, depending on your collaboration, as usual. So please do not forget to share in this pages whatever you may consider relevant for our community.

Let the New Year come to all of us full of happiness, and let us be able to bring that happiness to all the people around through our work.

Happy New Year 2023.

Humberto Bustince
Editor-in-chief

Message from the President (January 2021)

SUSANA MONTES

Dear members of our society,

We are just beginning 2023, and we hope it will be a year of total normality, where we can enjoy normal life and, especially, our scientific meetings.

In particular, a very important event is taking place this year: the 13th conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2023), the conference of our society. The conference will be held in Palma de Mallorca, September 4-8.

Beyond doubt, it will be an excellent opportunity to exchange and discuss ideas, as well as to establish new professional relationships. This is probably one of the best aspects of in-person conferences.

Apart from the different events where our society collaborates, we have some other very important objectives. As a result, we must collaborate to ensure that our community receives deserved international recognition, not just within the fuzzy area where our contribution is well appreciated. We need to prove our fortress in the general scientific community. For this purpose, we have to be able to develop competitive tools that are eligible for publication both in our usual magazines and in others that are more distant from us but have significant academic relevance. Thus, we will disseminate our capacity to solve interesting problems more efficiently. We must be able to merge our methodologies with others to prove the value of our proposals and show the invaluable potential of our research. It will not always be an easy task, but it will definitely result in a greater recognition of all the involved agents.

I look forward to meeting you in Palma de Mallorca and at any other event or meeting where we can exchange ideas.

Susana Montes
President of EUSFLAT

INTERVIEW

Interview with László Kóczy

László Kóczy

The **Mathware & Soft Computing** magazine provides here a space for these researchers to talk about the evolution of fuzzy logic in Brazil, based on data and materials provided by important names who introduced the area in such country, such as Prof. Dr. space for these researchers to talk about the evolution of fuzzy logic in Brazil, based on data and materials provided by important names who introduced the area in such country, such as Prof. Dr. Fernando Gomide (University of Campinas, São Paulo, Brazil) and Prof. Dr. Ricardo Tanscheit (Pontifical Catholic University of Rio de Janeiro, Brazil).

Vladik Kreinovich: How did you become interested in fuzzy?

László Kóczy: That I became interested in fuzzy is rather a coincidence. When I was in the 3rd year of my 5-year study program, a young lecturer from our university invited students with good marks in math to form a scientific circle for studying AI and related stuff. 7 or 8 of us agreed. For the first meeting, he brought us a bunch of papers of recent developments in AI. Among these papers, there was Lotfi's first paper on fuzzy logic, a paper by Goguen, and one more fuzzy-related paper. I took all three papers and started reading them. To be honest, I read them with some difficulty – they were very different from what I knew before that. But as I better understood these papers, I started to think about them. In 1974, when I was still a student, I had an idea – that it is better to apply algebraic operations, different from minimum, to describe intersection, and how exactly to do it. I put these ideas down as a technical report, and I send it to Lotfi. I was somewhat afraid that a famous professor will become unhappy with a student trying to change some of his ideas – or, at best, would simply ignore it. But he wrote a very kind answer, and he sent me many papers of his and of others on fuzzy. Moreover, when he – with B. Gaines – formed a bibliography on fuzzy techniques, he included my technical report in this bibliography.

VK: This is very interesting. So when and how did you actually meet Lotfi Zadeh?

LK: In 1975, Lotfi himself visited Hungary, to attend a joint US-Hungary Seminar on Pattern Recognition. To each visiting researcher, the Academy of Science provided a young researcher or student companion to help the visitor with sightseeing. Usually, the student was selected based on his/her research interests, so naturally I was selected as a companion student for Lotfi.

This, by the way, was a very interesting meeting. Researchers from US talked on many topics, semantics, pattern recognition, statistics, while most Hungarian talks were only about statistics: pattern recognition was a new topic for Hungarian researchers, but in terms of statistics, Hungary had a very strong school founded by Alfred Renyi. At this seminar, Professor Shen from Pennsylvania offered me a PhD scholarship, but the Hungarian government did not allow me to go.

Later on, when travel became easier, several times, I invited Lotfi to visit Hungary, and several times, he invited me to visit him. We became good friends, and our families became friendly, many times we stayed at each other's homes.

I still have a memory of Lotfi's first visit to me – Fay mentions this in her book about Lotfi. When Lotfi first came to visit me in Hungary, he brought me, as a present, a colorful tie. In those days, colorful ties were very unusual in Hungary, ties were usually of a single color or, at best, striped. I still have Lotfi's tie, it has always helped me feel more out-of-the-box, more creative.

VK: This is very unfortunate that you were not allow to travel in 1975, but good news is that now you are actively traveling, participating in many fuzzy conferences. How did the change happen?

LK: In general, until 1989, I had very few chances of attending conferences abroad, I rarely got support for conferences. Two conferences I remember: I was allowed to go to Linz, Austria, to attend a conference organized by Peter Klement. Peter wanted me to stay for some time after the conference, but I was only allowed to be there for a maximum of five days. I also attended a conference in East Germany, where I met a few fuzzy researchers: Siegfried Gottwald and Wolfgang Wechler.

After 1990 I could travel freely, I attended all IPMU, WCCI, FUZZ-IEEE, and IFSA conferences, and many EUSFLAT conferences. I officially retired this year, but I still have grants, so I can still travel to conferences.

Lotfi Zadeh and László Kóczy in Budapest in 1975

VK: You mentioned EUSFLAT. You have been an active member of EUSFLAT for many years, tell us how you got involved with this organization.

LK: I was actually a Founding Board member of EUSFLAT. The idea for this society emerged, if I remember correctly, in February 1999 during a meeting in Slovakia organized by Radko Mesiar. Bernard de Baets was there, as well as leaders of the Spanish fuzzy research group. We decided to join our efforts and to form a Europe-wide fuzzy research society, and this is how EUSFLAT started.

EUSFLAT is still a very strong society. We lost our weight in IFSA – many researchers and many papers now come from outside Europe, especially from Asia – but most mathematical fuzzy-related results still come from Europe.

Practically all fuzzy researchers from Europe – and practically all European fuzzy societies – decided to join EUSFLAT. To my disappointment, one of the exceptions was the Hungarian Fuzzy Association – which is now Hungarian Fuzzy Society. I was one of the founders of this organization, I was its Founding President in 1990, and in 1998-99 I was appointed Life Honorary President. I was all for joining EUSFLAT, but another group took over, they did not want to merge, they kept their autonomy. One of their arguments was that if we keep the autonomy, we will have an extra vote in IFSA, so Europe will have more votes.

At present, only the Hungarian and the Northern European fuzzy communities (such as Finland, Sweden, Iceland) are not part of EUSFLAT. The Hungarian Society organizes four series of annual conferences, one of them in collaboration with Jesus Medina and other Spanish researchers – which is not surprising, since Spain has the largest number of fuzzy researchers among all European countries. Imre Rudas is one of the main organizers of the other conferences.

I myself try my best to support both EUSFLAT-organized and Hungarian fuzzy conferences.

VK: At all the conferences, you present many interesting results. Tell us about your fuzzy-related research.

LK: As I mentioned earlier, I started with operations – t-norms and t-conorms – tried to do applications based on them.

In 1987, with Kaoru Hirota from Japan, we did a paper on fuzzy flip-flops, one of the attempts to implement fuzzy techniques in hardware. Later, with a PhD student, we further developed this idea. I still believe that there is a lot of potential in fuzzy hardware.

Another of my areas of interest in fuzzy interpolation. Usually, the cover of available fuzzy rules is sparse, it is not clear how to do fuzzy inference when the input is not covered by any rule. In this case, a natural idea is to interpolate between existing fuzzy rules. This research is still going one. My related papers gained several thousand citations, unfortunately very few of them are about real applications, I think that the ideas of fuzzy interpolation are under-utilized – and I also think that without using such ideas, we may reach the limit of usefulness of fuzzy techniques.

Interpolation techniques have another advantage: their use enables us to reduce the number of rules, and thus, to reduce the amount of computational resources needed for processing these rules. This is important, since in the traditional fuzzy approach, the number of rules grows exponentially as we take into account more inputs – so that the number of rules becomes unrealistically large: we cannot elicit that many rules from the experts.

One of the possible solutions to this problem is to extract fuzzy rules from data. When I was in Japan in 1993-94, I became acquainted with Michio Sugeno's method of building fuzzy models from observations and measurements, this has become one of my areas of interest. This is closely connected with interpolation of fuzzy rules – since extracting rules from data means, in effect, interpolating the general dependence from several known examples.

Budapest 1975

Another way to decrease the number of rules is to consider hierarchical rule bases. This is how we reason, this helps to decrease the number of rules and to make computations feasible. We have successfully applied this idea to applied geological problems, we are currently trying to apply it to biomedical engineering.

Yet another idea of how to make fuzzy techniques more effective is to combine them with other techniques. With my coauthors, we successfully combined fuzzy and wavelet techniques. I think there is a great future in such combinations, I wish these ideas would spread more.

Need for such combinations is well understood, e.g., in clustering, where different techniques – including fuzzy – have their own advantages and disadvantages, and it is desirable to combine these advantages. I therefore started trying to go beyond fuzzy c-means, the most widely used method of fuzzy clustering. I realized that adding evolutionary computation ideas may be very helpful. In the 1990s, Furuhashi and coauthors published several papers on Bacterial Evolutionary Algorithms, an essential modification of the traditional genetic algorithms. In particular, they used this new algorithm to optimize fuzzy rules.

Evolutionary algorithms are very good for global optimization, but on the local level, they are much slower than traditional mathematical – e.g., gradient-based – techniques. So, with a PhD student, we combined global evolutionary techniques with Levenberg-Marquardt, a second-order version of gradient descent – and ended up with algorithms which are most efficient: fast and accurate.

This general idea of combining global and local techniques enables us to come up with efficient algorithms for solving discrete problems, in particular, NP-hard problems, where we use discrete local search as a local options. Some of these algorithms I learned when I gave problem sessions under the guidance of Laszlo Lovasz and Laszlo Babai, prominent specialists in discrete problems.

Our algorithm for solving the traveling salesman problem is still slightly slower than the best known algorithm, but our algorithm is the most predictable: we can much better predict how much time our algorithm will reach the desired level of optimality and thus, better plan the use of our computational resources. Most existing algorithms for solving NP-hard problems are capricious: their execution time may change wildly if we make a very small modification to the input. In contrast, the proposed local-global algorithms are more stable: their average execution time is about the same as for the best known methods, but the variability of the execution time is much smaller.

Balaton 1975

Our algorithm for the job scheduling problem is the best so far, we tried it on many benchmark problems. We published one paper on this in the Symmetry journal some time ago. Now we have even better results. We have several other applications of such hybrid algorithms – that combine local search with bacterial evolutionary algorithms. Some of the resulting papers are in review, some still need to be finalized and submitted.

An important feature of the memetic and hybrid algorithms we developed is their universality. There are many different NP-hard problems. Efficient algorithms are known only for a few well-known NP-hard problems. These algorithms do not work so well when we apply them to other problems. In contrast, our global-local algorithms work as well on other NP-hard problems as on well-known problems like traveling salesman.

Many of these results are not about fuzzy. However, my interest in fuzzy remains, and for discrete problems, this interest is motivated by the fact that in practice, we often have fuzzy information. For example, in the traveling salesman problem, we rarely know the exact travel times, the exact travel costs – often, we only have fuzzy knowledge about them. It turned out that our global-local algorithms work well for such fuzzified discrete problems as well. At NAFIPS'2019 in Lafayette, Louisiana, we showed how such a fuzzified traveling salesman problem works when we use the usual, intuitionistic, and interval-valued fuzzy techniques. This paper won the outstanding paper award at this conference. The possibility to extend hybrid local/bacterial evolutionary methods to the fuzzy case is one more evidence of the universality of such methods.

By the way, evolutionary computations is not the only techniques that can be combined with other methods – we showed that adding the simulated annealing method can also lead to further improvements.

Another area of my interest is fuzzy cognitive maps, an idea originated by Bart Kosko. This was a very good idea, leading to good first-approximation models, but these models are often too simple and, as result, too approximate to be practically helpful. Many complex versions of this idea have been proposed to make the models more accurate, but most of these versions are too complex to be practically useful. I felt that a compromise was needed between the over-simplicity of the original model and the over-complexity of its more complex versions. And we managed to come up with such a compromise – we came up with what we called Concept Reduction algorithm, it was published in good journals. The main ideas behind this algorithm came to my mind from digital design, where one of the criteria is robustness: a computer should continue functioning if one of the billions memory cells fails. Similarly, we took into account that the results should not become different when the input fuzzy degree changes a little bit – especially since the fuzzy degrees that we start with are not precise numbers: the same expert who one time places the degree 0.3 on a connection between the two quantities can next time place 0.35 or 0.25. We applied our new technique to waste management, to financial investment problems, and, judging by the fact that we got a lot of nice citations, others are successfully using our techniques as well.

The last topic that I want to mention is the idea of fuzzy signature. This idea goes back to my joint work with Tibor Vamos. Tibor met with Lotfi Zadeh, got excited about the possibilities of the fuzzy approach, so we started thinking of how to use these techniques to compare economies of different countries. In attempting to do this, we encountered a problem – and we realized that this problem is not limited to economics applications, a similar problem appears in medical applications and in many other application areas.

The problem is that in the traditional fuzzy approach, for each object and for each property, we have a degree to which this object satisfies the given property. For example, we can have a degree to which a patient has fever, a degree to which the patient's cough is strong, a degree to which the doctor is confident that this is, e.g., cold and not flu, etc. When we have all this data for each patient, and we have rules that doctors use to diagnose the disease and to recommend cure, then we can use the usual fuzzy methodology to provide a reasonable diagnosis and cure for the given patients. In practice, however, we may have different amounts of information for different objects: for example, for some patient, we have all the information, while for other patients, we only have a few degrees. The situation is made even more complicated by the fact that while in the traditional fuzzy approach, it is assumed that the input degrees are, in effect, independent, in practice, some of them depend on others: e.g., the doctor's degree of confidence that the patient has cold is determined based on the degrees to which the patient shows different symptoms of this disease.

How can we make decisions in such a non-uniform situation, when for different situations, we have different sets of values? It is not practical to have rules describing each possible subset of the ideal set of values – since there are exponentially many such subsets. This problem is somewhat similar to the interpolation problem. In interpolation, we have rules about, e.g., cases when the input is small and cases when the input is large, but we do not have rules for the case when the input is medium. So, we need to "interpolate" the original rules and come up with new rules that would cover such situations. Similarly, in some situations, we do not have all the degrees needed to apply the existing rules, so we need to "interpolate" the original rules and come up with rules for situations when not all inputs are available.

A solution we came up with – and that we called fuzzy signature – is to explicitly take into account the hierarchical, tree-like character of the inputs. We have detailed characteristics of fever – is it high in the morning, is it high at night, etc. We have detailed characteristic of cough – how severe is it, is it dry or wet and if wet, to what extent, etc. Based on these detailed information, a doctor gets a general aggregated impression of fever, of cough, etc. Then, based on these general descriptions of fever, cough, etc., the doctor makes a diagnostic decision. So, we do not get just a list of fuzzy degrees, we get a tree of degrees, where each parent node aggregates information from the children nodes – and this tree is what we called a fuzzy signature.

Correspondingly, rules are selected that describe how to aggregate the values corresponding to the children nodes into a value describing the parent node – e.g., the aggregated degrees to which the patient has fever and cough into a degree to which the patient has cold. In this case, we do not need to consider all possible subsets of the original large set of possible symptoms: at each node of the tree, we only need to consider a few subsets of the set of all of this node's children. This makes the procedure feasible.

With Kevin Wong, we applied this techniques to SARS. Other researchers applied it to construction engineering, to management science, and to other disciplines. For example, one of my former students successfully applied fuzzy signature techniques in civil engineering; for this purpose, he came up with special aggregation rules that best describe the construction practice. We worked out the algebraic properties with Jesus Medina and his team, and Jesus currently leads a European group that – among others – applies this technique to forensic science, where also there are many potentially useful inputs, but the number of inputs differs from one situation to another. I believe that this is a very promising application direction, and I wish more people will become familiar with this approach and start using it.

Anchorage 1998

VK: Several times, you mentioned your current and former students. Many of your papers are coauthored by your current and former students. You seem to be very successful in mentoring students. What advice can you give to students and to young researchers in general?

LK: What is research: First, you get research results, and then you try to present them in a publishable form.

In terms of getting the research results, my advice is that we need to be systematic and we need to be patient. We need to be systematic: good ideas need to be written down (and published) in well-argued form. We need to be systematic and patient: when you are systematic, the problem becomes clearer, and then there is a good chance that a new idea will come.

For example, I was struggling, for a long time, with a problem of optimizing a telecommunication network under fuzzy uncertainty. He has patiently and systematically analyzed the problem – and realized that, from the mathematical viewpoint, this problem does not depend on specifics of namely telecommunications: the same mathematical formulation arrives when we consider other types of networks. It turns that one of such cases – a network of spring-type connections – has already been well studied in civil and materials engineering, and the results of this analysis helped him to come up with a good optimization technique for communication networks as well.

Sometimes the solution comes in a dream, because your brain is working on the problem even when you are asleep: this is the way my late father, who was a professor of materials engineering, had one of his ideas. There are many examples of such sudden sparks of intuition in the history of science. That an idea comes to you is your luck, but this luck only happens when you have been systematically and patiently thinking about your problem.

Once you get good results, comes another important stage: preparing your results for publication. Many young researchers start by sending their research results to a journal, usually a good journal. In many cases, the technical result is interesting and good, so the author(s) have a great hope that the paper will be accepted and published. However, these papers are often not well written: they are not clear enough, not well organized, and, as a result, the paper is often rejected – or, if accepted, published in the original difficult-to-understand form, which is not the best way to convey the paper's important results.

I think that submitting to a journal first is a wrong strategy. When you get results, start with giving a talk in your department, invite folks from other departments who may be interested. They will give you good advice, you can you use this advice to improve your paper. Then, present this result at a regional conference, get some new advice, then go to a major conference, and only after that, submit a paper to a journal. This is what I do, and my papers are usually accepted and well-cited.

In general, the most important thing is to write clearly. The material should be explained step-by-step, everything should be well documents and well organized.

It is also important to explicitly explain novelty. Novelty may be very clear to the paper author who knows a lot about the specific research problem, but novelty is often not that clear to a reviewer.

László Kóczy, Kaoru Hirota, Lotfi Zadeh, Annamaria Valkony-Koczy and Motoko Hirota. 11th IEEE International Conference on Fuzzy Systems FUZZ-IEEE'2002, Honolulu, Hawaii, May 12-17, 2002

VK: As we all know, in research not only it is important to find a good solution to a problem, it is probably even more important to select an important problem, an important challenge. What are, in your opinion, the main challenges facing Computational Intelligence?

LK: In my opinion, the most important challenge is that we need to design a Computational Intelligence (CI) toolkit that would help the users select an appropriate CI (or more traditional non-CI) tool and thus, solve practical problems – including highly complex ones – with high efficiency.

Lotfi Zadeh often mentioned that if all one has is a hammer, then everything start looking like nails, and this is a problem in computational intelligence: many researchers and practitioners are very familiar with only one technique, and they apply this technique everywhere – sometimes successfully, but often not. It is important to always compare different tools, and to select the most appropriate tool.

For example, if a problem has an explicit mathematical solution, there is no need to use complex CI tools. If we have a system of linear equations, for which a simple algorithm is well-known – use this algorithm. If we have a system of nonlinear equations, then there is no reason in forcing a linear model into it. If the inputs come with high uncertainty, there is no sense to treat the corresponding expert estimates as exact numbers, fuzzy techniques are more appropriate here. If we already have good solutions for several cases, machine learning is appropriate.

Many practical problems are optimization problems. When the problem is not very complex, mathematical optimization tools work well, but for complex problems they often require too much computation time. In most real-life problems, we do not need to find the exact optimum, it is sufficient to get close to the optimal solution, and in this case CI methods – like the methods we developed for discrete problems – work well and work fast.

It is desirable to utilize such commonsense recommendation and develop a toolkit that will help practitioners select a tool which is most appropriate for their problem.

A second important challenge is that many people do not believe in CI. To convince them, we need to spend more efforts publicising our successes to the practitioners. For example, many practitioners are sceptical about fuzzy logic, because they are not very familiar with its successful applications. In Hungary, there was recently a competition for designing the best control for a Martian rover – that would drive around and collect samples. Our team proposed a control based on fuzzy techniques – combining the traditional fuzzy control with fuzzy transform techniques – and won the competition: in a simulated environment, this fuzzy-based controller performed the best. This was a great success of fuzzy techniques, but even most fuzzy-related researchers are not aware of it. And this is just one example.

We need to participate more in such competitions, and we need to better promote our successes.

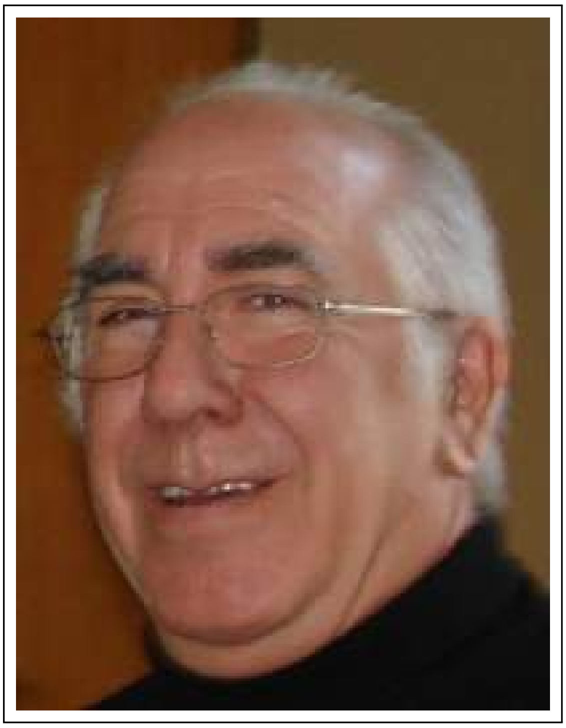
VK: Dear Laszlo, Many thanks for your very interesting ideas and useful stories. Any parting words?

LK: It is interesting that you mentioned stories, I never felt good with words. When I was in elementary school, my teacher wrote in his report to my parents that I was szofukar – a Hungarian word that can be loosely translated as "tight-lipped" or, alternatively, "miser of words." I often still feel that what I said or wrote can be said or written much better, in a much clearer way – and sometimes, when I re-read my old papers, I understand how their ideas can be explained more clearly. In this, I am always trying to improve, and I am always telling my students that they need to strive to write better. This is similar to complex NP-hard optimization problems in which I am so much interested these days: NP-hardness means that no feasible algorithm can always provide an optimal solutions, but we should always try to come with a solution which is as good as possible. In this struggle for understandability of our ideas we are all together, let us do more in sharing our experiences and in helping each other!

MEMORIAL

Joan Jacas Moral (Barcelona, 27.8.1944 - Sant Just Desvern, 28.2.2022)

Jordi Recasens Ferres
Dept. of Architecture Technology
Universitat Politècnica de Catalunya
Barcelona
SPAIN



Joan Jacas left us at the beginning of this year at the age of 77. He was a Professor at the School of Architecture of Barcelona and Head of the Structural Department for many years. He was also founding Editor-in-chief of Mathware & Soft Computing (1994) and edited the journal for several years. His main research interest was on Fuzzy Relations and was a pioneer on the study of Indistinguishability Operators (Fuzzy Equivalence Relations).

He was married to Susanna and had two daughters, Sara and Anna.

A person of good character he loved old cars and after his retirement he started farming and producing olive oil.

He was a good man.

My days with Joan

I met Joan in 1981 at Sant Just Desvern High School where I also was teaching Mathematics. Besides being a good Math teacher, he was a skilled handyman: he would fix the High School tables and chairs, put curtains on the classrooms... He had every sort of incredible tools and devices. For example, one year I had to organize a Christmas lunch for a large number of people and Joan lent me a huge pot that solved the problem.

At that time, he was finishing his PhD thesis and was talking about incomprehensible things for me such as the dimension of an indistinguishability operator. Due to the thesis'stress there were times in which he was a heavy smoker followed by periods of total abstinence. These radical an unexpected changes also used to occur on his face, now with beard, now, without.

At the High School I also met his daughters, Sara and Anna. When I taught Mathematics and ICT to Sara the students gave me the nickname El Raca (The Stingy, because I was rigid with the students marks), which is how they affectionately called me at Joan's home.

When Joan finished his PhD thesis he asked me to join his Department at the University. I agreed and he became my thesis supervisor. The first day he proposed two problems: characterizing the alpha-cuts of an indistinguishable operator and finding a method to calculate the dimension of these relations in the Archimedean case (he had solved the problem for the t-norm of the minimum). Regarding the first problem, I gave a topological interpretation to the alpha-cuts that we presented at the IPMU Conference in 1990 in Paris. This was the beginning of a long scientific collaboration. The second problem is solved in my thesis and we published it in the Journal Fuzzy Sets and Systems.

I also remember his time as Head of Department and editor of Mathware & Soft Computing. In spite of being sometimes overwhelmed by so many responsibilities, he was a man of good character and was a pleasure to be with him.

I will always remember the gatherings at El Molinet, his country house in Penedes. He used to invite the whole Department with families year after year for a calçotada (a typical Catalan meal). He and his wife Susanna were the perfect hosts.

Joan was a good man. Or, as Paco Herrera told me, he was not only a good man, but he also was a great person.

Jordi Recasens

Scientific work

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RECOGNITION

Rodney Carlos Bassanezi and fuzzy modelling

Marina T. Mizukoshi
Institute of Mathematics and Statistic, University Federal of Goiás
74690-970, Goiânia, GO, Brazil. email: tuyako@ufg.br

Rodney Carlos Bassanezi was central to developing fuzzy mathematical analysis and modeling. He contributed significantly to this research field with high-quality texts and advising many undergraduate and graduate students. In addition, his role in stimulating and organizing conferences related to fuzzy sets and systems, such as the Brazilian Conference on Fuzzy Systems, mattered to the proliferation of research and applications of fuzzy set theory in Brazil. Professor Bassanezi is one of Brazil's first researchers to publish on fuzzy mathematical analyses. In the following, we recall his academic trajectory, highlighting some of his contributions to fuzzy systems.

Rodney obtained his bachelor's degree in mathematics from the *São* Paulo State University Júlio de Mesquita Filho (UNESP), Brazil, in 1965. He earned his master's degree from the University of Campinas (Unicamp), Brazil, with a dissertation entitled "Complete Orthonormal Systems", supervised by Ayrton Badelucci in 1971. Rodney completed his Ph.D. in mathematics, also from Unicamp, in 1977. His Ph.D. dissertation, entitled "N-dimensional Dirichlet Problem for minimal surface equations in domains with singular boundary", was supervised by Ubiratan D'Ambrosio and Umberto Massari[1]. Besides his formation at UNESP and Unicamp, he did a post-doctorate in mathematics at Libera Universita Degli Studi Di Trento in 1981, 1985, and 1995.

Professor Bassanezi began his long and formative teaching career in 1968 as an assistant professor at the University of Brasília (UnB), Brazil. He was a professor and researcher at IMECC from 1969 until his retirement in 2001. He continued collaborating at IMECC but joined the Federal University of ABC as a professor and researcher volunteer from 2001 to 2006. In addition, Bassanezi was a visiting professor at the University of Londrina (UEL), Libera Universita Degli Studi Di Trento, and the Pontifical Catholic University (PUC) of Campinas. Bassanezi is currently a collaborating researcher in the Institute of Mathematics, Statistics, and Scientific Computation (IMECC)[2] at Unicamp.

His research includes mathematical analysis (minimal surfaces), biomathematics, mathematical modeling, optimal control, mathematics education, and fuzzy dynamical systems. He has published books in Portuguese, notably one textbook on differential equations (1988), one textbook on mathematical modeling (2002), fuzzy logic and applications (2010), as well as an introduction to calculus and applications (2015). He has been the president of the Sociedade Latino-Americano de Biomatemática (1999-2001).

Bassanezi contributed significantly to fuzzy mathematical analysis and modeling. To begin with, we would like to recall that he and Greco addressed necessary and sufficient conditions for a function to be representable by a fuzzy measure in a paper published in 1988 in the Journal of Mathematical Analysis and Applications [4]. In 1993, Greco and Bassanezzi showed that convergence in a measure is a sufficient condition for the continuity of the fuzzy integral [12]. Together with Roman-Flores, [5], he published the paper entitled "Embedding of level-continuous fuzzy sets in a Banach space and applications," where they analyzed the continuity properties of fuzzy entropies and established conditions that guarantee the convergence of a sequence of entropies. According to himself, Bassanezi was encouraged by Roman-Flores to study fuzzy theory.

In 1979 professor Bassanezi established the relations between mathematical theory and applications using deterministic/fuzzy theory to teach teachers to improve pedagogical methods. In the late 1990s, Bassanezi and collaborators analyzed the continuity of Zadeh's extension, putting forward the connection between fuzzy and interval theories [8, 19]. Moreover, Bassanezi and collaborators addressed the initial

[1] https://www.ime.unicamp.br/bimecc.
[2] https://www.ime.unicamp.br/departamentos/Matematica-Aplicada/corpo-docente

value problem (IVP):

$$\begin{cases} x'(t) = f(t,x) \\ x(t_0) = x_0, \end{cases} \quad (1)$$

where the initial condition and coefficients are uncertain. Uncertain IVP like (1), called a fuzzy differential equation, dates back to the 1980s when Kandel and Byatt considered that the right side of (1) is a fuzzy-set-valued function with fuzzy set variables [14]. Accordingly, the fuzzy solution is obtained considering fuzzy integrals and Minkowski sum, through which one gets the solution with a nondecreasing diameter.

Bassanezzi started his studies on fuzzy initial value problems in 1995 with Barros [2]. In a few words, Bassanezi and Barros compared the solutions obtained from the classical stochastic Boltzman model with the ones obtained by fuzzy techniques. In the middle 1990s, Bassanezi and collaborators also started researching fuzzy discrete dynamical systems [3]. They studied solutions and stability of discrete dynamical systems because the derivative concept had not yet been well established. The solutions of a discrete dynamical system can is obtained by iterating the initial solution, and the derivative definition is unnecessary.

In the 2000s, new concepts of derivatives were introduced [6, 11, 10]. Bassanezi and Mizukoshi investigated fuzzy differential equations with one fuzzy initial condition:

$$\begin{cases} x'(t) = ax(t), \\ x(t_0) = X_0, \end{cases} \quad (2)$$

where X_0 is a fuzzy number but $a, x \in \mathbb{R}$. They showed that the solution of (2) is the Zadeh's extension of the solution of (1). Moreover, the solution obtained is the same as the one obtained by differential inclusion theory when the initial condition X_0 is fuzzy [16]. In 2004, with Silva [20, 21], he introduced the idea of p—fuzzy dynamical systems, where the derivative concept is considered using a Mandani type fuzzy rule-based system. They established conditions of existence and the uniqueness of stationary points of p-fuzzy dynamical systems. In 2006, together with Cecconello [7], they investigated the presence of periodic solutions for fuzzy initial value problems. In 2008, together with Leite [15], they defined a fuzzy solution of a fuzzy diffusion-reaction-advection equation, its uniqueness, and its stability.

One principal characteristic of Bassanezi's research is that ideas come from real problems he tries to understand. Moreover, Bassanezi is concerned with how concepts and ideas can be passed quickly to people who are not so mathematically literate (see, for example[1]). For instance, Bassanezi and his collaborators helped physicians from the Clinical Hospital of the University of Campinas to understand the evolution of HIV and its effects on patients under treatment with the introduction of a fuzzy set-based model [13, 17].

Another applied problem dealt with fly aphids that caused disease in the orange trees, resulting in the death of the trees and causing significant economic losses to the orange juice industry. A cellular automata (CA) model with a rule-based fuzzy system was used to study the disease's temporal evolution. The CA model, which considered the relationship between the area covered by the aphid and the intensity of the wind, proved helpful for the orange juice industry [18].

A third successful application for cancer treatment was given by Souza and Bassanezi[22]. They studied the proliferation of tumor cells under chemotherapeutic treatment and the influence of integrative oncology. The Gompertz model with and without chemotherapeutic treatment was considered in their study. They observed that the variation of the immune system and tumor fighter cells is directly associated with the integrative treatment through the noetic threshold. The uncertainties of the parameters led them to use fuzzy theory to define a fuzzy noetic threshold, initial conditions, node threshold, and the resistant cells by fuzzy numbers.

The most recent research by Bassanezi is about fuzzy optimization problems using Zadeh's extension concerning parameters and independent variables [9] and fuzzy initial value problems with dependent variables.

Concluding, the way Rodney Bassanezi addresses applied issues with straight but rigorous mathematical concepts impacted several generations of students and researchers and has enriched fuzzy set theory and applications worldwide.

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RECOGNITION

EUSFLAT Congratulates

Francisco Herrera and José Luis Verdegay

Francisco Herrera

José Luis Verdegay

On 28 August 2022, Professors Francisco Herrera and José Luis Verdegay, from the Department of Computer Science and Artificial Intelligence of the University of Granada (Spain) were elected corresponding members of the Cuban Academy of Sciences (ACC) by direct and secret ballot of the members of the Academy.

ACC was founded under the Spanish Crown on 19 May 1861 as the Royal Academy of Medical, Physical and Natural Sciences of Havana. In 1902, the Academy maintained its structure and organisation, but dropped the word "Real" from its name. In April 1996, the ACC acquired by law its current status as an official institution of the Cuban State, with an independent and consultative character in the field of science. The objectives of the ACC are to promote Cuban science, the dissemination of national and universal scientific progress, the recognition of the prestige of excellent scientific research, the raising of professional ethical standards and social recognition of science, and the strengthening of the links between scientists and their organisations, with each other, with society in general and with the rest of the world.

On the other hand, Professor José Luis Verdegay was appointed Dr. Honoris Causa by the Universidad Central "Marta Abreu" de las Villas (UCLV), Cuba, and Professor Francisco Herrera was awarded the special teaching category of "Visiting Professor" of the same University. UCLV is a public university located in the Cuban city of Santa Clara. Founded on 30 November 1952, this institution is the most important centre of higher education in the central region of Cuba and the most multidisciplinary, highlighting its research activity in the field of Artificial Intelligence and particularly in the field of Fuzzy Sets and Systems.

Likewise, at the Technological University of Havana (CUJAE), Cuba, at the proposal of its Faculty of Computer Engineering, Professor Verdegay received the CUJAE seal, the highest distinction awarded by that University. The CUJAE, originally "Ciudad Universitaria José Antonio Echeverría", hence its initials in Spanish, is a university with more than a century of tradition in the teaching of engineering and architecture in Cuba and is made up of more than forty buildings, covering an area of 398,000 square metres, in which 9 faculties are located, where thirteen degrees are studied, and it has 12 research centres. In recent years, many CUJAE members have defended their doctoral theses on different topics related to Fuzzy Sets and Systems.

RECOGNITION

Kmita et al. won the FUZZ-IEEE 2022 Best Paper Award during World Congress on Computational Intelligence in Padua

Kamil Kmita

January 10, 2023

1 Introduction

A paper authored by: Kamil Kmita, an early career researcher from Systems Research Institute, Polish Academy of Sciences (SRI PAS), Gabriella Casalino and Giovanna Castellano from University of Bari Aldo Moro, and Olgierd Hryniewicz and Katarzyna Kaczmarek-Majer from SRI PAS won the IEEE International Conference on Fuzzy Systems 2022 Best Paper Award! The title of the paper is “Confidence path regularization for handling label uncertainty in semi-supervised learning: use case in bipolar disorder monitoring” (link to the full article). The Accepter Paper version of the article can be accessed freely at Kamil Kmita personal website https://kamilkmita.com/research/.

2 Short description

These are the labels that empower any supervised (or semi-supervised) learning model. However, one tends to take them for granted, without posing questions about the annotation process that must have taken place for the labels to become attached to the training examples. In the paper, a specific annotation process that leads to inevitable label uncertainty is analyzed. The authors proposed a novel Confidence Path Regularization (CPR) procedure to estimate the uncertainty of each annotation. As a result of CPR, one can distinguish between *highly certain* and *less certain* supervised observations.

In general, the method can be applied to any annotation process that annotates unsupervised data indirectly, using labels from a different information source based on some common identifiers. However, it is the specific problem of bipolar disorder (BD) monitoring, a challenging task that motivated this research.

Bipolar disorder is a chronic illness characterized by fluctuations between different mood phases [1]. Early intervention is crucial for the efficient handling of BD, hence the need for monitoring the patient's status. Nowadays one can monitor it with remote sensors, for example by installing a dedicated application on the patient's smartphone that collects a wide range of voice characteristics from phone calls. Note that the actual conversation is not persisted due to privacy reasons, and accordingly, the p—dimensional summary of j—th call $x_j \in R^p$ is treated as unsupervised.

Nonetheless, patients undergo regular psychiatric checkups during which the psychiatrist is providing a label $y \in \{y_1, \ldots, y_c\}$ describing the current disease phase. For modeling purposes, this label is extrapolated onto observations x_j that fall into the specific time frame around the visit (e.g. from 7 days before to visit up to 2 days after the visit). This time frame is called a *ground truth period*. Observations that are not annotated with any label (because of being recorded too far from all the visits) remain unsupervised.

The indirect character of the annotation process leads to inherent uncertainty about whether all calls should be equally treated as supervised to the same extent. Confidence Path Regularization addresses this uncertainty by estimating an observation-wise *adjusted confidence factor* $\mathrm{conf}_j^\star \in (0,1]$. Note that by default, $\mathrm{conf}_j = 1\ \forall j$. The higher the adjusted $\mathrm{conf}_j^\star$, the more certain the j—th observation label and the higher its impact on the modeling results. The CPR procedure wraps a well-known Semi-Supervised Fuzzy C-Means model [2] and uses its key mechanisms: (i) a soft assignment rooted in fuzzy logic, namely a concept of a membership value $u_{jk} \in [0,1]$ of j—th observation to k—th cluster, and (ii) the scaling factor $\alpha > 0$ that controls the strength of supervised observations' impact on the outcomes of clustering.

The idea of Confidence Path Regularization is based on the following Regularization Assumption: “highly certain supervised observations should be consistently assigned high degrees of membership to the cluster associated with their supervised class by the learning algorithms across varying values of confidence factor”. This assumption is implemented by fitting a path of models (each model with a different regularization of the confidence factor), denoting the resulting memberships, and finally appropriately summarizing the outcomes from the entire path of models to arrive at $\mathrm{conf}_j^\star$.

The research was conducted as a part of the BIPOLAR project (see the project's website).

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RECOGNITION

Professor Verdegay (University of Granada, Spain) appointed Distinguished Researcher of the International Institute for Research in Artificial Intelligence of Hebei Foreign Studies University

In a ceremony presided over by Professor Sun Jianzhong, President of the Hebei Foreign Studies University (HFSU) of the People's Republic of China, last 26th November Professor José LuisVerdegay (University of Granada, Spain) was appointed as Distinguished Researcher of the International Institute for Research in Artificial Intelligence of HFSU.

HFSU, as the only independent undergraduate international language university in Hebei Province, has developed into a comprehensive and international undergraduate university with 28 secondary colleges (15 national education-oriented colleges, 13 society-oriented colleges), more than 21,000 students, 1,358 faculty and staff, 7 foreign academicians, 210 foreign experts, offering 75 foreign languages, and 136 majors.

The International Institute for Research in Artificial Intelligence is conceived for research, development, innovation, doctoral, master and undergraduate training and is focused on research and development of applications in Biotechnology and Biomedicine, Environment, Quality of Life of the Elderly, Smart Cities, Precision Agriculture and Smart Manufacturing (Industry 4.0).

The Institute hosts research labs in Artificial Intelligence, Big Data, Internet of Things, Blockchain, Cloud Computing and Software Development and includes a Data Centre for High Performance Computing and Big Data, providing services to the Institute's research and to companies.


聘书

LETTER OF APPOINTMENT

Esteemed Professor José Luis Verdegay:

This is to certify that you are awarded Distinguished Researcher of the International Institute for Research in Artificial Intelligence, Hebei International Studies University, China

Hebei International Studies University

NEWS

Ph.D. Thesis defended by Gustavo Rivas Gervilla

University of Jaén, Jaén, Spain

Gustavo Rivas Gervilla defended his PhD Thesis, entitled "Formal frameworks for the representation and extraction of referring expressions in data-to-text systems" ("Mecanismos formales para la representación y extracción de expresiones de referencia en sistemas data-to-text", is the original title of the PhD Thesis which was written in Spanish), within the Doctoral Program in Information and Communication Technologies of the University of Granada on April 20th, 2022. His supervisors were Nicolás Marín Ruiz and Daniel Sánchez Fernández from the University of Granada.

This thesis in enclosed within the Natural Language Generation (NLG) field. In particular, it focuses on the Referring Expression Generation problem (REG), one of the main problems in NLG. This problem can be described as, given a set of objects with different properties, find a description that identifies an object among them unequivocally.

Since we want to generate a description similar to the one produced by a human being, we have to deal with the complexity of natural language. In particular, we have to deal with properties such as the *size* or the *color*. These properties could be called *gradual properties* in the sense that their definition is imprecise. For example, we can consider a person *tall* with a certain degree. To deal with this gradualness, we can use Fuzzy Logic.

The main contributions of this PhD thesis to the REG problem can be divided into two groups:

- On the one hand, we have studied and made new proposals in the referential success and specificity measure fields. These two kinds of measures, which are closely related, have an important role within the problem of REG.
- On the other hand, we have proposed the use of Formal Concept Analysis as a new formal framework to deal with the problem of Plural Referring Expression Generation. Plural referring expressions are referring expressions that point to a set of objects rather than a single object. The problem of referring to groups has received less attention in the literature than the problem of individual referring.

In the study of measures of specificity and their relation with the measures of referential success, we collaborated with Prof. Ronal R. Yager, who introduced the specificity measures for the first time. So, we analyze the relation between these two groups of measures [4], and propose a new method to construct measures of specificity from measures of similarity between fuzzy sets [5]. We have also studied different families of specificity measures and analyzed their properties, for example their bounding conditions [3].

We have also collaborated with Prof. Albert Gatt [1], an authority in the field of Natural Language Generation, since our work is extremely related with the Linguistics field. With the guidance of Prof. Gatt, we designed different experiments that let us study the adequacy of different measures of specificity in the referring task with real users.

Finally, the other main contribution of this PhD thesis is the proposal of Formal Concept Analysis (FCA) as a suitable formal framework to deal with the Plural Referring Expression Generation problem [2]. In addition, we have proposed an extension of FCA to deal with gradual attributes, so we can work with natural language properties. This extension, that has been made through the Theory of Representation by Levels, has relevant theoretical and computational properties with respect to other gradual FCA extensions that can be found in the literature.

The thesis is organized as follows: the first two chapters are devoted to introduce the ideas of the thesis, as well as some preliminary results necessary for the rest of the thesis. Then, in Chapter 3 we present the Referring Expression Generation problem, emphasizing the role of vagueness in this problem, and we introduce our results about measures of specificity and referential success. The next two chapters present our proposal to use Formal Concept Analysis as a formal framework for the REG problem, and our extension of FCA to the gradual case. Finally, the last two chapters speaks about some practical results of our investigation, and conclusions.

We will continue the research in this field to expand and complement the results obtained in this PhD thesis.

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NEWS

Ph.D. Thesis defended by Boris Pérez-Cañedo

University of Cienfuegos, Cienfuegos, Cuba

Boris Pérez-Cañedo defended his PhD Thesis, entitled "Algorithm for solving fuzzy multi-objective linear programming problems using lexicographic ranking criteria". His supervisors are José Luis Verdegay (University of Granada, Spain), Alejandro Rosete (Technical University of Havana, Cuba) and Eduardo-René Concepción (University of Cienfuegos, Cuba). The thesis was defended on September 8, 2021 at Technical University of Havana (CUJAE) Cuba, with the following jury:

- Dr. Rafael Bello, Central University of "Las Villas", Cuba.
- Dr.Yailé Caballero, University of Camagüey, Cuba.
- Dr. Manuel Cortés, University of Cienfuegos, Cuba.
- Dr. Julio Madera, University of Camagüey, Cuba, and
- Dr. Carlos Morell, Central University of "Las Villas", Cuba.

The increasing complexity of decision-making problems has made Fuzzy Linear Programming an area that exhibits considerable theoretical, methodological and algorithmic development. In this thesis, an algorithm for the solution of Fuzzy Multi-objective Linear Programming problems was proposed. In these problems, all parameters and variables are assumed to be fuzzy numbers. The proposed algorithm yields Pareto optimal solutions with respect to lexicographic criteria for the ranking of fuzzy numbers. The solution strategy consists of transforming the Fuzzy Multi-objective Linear Programming problem into a mono-objective one. To achieve this, the author proposes a fuzzy scalarisation method of the epsilon-constraint type, and a lexicographic method to solve the mono-objective case that guarantees the uniqueness of the optimal values. It is shown that the optimal solutions of the single-objective problem are Pareto optimal ones for the multi-objective problem. A berth allocation problem and a project crashing problem illustrate the practical application of the proposed algorithm and the advantages over other methods and algorithms described in the literature.

The thesis was prepared in the format known as a compendium, i.e. it includes all the papers on the topic addressed in the thesis that have already been published by the author and his doctoral tutors. Among these papers, the following stand out

1. B. Pérez-Cañedo and E. R. Concepción-Morales. A method to find the unique optimal fuzzy value of fully fuzzy linear programming problems with inequality constraints having unrestricted L-R fuzzy parameters and decision variables. Expert Systems with Applications 123 (2019), 256-269. ISSN : 0957-4174. DOI :10.1016/j.eswa.2019.01.041

2. B. Pérez-Cañedo and E. R. Concepción-Morales. On LR-type fully intuitionistic fuzzy linear programming with inequality constraints: Solutions with unique optimal values. Expert Systems with Applications 128 (2019), 246-255. ISSN :0957-4174. DOI :10.1016/j.eswa.2019.03.035.

3. B. Pérez-Cañedo, J. L. Verdegay and R. Miranda Pérez. An epsilon-constraint method for fully fuzzy multiobjective linear programming. International Journal of Intelligent Systems 35.4 (2020), 600-624. ISSN : 0884-8173. DOI :10.1002/int.22219.

4. B. Pérez-Cañedo and E. R. Concepción-Morales. A Lexicographic Approach to Fuzzy Linear Assignment Problems with Different Types of Fuzzy Numbers. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 28.3 (2020), 421-441. ISSN : 0218-4885. DOI :10.1142/S0218488520500178.

5. B. Pérez-Cañedo, J.L. Verdegay, A. Rosete and E. R. Concepción-Morales. Fully Fuzzy Multi-objective Berth Allocation Problem. Hybrid Artificial Intelligent Systems. HAIS 2020. E. A. de la Cal et al. (Eds). Vol. 12344. Lecture Notes In Computer Science. Cham: Springer, 2020, 261-272. ISSN : 0302-9743. ISBN : 978-3-030-61705-9. DOI :10.1007/978-3-030-61705-9_22.

6. B. Pérez-Cañedo, A. Rosete, J.L. Luis Verdegay and E.R. Concepción-Morales. A Fuzzy Goal Programming Approach to Fully Fuzzy Linear Regression. Information Processing and Management of Uncertainty in Knowledge-Based Systems. IPMU 2020. M.-J. Lesot

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NEWS

Polish Society of Fuzzy Sets (POLFUZZ)

On March 11, 2022, the **Polish Society of Fuzzy Sets (POLFUZZ)** was established during the Founding Reunion in Warsaw https://polfuzz.org/ The objectives of POLFUZZ are the development, promotion, and dissemination of the theory and applications of fuzzy sets and related topics; organizing scientific and training meetings; representing the POLFUZZ community in society, towards state and local government authorities, as well as other public and private organizations in Poland and abroad; cooperation with national and international scientific organizations.

Board of POLFUZZ

For the 2022-2024 term, the following Board was elected:

- Barbara Pekala - president,
- Krzysztof Dyczkowski - vice president,
- Przemyslaw Grzegorzewski - vice president,
- Michal Baczyński - secretary,
- Katarzyna Miś - treasurer.

Founding Reunion of POLFUZZ

In 2022, POLFUZZ organized two special sessions at the following conferences:

- "Mathematical modeling and fuzzy sets" at the conference On the Trail of Women in Mathematics - Contemporary Women in Mathematics, Gdańsk, Poland, September 15 - 17, 2022, https://otowim22.ptkwm.pl/ index.html
- "Uncertainty and imprecision in decision making" (POLFUZZ special session) as part of the Twentieth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets, Warsaw, Poland, October 14, 2022, https://www2.ibspan.waw.pl/ifs2022/

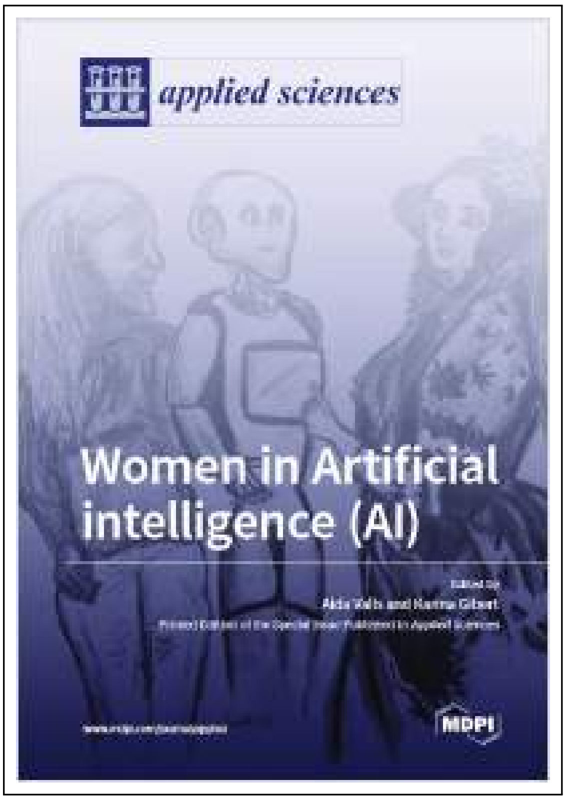
POLFUZZ is planning to organize the **International Symposium of Fuzzy Sets (ISFS 2023)** in Rzeszów, Poland, May 19-20, 2023. More details will be provided successively at the website https://polfuzz.org/category/wydarzenia/ We are waiting for you in Rzeszów at ISFS 2023!

SS IWIFSGN 2022

NEWS

Special issue "Women in Artificial Intelligence"

Applied Sciences Journal, MDPI, Appl. Sci. 2022, 12(19), 2022
https://www.mdpi.com/journal/applsci/special_issues/Women_in_AI
Printed edition as book: https://doi.org/10.3390/books978-3-0365-5532-4



On the Ada Lovelace Day (Oct 11th 2022) the special issue “Women in Artificial Intelligence” was published by MDPI. It has been edited by Dr. Aida Valls from Universitat Rovira i Virgili and Prof. Karina Gibert from Universitat Politècnica de Catalunya (Catalonia, Spain). Both of them are members of the gender groups of the Catalan Association for Artificial Intelligence (ACIA) and the Official Professional Chamber of Informatics Engineering in Catalonia (COEINF). Dr. Valls is member of EUSFLAT too.

This Special Issue includes 17 papers whose first author is a woman, from many countries worldwide (France, Czech Republic, United Kingdom, Australia, Bangladesh, Yemen, Romania, India, Cuba, Bangladesh and Spain).

The papers cover a broad scope of research areas within Artificial Intelligence, including Machine Learning, Perception, Reasoning or Planning, among others. The papers present applications to many relevant fields such as Human Health, Finance, or Education. Additionally, the issue includes three papers that deal with different aspects about gender bias in Artificial Intelligence.

Focusing on EUSFLAT related topics, we can find four papers in the area of management of uncertainty or qualitative reasoning. Jankova and Rakovska compare different types of membership functions in Type-2 Fuzzy Sets in a case study on the international financial market. The paper shows which methods should be used to be successful in stock market analysis. The paper by Maaroof et al. presents two methods to generate rules that explain black box classification models. One of them uses the technique of dominance-based rough sets. A case study in the diagnosis of diabetic retinopathy is used to evaluate and compare these two explainability methods. The paper of Lopez de Aberasturi et al. analyses two models of automated assessment of open-response assignments by means of probabilistic models such as Bayesian networks. They propose a new approach also based on probabilities that work in a multiagent system. Lastly, Alsinet et al. introduce a quantitative model for measuring polarization in an online debate by means of creating a bipartite graph between supporters and opponents to an argument. The proposed quantitative model is based on the maximum polarization we have in all the possible bipartitions of the graph, which is computed with a greedy local search optimization algorithm. Interactions between different users can give rise to circular agreement and disagreement relationships during the debate.

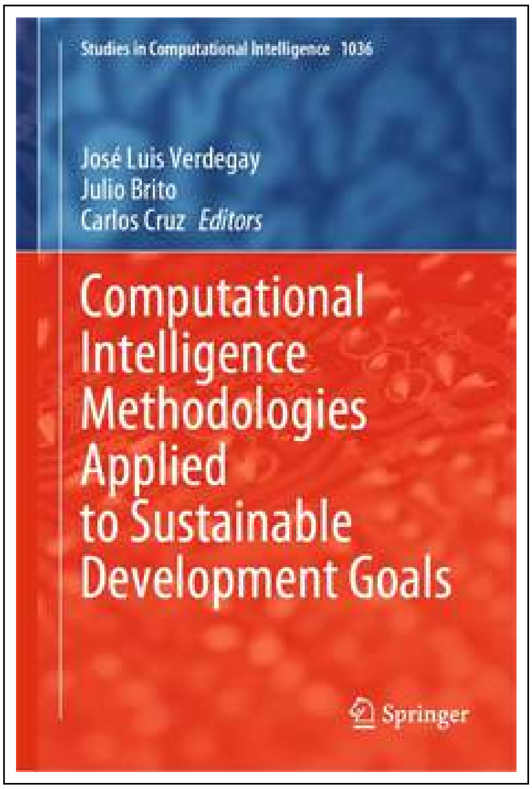
Apart from its intrinsic scientific value as a special issue itself, the special issue intends to help to the visibilization of where AI women are, what they do, and how they contribute to Artificial Intelligence developments from different places, positions, research branches and application fields in AI. Book cover illustrates this connection between past and current computer science women and Artificial Intelligence.

NEWS

Computational Intelligence Methodologies Applied to Sustainable Development Goals

Editors: José Luis Verdegay, Julio Brito, and Carlos Cruz
Series: Studies in Computational Intelligence
Volume: 1036
Editorial: Springer Nature Switzerland AG
Year: 2022
ISBN: 978-3-030-97343-8
DOI:https://doi.org/10.1007/978-3-030-97344-5

Book review by Boris Pérez-Cañedo
Department of Mathematics, University of Cienfuegos, Cienfuegos 55100, Cuba
bpcanedo@gmail.com



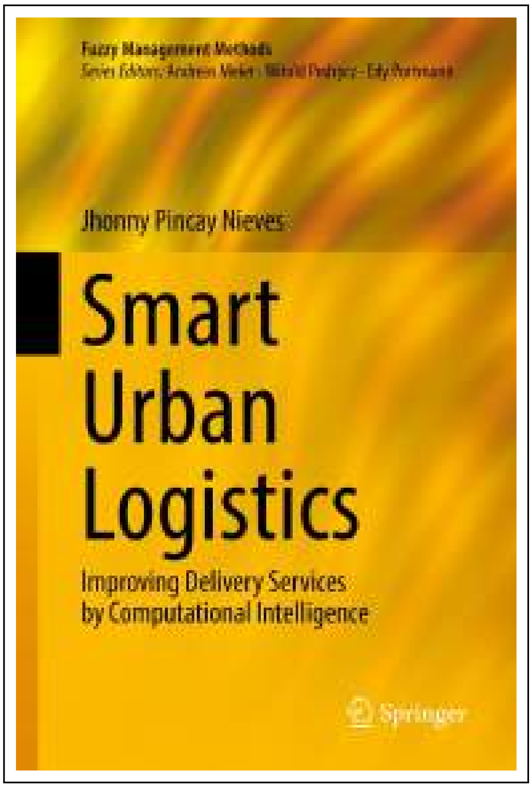
There is no doubt about the fundamental role of the Sustainable Development Goals (SDGs) for the development of present and future generations. This book, written and edited by renowned academics from 13 countries, is a must-read that addresses issues on health control, emissions control, and transportation and distribution in the framework of the SDGs, from the perspective of Computational Intelligence (CI) paradigms. The first part (Computational Intelligence and SDG) contains three papers that serve as an introduction to the intersection between the SDGs and the field of Computational Intelligence, and provides the reader with an overview of fundamental topics and applications of CI to address real-world problems. The second part (Health Control Models) discusses applications of CI techniques to predict the behavior of the Covid-19 pandemic, assess the risk of African swine fever outbreak, allocate healthcare facilities across geographically distributed areas, and optimize the Tikhonov regularization parameter for solving inverse problems in bioengineering, renewable energy and climate. The third part (Emission Control Models) deals with models for planning delivery routes and for cross-docking, emphasizing on environmental criteria, leak detection in water distribution networks, tropospheric ozone forecasting, fuzzy regression for power curve estimation, and analyzing the relationship between economic growth and environmental deterioration. Finally, the fourth part (Transportation and Distribution Models) is concerned with models for classical transportation, organizational transportation and tourist route design under fuzziness, as well as the generation of alternative solutions to a perishable food distribution problem. In all cases, the problems were addressed in close relation to their impact on the achievement of the SDGs. In closing, I must say that the book succeeds in giving a broad overview of current trends at the intersection between CI and a variety of SDGs-related topics. I am pleased to recommend its reading and I believe that the reader will find in it not only a source of up-to-date literature, but also a starting point and inspiration to develop new techniques within the CI paradigms that effectively address fundamental problems in sustainable development.

NEWS

Smart Urban Logistics Improving Delivery Services by Computational Intelligence

authored by Jhonny Pincay Nieves, released in the late autumn 2022 in the book series Fuzzy Management Methods of Springer, Cham.

Book review by Miroslav Hudec
University of Economics in Bratislava, Slovakia and
VSB - Technical University of Ostrava, Czech Republic.



The book is based on the PhD thesis defended in March 2022 at the University of Fribourg, Switzerland. It is a valuable contribution in improving living conditions in cities considering environmental aspects and citizens privacy in the demanding field of the last-mile delivery. This is a complex task that encompasses optimization in an uncertain and dynamic environment. In the recent years, the concept known as smart city was mainly focused on ensuring the efficiency from the mathematical and IT perspective. This is a significant way of improving urban systems, however, simply put, not sufficient. We should not neglect citizens and their needs. Based on these facts, Jhonny Pincay Nieves won the Faculty price for computer science of the Faculty of Science and Medicine of the University of Fribourg.

Having in mind a large amount of data as well as the uncertainty in data and in the environment, the concept based on computational intelligence is a reasonable solution. Generally, to be acceptable for all urban stakeholders' categories, the proposed solution should be explainable and interpretable. In addition, when considering citizens, the solution should respect the citizens' privacy. This is a hard task and challenge, which did not discourage Jhonny Pincay Nieves. Contrarily, the author has solved the problem excellently.

Fuzzy concepts and evolutionary algorithms are applied in many fields, for example, in medical science for diagnostic systems, in civil engineering for risk analysis of structures, in information science to describe fuzzy information and to formulate requirements for classification. The monograph proposes a new solution in the last-mile-delivery problem. The main requirements are understanding traffic situations and reaching customers in their homes on the first try, keeping their privacy (in this case, location) non-disclosed. Hence, the model should cover traffic situations, explain them, predict them, and predict the citizens' behavior, whether they will be at the delivery place without tracing them. A certain conception of interpretability and explainability is dealt with via fuzzy systems since they treat classes and concepts similarly to the human way of inference. It is welcome in tasks, like this, which deals with imprecise data and unsharp rules.

The monograph covers all relevant aspects of a smart last-mile-delivery. Firstly, it introduces the topic, background, motivation, research questions, and research methods. The next section discusses the latest developments related to smart cities, smart mobility, and smart logistics of the last-mile delivery. The successive section explains computational intelligence methods, namely fuzzy logic and swarm intelligence. The following section is dedicated to managing and fuzzifying geospatial traffic data in order to extract relevant information, construct a fuzzy rule-based system, and explain traffic information by the linguistic summaries, which are understandable at first glance. The results are also visualized on maps, which further contributes to legibility. The successive section is focused on the privacy-by-design requirement of public services, in this work, how to classify the behavior of delivery customers without disclosing their locations. For this task, traditional and fuzzy clustering and classification are developed and applied. Next, the novel framework of fuzzy ant routing is proposed, which is evaluated by simulation. Then, the evaluation of the prototype on the real data is developed and discussed. Finally, the author gives a critical reflection indicating benefits and weak points of the proposed solution.

To sum up, the monograph conveys a comprehensive view of the last-mile-delivery problem and merges approaches inspired by nature (fuzzy logic and ant colony optimization) with urban logistics. There is certainly a lot of work ahead. But this work paves the way for greater involvement of fuzzy approaches and ant colony optimization in smart cities and can inspire future research in the applicability of explainable and interpretable computational intelligence for making urban areas smarter. I am pleased to recommend this monograph to be used by communities of computational intelligence and smart cities.

CONFERENCE REPORT

15th +FuzzyMAD meeting

December 2, 2022
Faculty of Mathematics, Complutense University, Madrid, Spain
https://eventos.ucm.es/90010/detail/fuzzymad-2022.html

Sponsored by:
Faculty of Mathematics, Complutense University of Madrid
Interdisciplinary Mathematics Institute, Complutense University of Madrid (IMI Data Science Club)
Ph.D. program on Mathematical Engineering, Statistics and Operational Research (IMEIO)
Project PGC2018-096509-B-I00 of the Government of Spain (FORAid)

The 15th edition of the FuzzyMAD meeting, +FuzzyMAD since 2018, has achieved once again its objective of allowing researchers in Madrid, Spain, to share activities and plans within fuzzy logic and soft computing.

This edition of +FuzzyMAD meeting has been possible thanks to the support of the Faculty of Mathematics at Complutense University of Madrid, the Interdisciplinary Mathematics Institute (particularly its Data Science Club program), the Ph.D. Program on Mathematical Engineering, Statistics and Operational Research (IMEIO, a joint program between Complutense University and the Technical University of Madrid), plus the Spanish research group led by profs. Javier Montero and Daniel Gómez (which together with other partners conform the FORaid team). The members of +FuzzyMAD 2022's Scientific Committee are profs. Javier Montero, Tinguaro Rodríguez, Daniel Gómez and Pablo Flores-Vidal.

Following the traditional structure, +FuzzyMAD 2022 started with a course mainly oriented for Ph.D. students. Three suggesting talks were given by prof. Oier López de Lacalle (*Few-shot Information Extraction: Pre-train, Prompt and Entail*), prof. Alba Olivares (*Regularized and MINLP Approaches to Increase Interpretability in Prediction Models*), and prof. Juan Antonio Guevara (*Polarization measures on social networks*).

Afterwards, some Ph.D students presented the current stage of their research: Ziwei Shu, (*Hotel segmentation based on the 2-tuple fuzzy linguistic model and OWA operators*), Carlos Giner (*MANOVA-based multiobjective decision trees*), Javier Bonilla (*Fuzzy clustering methods using Rényi entropy*), Gustavo Ito (*Exit times in conic varieties*), Carlos Sebastián (*A new variable selection methodology for time series forecasting based on Shapley values*), João Gabriel (*Stabilization techniques for pure convection problems using finite elements method*), Elías Plaza (*Management and intelligent control of in-flight fuel distribution in a commercial aircraft*), Sara Hamaizia (*Numerical methods and Chaos for fractional damped Duffing equation*), and César Guevara (*COVID-19 Spread Algorithm in the International Airport Network-DetARPDS*).

Finally, we had the +FuzzyMAD traditional poster session while having a friendly buffet meeting, where attendants from most universities of Madrid could explore future joint collaborations discussing the posters each group has prepared with this aim.

This +FuzzyMAD2022 had more than 70 participants, and it was possible thanks also to the dedication of Inmaculada Gutiérrez, Carlos Giner-Baixauli, Maria Jaenada, Javier Bonilla, Adolfo Urrutia and Paula Teran.

We are looking the next edition of +FuzzyMAD!

Javier Montero and Tinguaro Rodríguez
Complutense University
Madrid, Spain

+Fuzzy MAD

Instituto de Matemática Interdisciplinar

IMI-DSC Club

FORMID

CONFERENCE REPORT

Soft Methods in Probability and Statistics (SMPS 2022)

The 10th International Conference on Soft Methods in Probability and Statistics (SMPS 2022) took place in Valladolid, Spain, on September 14-16, 2022. Conference website: http://smps2022.uva.es/

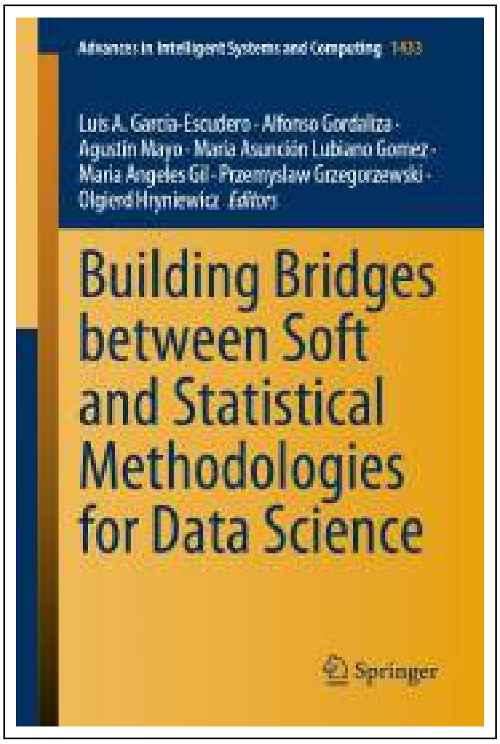
The SMPS conference is a scientific event organized every two years that gathers experts representing existing approaches used in soft modeling in probability, statistical reasoning, and data analysis. Past editions: SMPS 2002 - Warsaw, Poland; SMPS 2004 - Oviedo, Spain;SMPS 2006 - Bristol, United Kingdom; SMPS 2008 - Toulouse, France; SMPS 2010 - Oviedo and Mieres, Spain;SMPS 2012 - Konstanz, Germany; SMPS 2014 - Warsaw, Poland; SMPS 2016 - Rome, Italy; SMPS 2018 - Compiègne, France.

The 10th edition of the SMPS conference was initially scheduled for 2020, however, the SARS-COV-2 pandemic obligated to postpone twice the event. SMPS 2022 was organized by the Departamento de Estadística e Investigación Operativa and the Instituto de Investigación en Matemáticas (IMUVA) of the University of Valladolid, Spain, with General Chairs: Luis A. García-Escudero, Alfonso Gordaliza, Agustín Mayo-Íscar and Organizing Local Committee: Pedro C. Álvarez-Esteban and his collaborators. Thanks to them, we could spend a wonderful three days in Valladolid, listening to inspiring presentations, having interesting discussions, and as well taking part in some social events.

Seventy-five registered participants took part in the conference (a few of them online). Each day was begun with the keynote speaker's lecture: Francisco Herrera (University of Granada, Spain), "Deepening smart data: Quality data to enhancing Artificial Intelligence"; Christophe Croux (EDHEC Business School, France), "Networks in high dimensional time series using sparse VAR models"; and Frank Klawonn (Helmholtz Centre for Infection Research, Germany), "Using fuzzy cluster analysis to find interesting clusters".

During the conference, a traditional competition was held for the best paper given by Ph.D. students. The jury was composed of the conference chairs and members of the Editorial Board or IJAR and decided to split the prize into four awards, 2 Gold ones (300€ each) and 2 Silver ones (200€ each). The nominees were GOLD AWARDS: Arne Decadt (Ghent University), "Decision-making with E-admissibility given a finite assessment of choices" and Nicolas Dietrich (Universitat Salzburg), "Convergence of copulas revisited: Different notions of convergence and their interrelations"; SILVER AWARDS: Louisa Kontoghiorghes (King's College London), "Testing the homogeneity of topic distribution between documents of a corpus" and Patricia Ortega-Jiménez (Universidad de Cádiz), "A minimizing problem of distances between random variables with proportional reversed hazard rate functions".

As was the case during previous conferences, the proceedings were published by Springer in the following volume: Luis A. García-Escudero, Alfonso Gordaliza, Agustín Mayo, María Asunción Lubiano Gómez, María Ángeles Gil, Przemyslaw Grzegorzewski, Olgierd Hryniewicz (Eds.), Building Bridges between Soft and Statistical Methodologies for Data Science, Springer, 2023, https://doi.org/10.1007/978-3-031-15509-3. The volume contains fifty-two selected contributions that cover very different and relevant aspects such as imprecise probabilities, information theory, random sets and random fuzzy sets, belief functions, possibility theory, dependence modeling and copulas, clustering, depth concepts, dimensionality reduction of complex data and robustness. Thus, one may find them clearly useful in establishing those important bridges between soft and statistical methodologies for Data Science. Moreover, a special issue of the International Journal of Approximate Reasoning on "Statistical/soft approaches and computing for data analysis and classification" (with Alfonso Gordaliza, Agustín Mayo-Íscar, María Asunción Lubiano and Beatriz Sinova as the Special Issue Guest Editors) will be prepared. The deadline for submission is June 15th, 2023.



The next edition of the SMPS conference will take place in September 2024 in Salzburg and will be organized by Wolfgang Trutschnig (University of Salzburg) and his team.

Przemyslaw Grzegorzewski (Warsaw)

International Symposium on Fuzzy Sets
19-21 May 2023 Rzeszów

International Symposium on Fuzzy Sets (ISFS)

organized by

Polish Society for Fuzzy Sets (POLFUZZ) & University of Rzeszów

in cooperation with Slovak University of Technology (STU) in Bratislava

May 19 - 21, 2023

University of Rzeszów, Pigonia 1, Rzeszów, Poland

isfspolfuzz.ur.edu.pl

Aims and Scope:

The conference ISFS will provide an excellent international forum for sharing knowledge and results in theory, methodology, and applications of Fuzzy Sets and Systems.

Formation of a platform for discussion of critical research environments and disciplines. Indication of the need for changes in the cooperation of the scientific community and with business partners.

Research presentations are of interest in the following topics:

- Theoretical foundations of fuzzy logic and fuzzy set theory;
- Imprecise information modeling with fuzzy, rough, and other methods;
- Federated learning;
- Image processing and computer vision;
- Information retrieval;
- Knowledge representation and engineering;
- Decision-making models;
- Expert systems;
- Intelligent data analysis and data mining;
- Approximate reasoning.

Potential areas of application, among others, include:

- Medical and Healthcare systems;
- Business Process Modeling;
- Social and economic models.

Plenary speakers:

- **Susana Montes**, University of Oviedo, Spain & President of EUSFLAT
- **Janusz Kacprzyk**, Systems Research Institute Polish Academy of Sciences
- **Radko Mesiar**, Slovak University of Technology Bratislava, Slovakia
- **Józef Drewniak**, University of Rzeszów, Poland

Important dates:

Conference	May 19-21, 2023
Registration & Abstract submission	March 15, 2023
Abstract notification	March 31, 2023
Conference fee	April 15, 2023

Submission guidelines:

All submitted papers will be peer-reviewed.

Papers can be submitted in either 1-2 page(s) abstract.

Papers must show the scientific results, be original, and may not be published elsewhere.

After the Conference, accepted papers/abstracts will be published in the post-conference edition by the University of Rzeszów Publishing with ISBN.

The best papers and abstracts will be invited and recommended to extend their contributions to internationally recognized journals.

ISFS is a cyclic scientific meeting supported by the Polish Society for Fuzzy Sets, and by the following scientific organizations:

- **EUSFLAT - EUROPEAN SOCIETY FOR FUZZY LOGIC AND TECHNOLOGY**
- **PTM - Polish Mathematical Society**
- **PTKM - Polish Society of Women in Mathematics**

POLFUZZ
POLSKIE TOWARZYSTWO ZBIORÓW ROZMYTYCH

STU

MDAI 2023

The 20th International Conference on Modeling Decisions for Artificial Intelligence
Umeå, Sweden
June 19-22, 2023
http://www.mdai.cat/mdai2023

IMPORTANT DATES

Submission ISBN-only procedings:
April 30th, 2023

Acceptance notification:
May 20th, 2023

Final version:
May 25th, 2023

Conference:
June 19-22, 2023

CONTACTS

mdai@mdai.cat

ORGANIZATION CHAIRS

To be determined

GENERAL CHAIRS

Vicenç Torra (Umeå University)

PROGRAM CHAIRS

Vicenç Torra (Umeå University, Sweden),
Yasuo Narukawa (Tamagawa University, Japan)

ADVISORY BOARD

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AIMS AND GOALS

In MDAI we are particularly interested in the different facets of decision processes in a broad sense. This includes model building and all kind of mathematical tools for data aggregation, information fusion, and decision making; tools to help decision in data science problems (including e.g., statistical and machine learning algorithms as well as data visualization tools); and algorithms for data privacy and transparency-aware methods so that data processing processes and decisions made from them are fair, transparent, explainable and avoid unnecessary disclosure of sensitive information.

The MDAI conference includes the following tracks:

- **Human decision making,**
- **Aggregation functions,**
- **Data science,**
- **Machine Learning,**
- **Data privacy,**
- **Graphs and (social) networks,**

The conference has been since 2004 a forum for researchers to discuss last results into these areas of research. MDAI is rated as a CORE B conference by the Computing Research and Education Association of Australasia - CORE.

SUBMISSION AND PUBLICATION

Original technical contributions are sought. Contributions will be selected on the basis of their quality. Papers should not exceed 12 pages in total (using LNCS/LNAI style).

Proceedings with accepted papers will be published in the LNAI/LNCS series (Springer-Verlag) and distributed at the conference, as done in previous MDAI conferences. Besides, papers, that according to the evaluation of the referees, are not suitable for the LNAI but that have some merits will be published in a USB proceedings and scheduled in the MDAI program. Direct submission to these other proceedings is also possible (deadline: March 31st).

EUSFLAT 2023

The 13th Conference of the European Society for Fuzzy Logic and Technology
organized jontly with
AGOP 2023 - International Summer School on Aggregation Operators
FQAS 2023 - International Conference on Flexible Query Answering Systems

Palma, Mallorca, Spain
September 4-8, 2023

The topics addressed by the Conference cover all aspects of fuzzy logic and soft computing, namely (but not limited to):

Approximate reasoning
Clustering and classification
Cognitive modelling
Intelligent data analysis and data-mining
Data aggregation and fusion
Database management and querying
Theory and applications of decision-making
Forecasting and time series modelling
Fuzzy control
Theoretical foundations of fuzzy logic and fuzzy set theory
Imprecise probabilities and fuzzy methods in statistics
Image processing and computer vision
Information retrieval
Knowledge representation and knowledge engineering
Linguistic modelling
Machine learning
Natural language processing, generation and understanding
Neuro-fuzzy systems
Stochastic and fuzzy optimization
Possibility theory and applications
Rough sets theory
Semantic web
Uncertainty modelling

Important dates

Special session proposal
October 31, 2022
Paper & abstract submission
February 17, 2023
Early registration
April 17, 2023

EUROPEAN SOCIETY FOR FUZZY LOGIC AND TECHNOLOGY

Universitat de les Illes Balears

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SCIENTIFIC REPORT

VI Brazilian Conference of Fuzzy Systems (CBSF 2021)

Marcos Eduardo Valle[1], Regivan Hugo Nunes Santiago[2], and Francielle Santo Pedro Simões[3]

[1] Universidade Estadual de Campinas (UNICAMP), Campinas – Brazil.
[2] Universidade Federal do Rio Grande do Norte (UFRN), Natal – Brazil.
[3] Universidade Federal de São Paulo (UNIFESP), Osasco – Brazil

The Brazilian Conference of Fuzzy Systems (CBSF, in Portuguese Congresso Brasileiro de Sistemas Fuzzy) is the major South American event dedicated solely to fuzzy systems and their applications in mathematics, physics, engineering, health, and social sciences. Besides being an attractive forum on fuzzy systems and their applications, the CBSF aims to aggregate the South American community around this subject, favoring the exchange of ideas and increasing national and international collaborations. The CBSF has been held every two years since 2010, and the VI CBSF was scheduled to happen in 2020. However, because of the Covid-19 pandemic, it has been postponed to 2021, and it happened virtually at the São Paulo State University (UNESP) in São José do Rio Preto - Brazil, in November 2021. The event's general chair and co-chair were Valeriano A. Oliveira and Geraldo N. Silva, respectively.

The IV CBSF brought together 218 participants, including researchers, engineers, students, and professionals interested in fuzzy systems and their applications. The congress counted on six plenaries, three tutorials, and several technical sessions in which the works approved by the scientific committee were presented orally or by short video recordings. The plenary sessions featured the internationally prominent researchers Lluís Godo Lacasa (Artificial Intelligence Research Institute, Universitat Autonoma de Barcelona, Spain), Vladik Kreinovich (University of Texas at El Paso, USA), and Weldon Lodwick (University of Colorado Denver, USA) as well as the Brazilian exponential researchers Graciele P. Silveira (Universidade Federal de São Carlos, Brazil), Renata Hax Sander Reiser (Universidade Federal de Pelotas, Brazil), and Rodney Carlos Bassanezi (Universidade Estadual de Campinas, Brazil). The complete program, with links to all contributions presented at the technical sessions, is available at `https://sites.google.com/unesp.br/vicbsf-en/home`.

The contributions submitted to the VI CBSF, which were full papers or two-page abstracts, have been judged by at least two anonymous referees. Only those full papers that received a positive evaluation from all reviewers have been accepted to be presented orally. Similarly, the abstracts with a positive average grade were accepted for presentation. The 69 accepted contributions represent a view of the Brazilian contributions to the area of fuzzy systems. In particular, the 41 full papers have been organized as chapters of the book "Recent Trends on Fuzzy Systems," which is freely available for download at `https://drive.google.com/file/d/1KEjJ1xMcfqsRsZJhMSYelzYxBiu9OErJ/view?usp=sharing`.

The authors of the best full papers have been invited to submit an extended version of their contribution to Mathware and Soft Computing Magazine. All the invited manuscripts have been judged by anonymous referees, allowing us to select six papers for this issue.

During the VI CBSF, recognition was given to Rodney Bassanezi for his outstanding contributions and teaching excellence in the area of fuzzy systems. A brief account of Rodney's contributions to fuzzy systems is included in this issue.

We hope the participants had a great time during the VI CBSF, with many interesting and fruitful discussions and novel collaborations among fuzzy systems researchers. We would like to express our gratitude towards all individuals, including participants and members of the scientific and organizing committees, who have contributed to making the VI CBSF a great success.

Some finitary constructions on inflationary BL-algebras and their properties

José Ronaldo[1 [0000−0002−1143−3528]], Rui Paiva[1 [0000−0002−1775−8489]], and Benjamín Bedregal[2 [0000−0002−6757−7934]]

[1] Instituto Federal de Educação, Ciência e Tecnologia do Ceará – IFCE
ronaldoferreira.b@gmail.com
rui.brasileiro@ifce.edu.br
[2] Departamento de Informática e Matemática Aplicada – DIMAp
Universidade Federal do Rio Grande do Norte – UFRN
bedregal@dimap.ufrn.br

Abstract. Recently, in the article “Inflationary BL-algebras obtained from 2-dimensional general overlap functions”, published in Fuzzy Sets and Systems, Paiva *et al.* proposed a new generalization of BL-algebras dropping the requirements of associativity and existence of the neutral element for the fuzzy conjunction. Such a generalization was called inflationary BL-algebras. In this paper, we continue to consider this research topic and focus on operations of direct product, separated sum, and lifting for inflationary BL-algebras, as well as exploring some properties of these finitary constructions.

Keywords: Inflationary BL-algebras · Finitary operations · Fuzzy logical systems.

1 Introduction

Non-classical logic is useful in Computer Science to obtain specialized or decision support systems capable of dealing with uncertain information, fuzzy information, and intelligent systems. Various kinds of non-classical logical systems have been proposed and extensively investigated in this setting – for example, classes of some Fuzzy Logic that Peter Hájek algebraically modeled as BL-algebras.

The main example of a BL-algebra is the interval $[0, 1]$ endowed with the structure induced by a continuous t-norm [10], however, some generalizations thereof have been proposed as semantical counterpats of non-classical logical systems [1, 3, 5, 6, 14, 16, 19].

In order to obtain new classes of semantical counterpats of non-classical logical systems, properties related to the residuation principle of overlap functions as well as their extension to the context of lattices were investigated in [13, 15]. Recently, Paiva *et al.* proposed in [17] a new generalization of BL-algebras dropping the requirements of associativity and existence of the neutral element for the fuzzy conjunction. Such generalization, based on a subclass of bivariate general overlap functions, is called inflationary BL-algebras and it extends other non-associative generalizations of BL-algebras that have been proposed in [3, 16]. In this paper, we continue to consider this research topic and focus on operations of direct product, separated sum and lifting for inflationary BL-algebras and we explore some properties related to their finitary constructions.

The paper is organized as follows. In Section 2, after establishing the notation, we give an overview on cartesian product, separated sum and lifting on partially ordered sets as well as we expose a brief introduction of inflationary BL-algebras, some of its properties and some examples. In Section 3, we define Separated sum for the bounded lattices context and prove related lemmas. In Section 4, introduces external operations on pairs of inflationary BL-algebras and indicates a way to construct new inflationary BL-algebras from given ones. Finally, Section 5 contains some concluding remarks.

2 Basic notions and terminology

An algebraic structure (or algebra, for short) is usually represented as a nonvoid set together with a set of finitary operations on it. In this paper, the investigated algebras provide algebraic models for certain fuzzy logical systems. In the present section, some basic concepts and terminologies used throughout the paper are remembered. The reader is assumed to be familiar with some elementary notions of universal algebra and partially ordered sets, especially lattices. For more details it is indicated [2, 4, 7, 9]

2.1 Some finite operations on partially ordered sets

In this subsection, we will briefly review some basic concepts of operations cartesian product, separated sum and lifting on partially ordered sets necessary for the development this paper.

Definition 1 ([20]). *Let D and E be two partially ordered sets. Then the* cartesian product *of D and E, denoted $D \times E$, is the partially ordered set $D \times E = \{(x, y) \,|\, x \in D, y \in E\}$, with ordering $(x, y) \leq (z, w)$ if and only if $x \leq z$ and $y \leq w$.*

Definition 2 ([20]). *Let D_0 and D_1 be two partially ordered sets. The* separated sum *of D_0 and D_1, denoted $D_0 + D_1$, is the partially ordered set*

$$D_0 + D_1 = \{(0, x) \,|\, x \in D_0\} \cup \{(1, y) \,|\, y \in D_1\} \cup \{\perp\}$$

with ordering $\vec{u} \leq \vec{v}$ if and only if (i) $\vec{u} = \perp$ or (ii) $\vec{u} = (j, x), \vec{v} = (j, y)$ and $x \leq y$.

Here, it is assumed that $\perp$ is a distinguished element which does not belong to $(\{0\} \times D_0) \cup (\{1\} \times D_1)$. Its associated injections $\zeta_j : D_j \longrightarrow D_0 + D_1$, for $j = 0, 1$, are defined by $\zeta_j(x) = (j, x)$.

Definition 3 ([20]). *Let D be a partially ordered set and let $\perp \notin D$ be some distinguished element. Then the* lifting *of D, denoted by $D_\perp$, is the set $D_\perp = D \cup \{\perp\}$ ordered by $x \leq y$ if and only if $x = \perp$ or $x \leq_D y$.*

2.2 Inflationary BL-algebras

In this subsection, we expose a brief introduction of inflationary BL-algebras, some of its properties and some examples.

Definition 4 ([17]). *An algebra* $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \bot, \top \rangle$ *of type* $(2, 2, 2, 2, 0, 0)$, *such that the binary operator* $*$ *is inflationary, i.e., for all* $x \in A$, $x * \top \geq x$, *is called an* inflationary BL-algebra *if it satisfies:*

($\mathcal{IBL}$1) $\langle A, \wedge, \vee, \bot, \top \rangle$ *is a bounded lattice;*
($\mathcal{IBL}$2) $\langle A, * \rangle$ *is a commutative groupoid*[3]*;*
($\mathcal{IBL}$3) The pair $(*, \rightarrow)$ *is a Galois connection:*

$$\forall x, y, z \in A, \text{ we have } x * z \leq y \text{ if and only if } z \leq x \rightarrow y;$$

($\mathcal{IBL}$4) $x \wedge y = x * (x \rightarrow y)$ *(divisibility);*
($\mathcal{IBL}$5) $(x \rightarrow (y * \top)) \vee (y \rightarrow (x * \top)) = \top$ *(general prelinearity).*

Theorem 1 ([17]). *If* $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \bot, \top \rangle$ *is an inflationary BL-algebra, then for all* $x, y, z \in A$:

($\mathcal{IBL}$6) $x * (x \rightarrow y) \leq y$;
($\mathcal{IBL}$7) if $x \leq y$ *then* $x * z \leq y * z$;
($\mathcal{IBL}$8) if $x \leq y$ *then* $y \rightarrow z \leq x \rightarrow z$;
($\mathcal{IBL}$9) if $x \leq y$ *then* $z \rightarrow x \leq z \rightarrow y$;
($\mathcal{IBL}$10) $(x \vee y) \rightarrow z \leq (x \rightarrow z) \wedge (y \rightarrow z)$;
($\mathcal{IBL}$11) $x \rightarrow (y \wedge z) \leq (x \rightarrow y) \wedge (x \rightarrow z)$;
($\mathcal{IBL}$12) $(x \vee y) * z = (x * z) \vee (y * z)$;
($\mathcal{IBL}$13) $x \leq y \rightarrow (x * y)$;
($\mathcal{IBL}$14) $x * \bot = \bot$.

Theorem 2 ([17]). *Let* $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \bot, \top \rangle$ *be an inflationary BL-algebra. If* $\langle A, \wedge, \vee, \bot, \top \rangle$ *is a complete lattice then, for each* $x, y \in A$

$$x \rightarrow y = \bigvee \{ z \in A \mid x * z \leq y \} \tag{1}$$

and $\top \rightarrow x \leq x$.

Theorem 3 ([17]). *Let* $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \bot, \top \rangle$ *be an inflationary BL-algebra. Then, for all* $x, y \in A$ *we have*

(i) $x \leq y$ *whenever* $x \rightarrow y = \top$ *(right order property);*
(ii) $x \leq y$ *if and only if* x *and* y *are comparable and* $x \rightarrow (y * \top) = \top$ *(general order property);*
(iii) $x \rightarrow \top = \top$ *(right boundary).*

[3] In the literature there are many nonequivalent definitions for groupoid. Here, a groupoid is an algebraic structure $\langle A, * \rangle$ where A is a non-empty set and $*$ is a binary function defined on A [12].

In what follows we present some example of inflationary BL-algebras:

Example 1. Let S be a nonempty set, $\mathcal{P}(S)$ be the family of all subsets of S and $A \in \wp(S)^+ = \{X \subseteq S : X \neq \emptyset\}$. Define operations $*$ and $\to$ by

$$X * Y = \begin{cases} S & \text{if } X \cap Y = S \\ (X \cap Y) \cup A & \text{if } X \cap Y \neq S,\ X \neq \emptyset \text{ and } Y \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases} \qquad \text{and}$$

$$X \to Y = \bigvee \{W \subseteq S \,|\, X * W \leq Y\},$$

for all $X, Y \in \mathcal{P}(S)$. Then $\langle \mathcal{P}(S), \cap, \cup, *, \to, \emptyset, S \rangle$ is an inflationary BL-algebra. We call $\mathcal{P}(X)$ as the power inflationary BL-algebra of S.

Example 2. Consider the complete lattice $\mathbb{Z}_0^\infty$, of non negative integers extended with ∞ endowed with the natural usual order $\leq$. By convention, $0 \cdot \infty = \infty \cdot 0 = 0$ and $x \cdot \infty = \infty \cdot x = \infty$ for each $x \neq 0$. Define the binary operations $*$ and $\to$ on $\mathbb{Z}_0^\infty$ as follows:

$$x * y = k \cdot \min(x, y) \text{ for a positive integer constant } k, \quad \text{and}$$

$$x \to y = \begin{cases} \infty & \text{if } x \leq \frac{y}{k} \\ \lfloor \frac{y}{k} \rfloor & \text{otherwise} \end{cases}$$

Under these conditions, the structure $\langle \mathbb{Z}_0^\infty, \min, \max, *, \to, 0, \infty \rangle$ is an inflationary BL-algebra.

Example 3. If $\mathcal{A} = \langle A, \wedge, \vee, *, \to, \bot, \top \rangle$ is an inflationary BL-algebra and X is a nonempty set, then the functions space A^X becomes an inflationary BL-algebra $\langle A^X, \wedge, \vee, *, \to, \bot_{A^X}, \top_{A^X} \rangle$ with the operations defined elementwise. If $f, g \in A^X$, then

$$\begin{aligned} (f \wedge g)(x) &= f(x) \wedge g(x) \\ (f \vee g)(x) &= f(x) \vee g(x) \\ (f * g)(x) &= f(x) * g(x) \\ (f \to g)(x) &= f(x) \to g(x) \end{aligned}$$

for all $x, y \in X$ and $\bot_{A^X}, \top_{A^X} : X \to A$ are the constant functions associated with $\bot, \top \in A$.

3 Some finitary operations on bounded lattices

Notice that the separated sum operation, given in Definition 2, is not a closed operation for bounded lattices since it fails at the top element. So, the next results aim to solve this problem.

In the follows, $\bot$ and $\top$ are two distinguished elements which does not belong to $(\{0\} \times A) \cup (\{1\} \times B)$.

Definition 5 (Separated sum of bounded lattices). *Let $\mathcal{A}$ and $\mathcal{B}$ be two bounded lattices. The* separated sum *of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \oplus \mathcal{B}$, is the following partially ordered set $\mathcal{A} \oplus \mathcal{B} = \{(0, x) \mid x \in A\} \cup \{(1, y) \mid y \in B\}\} \cup \{\bot, \top\}$ with ordering $\vec{u} \leq \vec{v}$ if and only if (i) $\vec{u} = \bot$ or (ii) $\vec{v} = \top$ or (iii) $\vec{u} = (i, x), \vec{v} = (i, y)$ and $x \leq y$.*

Lemma 1. *The separated sum $\mathcal{A} \oplus \mathcal{B}$ of two bounded lattices $\mathcal{A}$ and $\mathcal{B}$ given in Definition 5 is a bounded lattice.*

Proof. Considering the partial order given in Definition 5, the following characterization is proposed for $\wedge_{A \oplus B}$ and $\vee_{A \oplus B}$:

$$\vec{x} \wedge_{A\oplus B} \vec{y} = \begin{cases} (0, x \wedge_A y), & \text{if } \vec{x} = (0, x) \text{ and } \vec{y} = (0, y) \\ (1, x \wedge_B y), & \text{if } \vec{x} = (1, x) \text{ and } \vec{y} = (1, y) \\ \vec{x}, & \text{if } \vec{y} = \top \\ \vec{y}, & \text{if } \vec{x} = \top \\ \bot, & \text{otherwise,} \end{cases}$$

$$\vec{x} \vee_{A\oplus B} \vec{y} = \begin{cases} (0, x \vee_A y), & \text{if } \vec{x} = (0, x) \text{ and } \vec{y} = (0, y) \\ (1, x \vee_B y), & \text{if } \vec{x} = (1, x) \text{ and } \vec{y} = (1, y) \\ \vec{x}, & \text{if } \vec{y} = \bot \\ \vec{y}, & \text{if } \vec{x} = \bot \\ \top, & \text{otherwise.} \end{cases}$$

Hence, as $\mathcal{A}$ and $\mathcal{B}$ are two bounded lattices, each pair of elements $\vec{x}, \vec{y} \in \mathcal{A} \oplus \mathcal{B}$ has infimum and supremum whenever $\vec{x} = (0, x)$ and $\vec{y} = (0, y)$ or $\vec{x} = (1, x)$ and $\vec{y} = (1, y)$. Moreover, for the other cases, there is the following $\vec{x} \wedge_{A\oplus B} \vec{y} = \bot$ and $\vec{x} \vee_{A\oplus B} \vec{y} = \top$. Therefore, $\mathcal{A} \oplus \mathcal{B} = \langle A \oplus B, \wedge_{A\oplus B}, \vee_{A\oplus B}, \bot, \top \rangle$ is a bounded lattice. □

Remark 1. The bounded lattice obtained by Lemma 1 can be pictured as in the Figure 1:

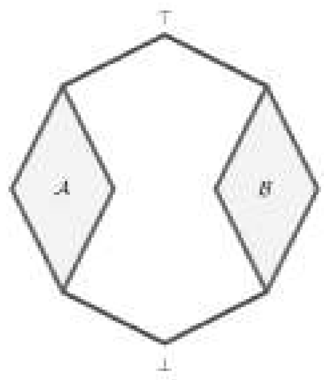

Fig. 1. Separated sum $\mathcal{A} \oplus \mathcal{B}$ of the bounded lattices $\mathcal{A}$ and $\mathcal{B}$.

It should also be recalled that the opposite (or dual) of a partially ordered set $\langle P, \leq \rangle$ is another partially ordered set denoted by $\langle P, \leq^{op} \rangle$, where the following equivalence holds $x \leq^{op} y$ if and only if $y \leq x$. In particular, based on the algebraic structure point of view, the opposite (or dual) of a lattice $\mathcal{L} = \langle L, \wedge, \vee \rangle$ is $\mathcal{L}^{op} = \langle L, \wedge_{op}, \vee_{op} \rangle$ and, for each $x, y \in L$, one has $x \wedge_{op} y = x \vee y$ and $x \vee_{op} y = x \wedge y$. Formally, $\mathcal{L}$ and $\mathcal{L}^{op}$ are dually isomorphic or anti-isomorphic, in the sense of Birkhoff in [2]. In this same algebraic point of view, if $\mathcal{L} = \langle L, \wedge, \vee \rangle$ is a lattice and $\perp \notin L$ is a distinguished element, then the lifting of $\mathcal{L}$ is a lattice denoted by $\mathcal{L}_\perp$ where for each $x, y \in L$, the characterizations of $\wedge_\perp$ and $\vee_\perp$ are given as:

$$x \wedge_\perp y = x \wedge y \tag{2}$$
$$x \vee_\perp y = x \vee y \tag{3}$$
$$x \vee_\perp \perp = x \tag{4}$$
$$x \wedge_\perp \perp = \perp. \tag{5}$$

Using the notion of opposite lattice, the next lemma provides a way to obtain bounded lattices from the separated sum and lifting given respectively in Definitions 2 and 3.

Lemma 2. *Let $\mathcal{A} + \mathcal{B} = \{(0, x) \,|\, x \in A\} \cup \{(1, y) \,|\, y \in B\} \cup \{\perp_{A+B}\}$ where $\mathcal{A}$ and $\mathcal{B}$ are two bounded lattices. If $\perp \notin (\mathcal{A}+\mathcal{B})^{op}$ is a distinguished element, then the* lifting *of $(\mathcal{A} + \mathcal{B})^{op}$, given by $(\mathcal{A} + \mathcal{B})^{op}_\perp = (\mathcal{A} + \mathcal{B})^{op} \cup \{\perp\}$ is a bounded lattice ordered by $\vec{x} \preceq \vec{y}$ if and only if $\vec{x} = \perp$ or $\vec{x} \leq_{(A+B)^{op}} \vec{y}$.*

Proof. Considering the partial orders given above, one has that $\vec{x} \leq_{(A+B)^{op}} \vec{y}$ if and only if $\vec{y} \leq^{op}_{A+B} \vec{x}$ if and only if $\vec{x} \wedge^{op}_{A+B} \vec{y} = \vec{y} = \vec{x} \vee_{(A+B)^{op}} \vec{y}$ and $\vec{x} \vee^{op}_{A+B} \vec{y} = \vec{x} = \vec{x} \wedge_{(A+B)^{op}} \vec{y}$. Therefore, the following characterization is proposed for $\wedge_{(A+B)^{op}_\perp}$ and $\vee_{(A+B)^{op}_\perp}$:

$$\vec{x} \wedge_{(A+B)^{op}_\perp} \vec{y} = \begin{cases} (0, x \vee_A y), & \text{if } \vec{x} = (0, x) \text{ and } \vec{y} = (0, y) \\ (1, x \vee_B y), & \text{if } \vec{x} = (1, x) \text{ and } \vec{y} = (1, y) \\ \vec{x}, & \text{if } \vec{y} = \perp \\ \vec{y}, & \text{if } \vec{x} = \perp \\ \top_{(A+B)^{op}}, & \text{otherwise,} \end{cases}$$

$$\vec{x} \vee_{(A+B)^{op}_\perp} \vec{y} = \begin{cases} (0, x \wedge_A y), & \text{if } \vec{x} = (0, x) \text{ and } \vec{y} = (0, y) \\ (1, x \wedge_B y), & \text{if } \vec{x} = (1, x) \text{ and } \vec{y} = (1, y) \\ \vec{x}, & \text{if } \vec{y} = \top_{(A+B)^{op}} \\ \vec{y}, & \text{if } \vec{x} = \top_{(A+B)^{op}} \\ \perp, & \text{otherwise,} \end{cases}$$

where $\top_{(A+B)^{op}} = \perp_{A+B}$. Hence, as $\mathcal{A}$ and $\mathcal{B}$ are two bounded lattices, each pair of elements $\vec{x}, \vec{y} \in \mathcal{A}$ or $\vec{x}, \vec{y} \in \mathcal{B}$ has infimum and supremum. Thus, each pair of elements $\vec{x}, \vec{y} \in (\mathcal{A} + \mathcal{B})^{op}_\perp$ has infimum and supremum whenever $\vec{x} = (0, x)$

and $\vec{y} = (0, y)$ or $\vec{x} = (1, x)$ and $\vec{y} = (1, y)$. Moreover, for the other cases, there is the following $\vec{x} \wedge_{(A+B)^{op}_{\perp}} \vec{y} = \perp$ and $\vec{x} \vee_{(A+B)^{op}_{\perp}} \vec{y} = \top_{(A+B)^{op}}$. Therefore, the algebra $(\mathcal{A} + \mathcal{B})^{op}_{\perp} = \langle (A + B)^{op}_{\perp}, \wedge_{(A+B)^{op}_{\perp}}, \vee_{(A+B)^{op}_{\perp}}, \perp, \top_{(A+B)^{op}} \rangle$ is a bounded lattice. □

Remark 2. The bounded lattice obtained by Lemma 2 can be pictured as in the Figure 2:

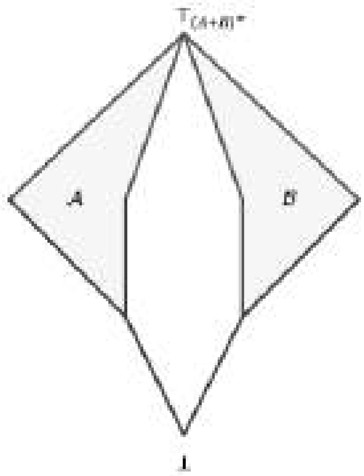

Fig. 2. Lifiting of $(\mathcal{A} + \mathcal{B})^{op}$, where $\mathcal{A}$ and $\mathcal{B}$ are bounded lattices.

Corollary 1. *The bounded lattices $\mathcal{A} \oplus \mathcal{B}$ and $(\mathcal{A} + \mathcal{B})^{op}_{\perp}$ are dually isomorphic.*

Proof. Clearly the map $\xi : \mathcal{A} \oplus \mathcal{B} \longrightarrow (\mathcal{A} + \mathcal{B})^{op}_{\perp}$ given by

$$\xi(\vec{x}) = \begin{cases} \top_{(A+B)^{op}}, \text{ if } \vec{x} = \perp_{A \oplus B} \\ \perp_{(A+B)^{op}_{\perp}}, \text{ if } \vec{x} = \top_{A \oplus B} \\ \vec{x}, \text{ otherwise,} \end{cases}$$

and such that $\xi(\vec{y}) < \xi(\vec{x})$ whenever $\vec{x} < \vec{y}$ it is a order reverse bijection. □

4 Extending finitary operations to inflationary BL-algebras

This section introduces external operations on pairs of inflationary BL-algebras and indicates a way to construct new inflationary BL-algebras from given ones.

First, with an approach similar to that given in [4, 11], given two inflationary BL-algebras $\mathcal{A}$ and $\mathcal{B}$, the external direct product $\mathcal{A} \times \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is the cartesian product $A \times B$, together with the operations defined componentwise – that is, $(a_1, b_1) \star_{A \times B} (a_2, b_2) = (a_1 \star_A a_2, b_1 \star_B b_2)$, where $\star_{\Phi} \in \{\wedge_{\Phi}, \vee_{\Phi}, *_{\Phi}, \rightarrow_{\Phi}\}$ and $\Phi \in \{A, B\}$.

Theorem 4. *If $\mathcal{A}$ and $\mathcal{B}$ are two inflationary BL-algebras, then the direct product $\mathcal{A} \times \mathcal{B}$ is also an inflationary BL-algebra.*

Proof. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ where A and B are underlying sets of inflationary BL-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Clearly $(a_1, b_1) \wedge_{A\times B} (a_2, b_2) = (a_1 \wedge_A a_2, b_1 \wedge_B b_2)$ is the greatest lower bound of (a_1, b_1) and (a_2, b_2). Analogously, $(a_1, b_1) \vee_{A\times B} (a_2, b_2) = (a_1 \vee_A a_2, b_1 \vee_B b_2)$ is the least upper bound of (a_1, b_1) and (a_2, b_2). Hence, the algebra $\langle A \times B, \wedge_{A\times B}, \vee_{A\times B}, \bot, \top\rangle$ is a bounded lattice with bottom element $\bot = (\bot_A, \bot_B)$ and top element $\top = (\top_A, \top_B)$. Thus, the property $(\mathcal{IBL}1)$ is satisfied. For the property $(\mathcal{IBL}2)$, one has that $*_A$ and $*_B$ are commutative, then

$$\begin{aligned}(a_1, b_1) *_{A\times B} (a_2, b_2) &= (a_1 *_A a_2, b_1 *_B b_2)\\ &= (a_2 *_A a_1, b_2 *_B b_1)\\ &= (a_2, b_2) *_{A\times B} (a_1, b_1).\end{aligned}$$

Hence, $\langle A \times B, *_{A\times B}\rangle$ is a commutative groupoid. Moreover, because these operators are inflationary, $(a_1, b_1) \leq_{A\times B} (a_1 *_A \top_A, b_1 *_B \top_B) = (a_1, b_1) *_{A\times B} (\top_A, \top_B)$. Therefore, $\leq_{A\times B}$ is inflationary. As for property $(\mathcal{IBL}3)$, for any $(x_A, x_B), (y_A, y_B), (z_A, z_B) \in A \times B$, since $\mathcal{A}$ and $\mathcal{B}$ are inflationary BL-algebras, by $(\mathcal{IBL}6)$ one has $(x_A \to_A y_A) \in \{t \in A \,|\, x_A *_A t \leq_A y_A\}$ and $(x_B \to_B y_B) \in \{w \in B \,|\, x_B *_B w \leq_B y_B\}$. Moreover, $(x_A, x_B) *_{A\times B} (z_A, z_B) \leq_{A\times B} (y_A, y_B)$ if and only if $x_A *_A z_A \leq_A y_A$ and $x_B *_B z_B \leq_B y_B$ if and only if $z_A \leq_A x_A \to_A y_A$ and $z_B \leq_B x_B \to_B y_B$ if and only if $(z_A, z_B) \leq_{A\times B} (x_A \to_A y_A, x_B \to_B y_B)$. But $(x_A \to_A y_A, x_B \to_B y_B) = (x_A, x_B) \to_{A\times B} (y_A, y_B)$ and so $(x_A, x_B) *_{A\times B} (z_A, z_B) \leq_{A\times B} (y_A, y_B)$ if and only if $(z_A, z_B) \leq_{A\times B} (x_A, x_B) \to_{A\times B} (y_A, y_B)$. For $(\mathcal{IBL}4)$, since $\mathcal{A}$ and $\mathcal{B}$ are inflationary BL-algebras, we have

$$\begin{aligned}(x_A, x_B) \wedge_{A\times B} (y_A, y_B) &= (x_A \wedge_A y_A, x_B \wedge_B y_B)\\ &= (x_A *_A (x_A \to_A y_A), x_B *_B (x_B \to_B y_B))\\ &= (x_A, x_B) *_{A\times B} ((x_A, x_B) \to_{A\times B} (y_A, y_B)).\end{aligned}$$

Finally, for property $(\mathcal{IBL}5)$, take $\vec{x} = (x_A, x_B)$, $\vec{y} = (y_A, y_B)$ and $\vec{\top} = (\top_A, \top_B)$. Then, as $\mathcal{A}$ and $\mathcal{B}$ are inflationary BL-algebras,

$$\begin{aligned}&\vec{\top} = (\top_A, \top_B)\\ &= \Big((x_A \to_A (y_A *_A \top_A) \vee_A (y_A \to_A (x_A *_A \top_A), (x_B \to_B (y_B *_B \top_B) \vee_B (y_B \to_B (x_B *_B \top_B)\Big)\\ &= \Big(\vec{x} \to_{A\times B} (\vec{y} *_{A\times B} \vec{\top})\Big) \vee_{A\times B} \Big(\vec{y} \to_{A\times B} (\vec{x} *_{A\times B} \vec{\top})\Big)\end{aligned}$$

Therefore, $\mathcal{A} \times \mathcal{B}$ is an inflationary BL-algebra. □

Given two inflationary BL-algebras, $\mathcal{A}$ and $\mathcal{B}$, there are two "coordinate projections" $\pi_1 : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A}$ and $\pi_2 : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{B}$ such that $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. Having made this observation, the following result can be presented.

Corollary 2. *Let $\mathcal{B}$ and $\mathcal{C}$ be two inflationary BL-algebras. Suppose in addition that $\mathcal{A}$ is also an inflationary BL-algebra and $f : \mathcal{A} \longrightarrow \mathcal{B}$ and $g : \mathcal{A} \longrightarrow \mathcal{C}$ are two mappings. Then there is a mapping $\xi : A \longrightarrow \mathcal{B} \times \mathcal{C}$ such that $f = \pi_1 \circ \xi$ and $g = \pi_2 \circ \xi$, making the diagram of Figure 3 commute.*

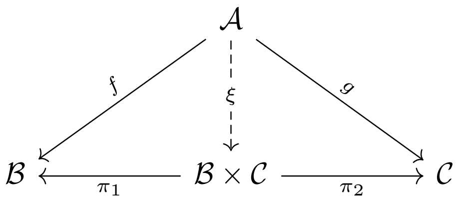

Fig. 3. Commutative diagram of the external direct product $\mathcal{B} \times \mathcal{C}$.

Proof. In fact, just define $\xi : A \longrightarrow \mathcal{B} \times \mathcal{C}$ by $\xi(a) = (f(a), g(a))$, for each $a \in \mathcal{A}$. Hence, $\pi_1(\xi(a)) = f(a)$ and $\pi_2(\xi(a)) = g(a)$ for each $a \in \mathcal{A}$. Therefore, there is a mapping $\xi : A \longrightarrow \mathcal{B} \times \mathcal{C}$ such that the diagram of Figure 3 commute. □

Corollary 3. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be inflationary BL-algebras and suppose in addition that $f : \mathcal{A} \longrightarrow \mathcal{C}$ and $g : \mathcal{B} \longrightarrow \mathcal{D}$ are two mappings. Then there is a mapping $\kappa : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C} \times \mathcal{D}$ such that $\pi_1 \circ \kappa = f \circ \widetilde{\pi}_1$ and $\pi_2 \circ \kappa = g \circ \widetilde{\pi}_2$, making the diagram of Figure 4 commute.*

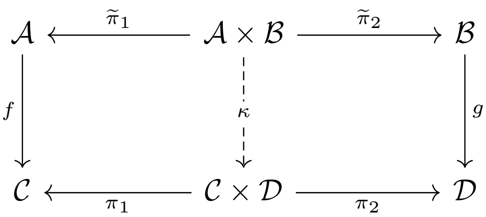

Fig. 4. Commutative diagram of the external direct products $\mathcal{A} \times \mathcal{B}$ and $\mathcal{B} \times \mathcal{D}$.

Proof. Let $\kappa : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C} \times \mathcal{D}$ be a mapping such that for each $a \in \mathcal{A}$ and for each $b \in \mathcal{B}$, one has $\kappa(a, b) = \Big(f(\widetilde{\pi}_1(a)), g(\widetilde{\pi}_2(b)) \Big)$. Therefore, there is a mapping $\kappa : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C} \times \mathcal{D}$ making the diagram of Figure 4 commute. □

In order to obtain arbitrary finite direct products we simply iterate the construction.

Corollary 4. *Let $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ be inflationary BL-algebras. Then the direct product of $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ is given inductively by*

$$\prod_{i<n} \mathcal{A}_i = \left(\prod_{i<n-1} \mathcal{A}_i \right) \times \mathcal{A}_{n-1}$$

where $\prod_{i<2} \mathcal{A}_i = \mathcal{A}_0 \times \mathcal{A}_1$.

Proof. Direct. □

Remark 3. Clearly $\prod_{i<n} \mathcal{A}_i$ is isomorphic to the set $\{(x_0, x_1, \ldots, x_{n-1}) \mid x_i \in \mathcal{A}_i\}$ with the coordinatewise ordering

$$(x_0, x_1, \ldots, x_{n-1}) \leq (y_0, y_1, \ldots, y_{n-1}) \iff (\forall i < n)(x_i \leq y_i).$$

Thus we identify $\prod_{i<n} \mathcal{A}_i$ with this inflationary BL-algebra. The appropriate projection mappings are

$$\pi_i : \prod_{j<n} \mathcal{A}_j \longrightarrow \mathcal{A}_i$$

defined by $\pi_i(x_0, x_1, \ldots, x_{n-1}) = x_i$. We also write $\mathcal{A}_0 \times \mathcal{A}_1 \times \ldots \times \mathcal{A}_{n-1}$ for $\prod_{i<n} \mathcal{A}_i$.

Similar to the external direct product, given two inflationary BL-algebras $\mathcal{A}$ and $\mathcal{B}$, in what follows $\mathcal{A} \oplus \mathcal{B}$, called external separated sum, denotes the separated sum of the bounded lattices $\mathcal{A}$ and $\mathcal{B}$, along with its finitary operations that characterize them as inflationary BL-algebras.

Theorem 5. *If $\mathcal{A}$ and $\mathcal{B}$ are two inflationary BL-algebras, then the external separated sum $\mathcal{A} \oplus \mathcal{B} = \langle A \oplus B, \wedge_{A\oplus B}, \vee_{A\oplus B}, \bot, \top \rangle$ is also an inflationary BL-algebra where*

$$\vec{x} *_{A\oplus B} \vec{y} = \begin{cases} (0, x *_A y), & \textit{if } \vec{x} = (0, x) \textit{ and } \vec{y} = (0, y) \\ (1, x *_B y), & \textit{if } \vec{x} = (1, x) \textit{ and } \vec{y} = (1, y) \\ \vec{x} \wedge_{A\oplus B} \vec{y}, & \textit{otherwise,} \end{cases}$$

$$\vec{x} \rightarrow_{A\oplus B} \vec{y} = \begin{cases} (0, x \rightarrow_A y), & \textit{if } \vec{x} = (0, x) \textit{ and } \vec{y} = (0, y) \\ (1, x \rightarrow_B y), & \textit{if } \vec{x} = (1, x) \textit{ and } \vec{y} = (1, y) \\ \bot, & \textit{if } \vec{y} = \bot \\ \top, & \textit{if } \vec{y} = \top \textit{ or } \vec{x} = \bot \\ \vec{y}, & \textit{otherwise.} \end{cases}$$

Proof. Property ($\mathcal{IBL}1$) follows directly from Lemma 1. For Property ($\mathcal{IBL}2$), since $\mathcal{A}$ and $\mathcal{B}$ are two inflationary BL-algebras, $*_{A\oplus B}$ is commutative whenever $\vec{x} = (0, x)$ and $\vec{y} = (0, y)$ or $\vec{x} = (1, x)$ and $\vec{y} = (1, y)$. Moreover, without loss of generality, suppose that $\vec{x} \in \{(0, u) \mid u \in A\}$ and $\vec{y} \in \{(1, v) \mid v \in B\}\}$, then $\vec{x} *_{A\oplus B} \vec{y} = \bot = \vec{y} *_{A\oplus B} \vec{x}$. In addition, if $\vec{x} = \top$ or $\vec{y} = \top$ then $\vec{x} *_{A\oplus B} \vec{y} = \vec{y} *_{A\oplus B} \vec{x} = \vec{x} \wedge_{A\oplus B} \vec{y}$. Also, if $\vec{x} \in A \oplus B$ then $\vec{x} \leq \vec{x} *_{A\oplus B} \top = \top *_{A\oplus B} \vec{x}$. As for Property ($\mathcal{IBL}3$), for any $\vec{x}, \vec{y}, \vec{z} \in A \oplus B$ such that $\vec{x} *_{A\oplus B} \vec{z} \leq \vec{y}$, i.e., $\vec{x} *_{A\oplus B} \vec{z} = \bot$ or $\vec{y} = \top$ or $\vec{x} *_{A\oplus B} \vec{z} = (i, u)$, $\vec{y} = (i, v)$ and $u \leq v$. Thus:

(i) Case $\vec{x} *_{A\oplus B} \vec{z} = \bot$, necessarily $\vec{x} = \bot$ or $\vec{z} = \bot$ or, without loss of generality, $\vec{x} = (0, x)$ and $\vec{z} = (1, z)$. Hence, if $\vec{x} = \bot$ then $\vec{x} \to_{A\oplus B} \vec{y} = \top$ and so $\vec{z} \leq \vec{x} \to_{A\oplus B} \vec{y}$. If $\vec{z} = \bot$ then $\vec{z} \leq \vec{x} \to_{A\oplus B} \vec{y}$. On the other hand, if $\vec{x} = (0, x)$ and $\vec{z} = (1, z)$ then, as $\mathcal{B}$ is inflationary BL-algebra, there is $\vec{y} = (1, y)$ such that $z \leq y$. Therefore, $\vec{z} \leq \vec{y} = \vec{x} \to_{A\oplus B} \vec{y}$.
(ii) Case $\vec{y} = \top$ then $\vec{x} \to_{A\oplus B} \vec{y} = \top$ and so, for any $\vec{z} \in \mathcal{A} \oplus \mathcal{B}$, $\vec{z} \leq \vec{x} \to_{A\oplus B} \vec{y}$.
(iii) Case $\vec{x} *_{A\oplus B} \vec{z} = (i, u)$, $\vec{y} = (i, v)$ and $u \leq v$ then $\vec{x}, \vec{y}, \vec{z} \in \{(0, u) \,|\, u \in A\}$ or $\vec{x}, \vec{y}, \vec{z} \in \{(1, v) \,|\, v \in B\}$. In both situations, $\vec{z} \leq \vec{x} \to_{A\oplus B} \vec{y}$.

On the other hand, assuming that $\vec{z} \leq \vec{x} \to_{A\oplus B} \vec{y}$, one has that $\vec{z} = \bot$ or $\vec{x} \to_{A\oplus B} \vec{y} = \top$ or $\vec{z} = (i, u)$, $\vec{x} \to_{A\oplus B} \vec{y} = (i, v)$ and $u \leq v$

(i) Case $\vec{z} = \bot$ then $\vec{x} *_{A\oplus B} \vec{z} = \bot$ and so $\vec{x} *_{A\oplus B} \vec{z} \leq \vec{y}$.
(ii) Case $\vec{x} \to_{A\oplus B} \vec{y} = \top$, necessarily $\vec{x} = \bot$ or $\vec{y} = \top$. Hence, if $\vec{x} = \bot$ then $\vec{x} *_{A\oplus B} \vec{z} = \bot$ and so $\vec{x} *_{A\oplus B} \vec{z} \leq \vec{y}$. However, if $\vec{y} = \top$ then, for any $\vec{z} \in \mathcal{A} \oplus \mathcal{B}$, $\vec{z} \leq \vec{x} \to_{A\oplus B} \vec{y}$.
(iii) Case $\vec{z} = (i, u)$, $\vec{x} \to_{A\oplus B} \vec{y} = (i, v)$ and $u \leq v$, then $\vec{x}, \vec{y}, \vec{z} \in \{(0, u) \,|\, u \in A\}$ or $\vec{x}, \vec{y}, \vec{z} \in \{(1, v) \,|\, v \in B\}$. In both situations, $\vec{x} *_{A\oplus B} \vec{z} \leq \vec{y}$.

Therefore, the pair $(*_{A\oplus B}, \to_{A\oplus B})$ is a Galois connection. For $(\mathcal{IBL}4)$, first suppose that $\vec{x}, \vec{y} \in \{(0, u) \,|\, u \in A\}$ or $\vec{x}, \vec{y} \in \{(1, v) \,|\, v \in B\}$. Since $\mathcal{A}$ and $\mathcal{B}$ are inflationary BL-algebras, one has that $(0, x \wedge_A y) = (x *_A (x \to_A y))$ or $(1, x \wedge_B y) = (x *_B (x \to_B y))$ and so $\vec{x} \wedge_{A\oplus B} \vec{y} = \vec{x} *_{A\oplus B} (\vec{x} \to_{A\oplus B} \vec{y})$ whenever $\vec{x}, \vec{y} \in \{(0, u) \,|\, u \in A\}$ or $\vec{x}, \vec{y} \in \{(1, v) \,|\, v \in B\}$. Secondly, if $\vec{x} = \top$ or $\vec{y} = \top$ then by the definition of the operators $\wedge_{A\oplus B}$, $*_{A\oplus B}$ and $\to_{A\oplus B}$, also one has that $\vec{x} \wedge_{A\oplus B} \vec{y} = \vec{x} *_{A\oplus B} (\vec{x} \to_{A\oplus B} \vec{y})$. Finally, since $\vec{x} \to_{A\oplus B} \vec{y} = \vec{y}$, $\vec{x} *_{A\oplus B} (\vec{x} \to_{A\oplus B} \vec{y}) = \vec{x} *_{A\oplus B} \vec{y} = \vec{x} \wedge_{A\oplus B} \vec{y}$ whenever $\vec{x} = (0, x)$ and $\vec{y} = (1, y)$. Then divisibility is satisfied. As for property $(\mathcal{IBL}5)$, obviously it is satisfied if $\vec{x} = (i, x)$ and $\vec{y} = (i, y)$ where $i \in \{0, 1\}$, since $\mathcal{A}$ and $\mathcal{B}$ are inflationary BL-algebras. It is also obvious that if $\vec{x} = \bot$, $\vec{y} = \bot$, $\vec{y} = \top$ or $\vec{x} = \top$, then

$$\Big(\vec{x} \to_{A\oplus B} (\vec{y} *_{A\oplus B} \top)\Big) \vee_{A\oplus B} \Big(\vec{y} \to_{A\oplus B} (\vec{x} *_{A\oplus B} \top)\Big) = \top.$$

Finally, if $\vec{x} = (i, x)$ and $\vec{y} = (j, y)$ with $i \neq j$, then $\Big(\vec{x} \to_{A\oplus B} (\vec{y} *_{A\oplus B} \top)\Big) = (\vec{x} \to_{A\oplus B} \vec{y}) = \vec{y}$ and $\Big(\vec{y} \to_{A\oplus B} (\vec{x} *_{A\oplus B} \top)\Big) = (\vec{y} \to_{A\oplus B} \vec{x}) = \vec{x}$. Therefore,

$$\Big(\vec{x} \to_{A\oplus B} (\vec{y} *_{A\oplus B} \top)\Big) \vee_{A\oplus B} \Big(\vec{y} \to_{A\oplus B} (\vec{x} *_{A\oplus B} \top)\Big) = \vec{y} \vee_{A\oplus B} \vec{x} = \top.$$

Thus, general prelinearity is satisfied. □

Corollary 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two inflationary BL-algebras. Suppose in addition that $\mathcal{C}$ is also an inflationary BL-algebra and $f : \mathcal{A} \longrightarrow \mathcal{C}$ and $g : \mathcal{B} \longrightarrow \mathcal{C}$ are two mappings. Then there is a mapping $[f, g]$ making the diagram of Figure 5 commute.*

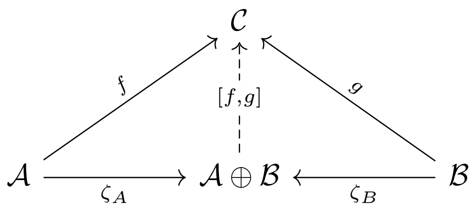

Fig. 5. Commutative diagram of the external separated sum $\mathcal{A} \oplus \mathcal{B}$.

Proof. In fact, since the mapping $\zeta_A : \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathcal{B}$ is given by $\zeta_A(x) = (0, x)$ and the mapping $\zeta_B : \mathcal{B} \longrightarrow \mathcal{A} \oplus \mathcal{B}$ is given by $\zeta_B(y) = (1, y)$, just define $[f, g] : \mathcal{A} \oplus \mathcal{B} \longrightarrow \mathcal{C}$ by

$$[f,g](\vec{w}) = \begin{cases} f(x), \text{ if } \vec{w} = (0,x) \\ g(y), \text{ if } \vec{w} = (1,y) \\ \bot_{\mathcal{C}}, \text{ if } \vec{w} = \bot_{A \oplus B} \\ \top_{\mathcal{C}}, \text{ if } \vec{w} = \top_{A \oplus B}. \end{cases}$$

□

Corollary 6. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be inflationary BL-algebras and $f : \mathcal{A} \longrightarrow \mathcal{C}$ and $g : \mathcal{B} \longrightarrow \mathcal{D}$ are two mappings. Then there is a mapping $\mu : \mathcal{A} \oplus \mathcal{B} \longrightarrow \mathcal{C} \oplus \mathcal{D}$ making the diagram of Figure 6 commute.*

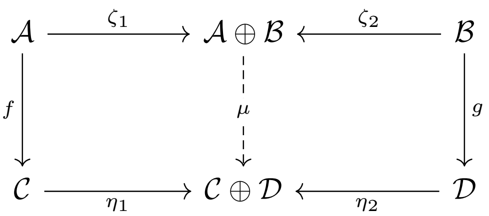

Fig. 6. Commutative diagram of the external separated sums $\mathcal{A} \oplus \mathcal{B}$ and $\mathcal{B} \oplus \mathcal{D}$.

Proof. Since the mapping $\zeta_1 : \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathcal{B}$ is given by $\zeta_1(x) = (0, x)$ and the mapping $\zeta_2 : \mathcal{B} \longrightarrow \mathcal{A} \oplus \mathcal{B}$ is given by $\zeta_2(y) = (1, y)$ and, in the same way, the mapping $\eta_1 : \mathcal{C} \longrightarrow \mathcal{C} \oplus \mathcal{D}$ is given by $\eta_1(f(x)) = (0, f(x))$ and the mapping $\eta_2 : \mathcal{D} \longrightarrow \mathcal{C} \oplus \mathcal{D}$ is given by $\eta_2(f(y)) = (1, f(y))$, just define $\mu : \mathcal{A} \oplus \mathcal{B} \longrightarrow \mathcal{C} \oplus \mathcal{D}$ by

$$\mu(\vec{w}) = \begin{cases} \eta_1(f(x)), \text{ if } \vec{w} = (0,x) \\ \eta_2(g(y)), \text{ if } \vec{w} = (1,y) \\ \bot_{\mathcal{C} \oplus \mathcal{D}}, \text{ if } \vec{w} = \bot_{A \oplus B} \\ \top_{\mathcal{C} \oplus \mathcal{D}}, \text{ if } \vec{w} = \top_{A \oplus B}. \end{cases}$$

Hence, the result follows. Therefore, there is a mapping $\mu : \mathcal{A} \oplus \mathcal{B} \longrightarrow \mathcal{C} \oplus \mathcal{D}$ making the diagram of Figure 6 commute. □

Theorem 6. *Let $\mathcal{A}+\mathcal{B}=\{(0,x)\,|\,x\in A\}\cup\{(1,y)\,|\,y\in B\}\cup\{\bot_{A+B}\}$ where $\mathcal{A}$ and $\mathcal{B}$ are two inflationary BL-algebras. If $\bot\notin(\mathcal{A}+\mathcal{B})^{op}$ is a distinguished element, then the* lifting *of $(\mathcal{A}+\mathcal{B})^{op}$, denoted by $(\mathcal{A}+\mathcal{B})^{op}_{\bot}$, is also an inflationary BL-algebra where the following operators are considered:*

$$\vec{x} *_{(A+B)^{op}_{\bot}} \vec{y} = \begin{cases} (0, x *_A y), & \text{if } \vec{x}=(0,x) \text{ and } \vec{y}=(0,y) \\ (1, x *_B y), & \text{if } \vec{x}=(1,x) \text{ and } \vec{y}=(1,y) \\ \vec{x} \vee_{(A+B)^{op}_{\bot}} \vec{y}, & \text{otherwise,} \end{cases}$$

$$\vec{x} \rightarrow_{(A+B)^{op}_{\bot}} \vec{y} = \begin{cases} (0, x \rightarrow_A y), & \text{if } \vec{x}=(0,x) \text{ and } \vec{y}=(0,y) \\ (1, x \rightarrow_B y), & \text{if } \vec{x}=(1,x) \text{ and } \vec{y}=(1,y) \\ \bot, & \text{if } \vec{y}=\bot \\ \top_{(A+B)^{op}}, & \text{if } \vec{y}=\top_{(A+B)^{op}} \text{ or } \vec{x}=\bot \\ \vec{y}, & \text{otherwise.} \end{cases}$$

Proof. Considering the Corollary 1 and the way as the operators $*_{(A+B)^{op}_{\bot}}$ and $\rightarrow_{(A+B)^{op}_{\bot}}$ were defined, here the proof follows in a similar way to Theorem 5. □

Corollary 7. *The mapping $\vartheta:\mathcal{A}\oplus\mathcal{B}\longrightarrow(\mathcal{A}+\mathcal{B})^{op}_{\bot}$ given by*

$$\vartheta(\vec{x}) = \begin{cases} \top_{(A+B)^{op}}, \text{ if } \vec{x}=\bot_{A\oplus B} \\ \bot_{(A+B)^{op}_{\bot}}, \text{ if } \vec{x}=\top_{A\oplus B} \\ \vec{x}, \text{ otherwise,} \end{cases}$$

where $\vartheta(\vec{y})<\vartheta(\vec{x})$ whenever $\vec{x}<\vec{y}$, is a dual isomorphism between the inflationary BL-algebras $\mathcal{A}\oplus\mathcal{B}$ and $(\mathcal{A}+\mathcal{B})^{op}_{\bot}$.

Proof. It follows directly from Corollary 1 and Theorems 5 and 6. □

Theorem 7. *If $\mathcal{A}=\langle A,\wedge,\vee,*,\rightarrow,\bot_A,\top_A\rangle$ is an inflationary BL-algebra and $\bot\notin A$ is a distinguished element, then the lifting of $\mathcal{A}$, denoted by $\mathcal{A}_{\bot}=\langle A\cup\{\bot\},\wedge_{\bot},\vee_{\bot},*_{\bot},\rightarrow_{\bot},\bot,\top_A\rangle$ is also an inflationary BL-algebra, where*

$$x *_{\bot} y = \begin{cases} x * y & \text{if } x,y\in A \\ \bot & \text{if } x=\bot \text{ or } y=\bot \end{cases}$$

$$x \rightarrow_{\bot} y = \begin{cases} x \rightarrow y & \text{if } x,y\in A \\ \bot & \text{if } x\in A \text{ and } y=\bot \\ \top_A & \text{otherwise} \end{cases}$$

Proof. Obviously, by the Equations (2)–(5), the property ($\mathcal{IBL}1$) is satisfied, since $\langle A\cup\{\bot\},\wedge_{\bot},\vee_{\bot},\bot,\top_A\rangle$ is a bounded lattice. Since $\mathcal{A}$ is an inflationary BL-algebra and $x*_{\bot}\bot=\bot*_{\bot}x=\bot$ for all $x\in A$, it follows that $\langle A_{\bot},*_{\bot}\rangle$ is a commutative groupoid. Therefore, the property ($\mathcal{IBL}2$) is satisfied. As for Property ($\mathcal{IBL}3$), for any $x,y,z\in A_{\bot}$ such that $x*_{\bot}z\leq y$, i.e., $x*_{\bot}z=\bot$ or $x*_{\bot}z=x*z\leq_A y$. Thus:

(i) Case $x *_\perp z = \perp$, necessarily $x = \perp$ or $z = \perp$. Hence, if $x = \perp$ then $x \to_\perp y = \top_A$ and so $z \leq x \to_\perp y$. If $z = \perp$ then $z \leq x \to_\perp y$.
(ii) Case $x *_\perp z = x * z \leq_A y$, the result follows from the fact that $\mathcal{A}$ is an inflationary BL-algebra.

On the other hand, assuming that $z \leq x \to_\perp y$, one has that $z = \perp$ or $z \leq_A x \to_\perp y = x \to y$.

(i) Case $z = \perp$ then $x *_\perp z = \perp$ and so $x *_\perp z \leq y$.
(ii) Case $z \leq_A x \to_\perp y = x \to y$, the result follows from the fact that $\mathcal{A}$ is an inflationary BL-algebra.

Therefore, the pair $(*_\perp, \to_\perp)$ is a Galois connection. For $(\mathcal{IBL}4)$, first suppose that $x, y \in A$. Then, since $\mathcal{A}$ is an inflationary BL-algebra, one has that $x \wedge y = x * (x \to y)$ and so $x \wedge_\perp y = x *_\perp (x \to_\perp y)$ whenever $x, y \in A$. Secondly, if $x = \perp$ or $y = \perp$ then by the definition of the operators $\wedge_\perp$, $*_\perp$ and $\to_\perp$, also one has that $x \wedge_\perp y = x *_\perp (x \to_\perp y)$. Then divisibility is satisfied. As for property $(\mathcal{IBL}5)$, obviously it is satisfied if $x, y \in A$, since $\mathcal{A}$ is an inflationary BL-algebra. It is also obvious that if $x = \perp$ or $y = \perp$, then

$$\Big(x \to_\perp (y *_\perp \top_A)\Big) \vee_\perp \Big(y \to_\perp (x *_\perp \top_A)\Big) = \top_A.$$

Therefore, general prelinearity is satisfied. □

Remark 4. From Theorem 7, $\mathcal{A}_\perp$ is an inflationary BL-algebra whenever $\mathcal{A}$ is. Its picture is as in the Figure 7:

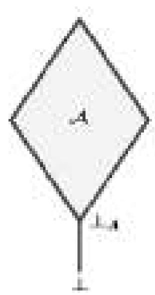

Fig. 7. Lifiting of $\mathcal{A}$, where $\mathcal{A}$ is an inflationary BL-algebra.

Corollary 8. *Let $\iota : \mathcal{A} \hookrightarrow \mathcal{A}_\perp$ be the inclusion mapping. If $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a mapping between the inflationary BL-algebras $\mathcal{A}$ and $\mathcal{B}$, then Then there is a function $h : \mathcal{A}_\perp \longrightarrow \mathcal{B}$ such that the diagram of Figure 8 commutes.*

Proof. In fact, just define the function h by:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ \perp_B & \text{if } x = \perp \end{cases}$$

□

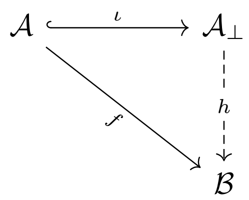

Fig. 8. Lifting of h to $\mathcal{B}$.

5 Final remarks

In the present paper, we have introduced external operations on pairs of inflationary BL-algebras and we have indicated a way to build new inflationary BL-algebras from given ones. Clearly, the dual isomorphism between $\mathcal{A} \oplus \mathcal{B}$ and $(\mathcal{A}+\mathcal{B})_\perp^{op}$ translates, from a categorical point of view, as a functor. This functor assigns to each space its dual space, and the pullback construction assigns to each morphism $\vartheta : \mathcal{A} \oplus \mathcal{B} \longrightarrow (\mathcal{A}+\mathcal{B})_\perp^{op}$ its dual $\vartheta^* : (\mathcal{A}+\mathcal{B})_\perp^{op*} \longrightarrow \mathcal{A} \oplus \mathcal{B}^*$. Therefore, one possibility of future work is to study the structures $\mathcal{A} \oplus \mathcal{B}$ and $(\mathcal{A}+\mathcal{B})_\perp^{op}$ from a categorical point of view, as discussed in [18]. Another research direction worth is to investigate the intervalization process of inflationary BL-algebras, as discussed in [14] or of generalizations thereof. There is still a third perspective of investigation, namely, the tensor product $A \otimes B$ for arbitrary semilattices A and B, as defined in [8]. As is known, the tensor product is used to model quantum logic. Therefore, it is interesting to investigate whether the tensor product preserves inflationary BL-algebras, as well as to investigate related properties.

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Interactive Riemann Integral in the Space of the Linearly Correlated Fuzzy Numbers $\mathbb{R}_{F(A)}$

Francielle Santo Pedro[1[0000−0001−5177−563X]], Beatriz Laiate [⋆2[0000−0002−7312−4843]], Estevão Esmi [⋆⋆2[0000−0001−6719−678X]], and Laécio Carvalho de Barros [⋆⋆⋆2[0000−0003−4322−5974]]

[1] Universidade Federal de São Paulo, Osasco SP 06110-295, Brazil
fsimoes@unifesp.br
[2] Universidade Estadual de Campinas, Campinas SP 13083-970,Brazil
beatrizlaiate@gmail.com,{eelaureano,laeciocb}@ime.unicamp.br

Abstract. In this work we study the Riemann integral of A-linearly correlated fuzzy functions in Banach spaces of fuzzy numbers. We compare the interactive integral with the non-interactive integral in Riemann sense, and present the product rule and the method of integration by parts. Finally, trapezoidal and Simpson rules were developed for approximate calculations of interactive Riemann integral of A-linearly correlated functions.

Keywords: A-linearly correlated fuzzy functions · Interactive Riemann integral · Numerical integration · Product rule· Method of integration by parts

1 Introduction

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [8]. A Riemann integral type approach was later presented by Goetschel and Voxman [11] and later by Matloka [15]. On the other hand, Puri and Ralescu [17], and Kaleva [12] choose to use the Lebesgue type concept of integration to define the integral of fuzzy-number value functions.

In this paper we will focus on the interactive integral called Ψ-Riemann integral. This integral was first presented in [18] and, together with the notion of Fréchet derivative, a theory of calculus for a class of fuzzy numbers was established in [9,18]. This theory allows the construction of a vector space of linearly correlated fuzzy numbers, which is composed by fuzzy numbers that are interactive with each other, so that the vector addition corresponds to an interactive operation. The fuzzy functions in this space are called A-linearly correlated fuzzy functions. This name is due a particular type of fuzzy interactive process presented in [3].

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2 Mathematical background

The membership function of the fuzzy number A is denoted by $A(\cdot)$ and $A(x)$ is the degree to which x belongs to A. The α-level sets of the fuzzy number A are non-empty, bounded, closed intervals and they are denoted by $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$, $\forall\alpha \in [0,1]$, where a_α^-, a_α^+ are respectively the left and right endpoints. A triangular fuzzy number is denoted by a triple $(a;b;c)$, where $a \leq b \leq c$, and its α-levels are given by $[a+\alpha(b-a), c-\alpha(c-b)]$, $\forall\alpha \in [0,1]$ [2].

We denote the set of all fuzzy numbers by $\mathbb{R}_\mathsf{F}$, and we say that a fuzzy number $A \in \mathbb{R}_\mathsf{F}$ is symmetric with respect to $x \in \mathbb{R}$ if $A(x-y) = A(x+y)$, $\forall y \in \mathbb{R}$. We say that A is non-symmetric if there exists no x such that A is symmetric. If $A \in \mathbb{R}_\mathcal{F}$ is a fuzzy numbers, the length of $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$ is defined as $len\left([A]_\alpha\right) = a_\alpha^+ - a_\alpha^-$, $\forall\alpha \in [0,1]$. The diameter of A is defined as the length of $[A]_0$, that is, $diam\,(A) = a_0^+ - a_0^-$. The diameter of a fuzzy number can be seen as a measure of uncertainty of the quantity modeled by it.

The Hausdorff distance between two fuzzy numbers is given by

$$d_\infty(B,C) = \sup_{\alpha\in[0,1]} \max\left\{\left|b_\alpha^- - c_\alpha^-\right|, \left|b_\alpha^+ - c_\alpha^+\right|\right\}, \tag{1}$$

where $B, C \in \mathbb{R}_\mathsf{F}$ are given levelwise by $[B]_\alpha = [b_\alpha^-, b_\alpha^+]$ and $[C]_\alpha = [c_\alpha^-, c_\alpha^+]$, $\forall\alpha \in [0,1]$. The usual norm in the class of fuzzy numbers is defined as $\|B\|_\mathcal{F} = d_\infty(B,0)$, for all $B \in \mathbb{R}_\mathsf{F}$, where $0 \in \mathbb{R}_\mathsf{F}$ is given as a singleton.

For each $A \in \mathbb{R}_\mathsf{F}$, the operator $\Psi_A : \mathbb{R}^2 \longrightarrow \mathbb{R}_\mathsf{F}$ associates each vector (q,r) to the fuzzy number $\Psi_A(q,r)$, whose α-levels are given by

$$[\Psi_A(q,r)]_\alpha = \{qx + r \,|\, x \in [A]_\alpha\} = q[A]_\alpha + r, \text{ for all } \alpha \in [0,1]. \tag{2}$$

The range of Ψ_A is denoted by the symbol $\mathbb{R}_{\mathsf{F}(A)} = \{\Psi_A(q,r) \,|\, (q,r) \in \mathbb{R}^2\}$. If a fuzzy number $B \in \mathbb{R}_{\mathsf{F}(A)}$ for some $A \in \mathbb{R}_\mathsf{F}$, then we call B an $A-$*linearly correlated fuzzy number*. It is worth noticing that $\mathbb{R} \subset \mathbb{R}_{\mathsf{F}(A)}$, since $r = \Psi_A(0,r)$, $\forall r \in \mathbb{R}$, that is, the set of real numbers is contained in the set of $A-$linearly correlated fuzzy numbers, for all $A \in \mathbb{R}_\mathcal{F}$.

The operator Ψ_A is injective if, and only if, A is non-symmetric [9]. In this case, the set $\left(\mathbb{R}_{\mathsf{F}(A)}, +_{\Psi_A}, \cdot_{\Psi_A}\right)$ is a 2-dimensional vector space over $\mathbb{R}$, where the addition $(+_{\Psi_A})$ and the scalar product $(\cdot_{\Psi_A})$ are defined by

$$i)\ B +_{\Psi_A} C = \Psi_A\left(\Psi_A^{-1}(B) + \Psi_A^{-1}(C)\right) \text{ and} \tag{3}$$

$$ii)\ \gamma \cdot_{\Psi_A} B = \Psi_A\left(\gamma \Psi_A^{-1}(B)\right), \tag{4}$$

for all $B, C \in \mathbb{R}_{\mathsf{F}(A)}$ and $\gamma \in \mathbb{R}$. Similarly, the difference $(-_{\Psi_A})$ is given by $B -_{\Psi_A} C = B +_{\Psi_A} (-1) \cdot_{\Psi_A} C$ for all $B, C \in \Psi_A$. It is worth noting that, for $q, r \in \mathbb{R}$, we have $q \cdot_{\Psi_A} A +_{\Psi_A} r = qA + r$, where $+$ and $\cdot$ denote, respectively, the standard sum and the scalar product of fuzzy numbers. Moreover, $B +_{\Psi_A} C = (q_B + q_C)A + (r_B + r_C)$, where $B = q_B A + r_B$ and $C = q_C A + r_C$. The norm in $\mathbb{R}_{\mathsf{F}(A)}$ is given by $\|B\|_{\Psi_A} = \|\Psi_A^{-1}(B)\|_\infty$, and it induces the metric $d_{\Psi_A}(B,C) = \|B -_{\Psi_A} C\|_{\Psi_A}$ for all $B, C \in \mathbb{R}_{\mathsf{F}(A)}$. The reasoning above leads us to conclude that $\left(\mathbb{R}_{\mathsf{F}(A)}, +_{\Psi_A}, \cdot_{\Psi_A}, \|\cdot\|_{\Psi_A}\right)$ is a two-dimensional normed space and, therefore, it is a Banach space.

The induced norm $\|\cdot\|_{\Psi_A}$ and the usual norm $\|\cdot\|_{\mathcal{F}}$ can be related as stated in the following:

Lemma 1. *Let $A \in \mathbb{R}_{\mathcal{F}}$ be non-symmetric. Then, there exists $c = c(A) > 1$ such that $\|B\|_{\mathcal{F}} \leq c\,\|B\|_{\Psi_A}$ holds for all $B \in \mathbb{R}_{\mathsf{F}(A)}$.*

Proof. Let $B \in \mathbb{R}_{\mathsf{F}(A)}$ be given by $B = q_B A + r_B$. Then,

$$\begin{aligned}\|B\|_{\mathcal{F}} = \|q_B A + r_B\|_{\mathcal{F}} &\leq \|q_B A\|_{\mathcal{F}} + \|r_B\|_{\mathcal{F}} \\ &= |q_B|\,\|A\|_{\mathcal{F}} + |r_B| \\ &\leq \max\{q_B, r_B\}\,(\|A\|_{\mathcal{F}} + 1) \\ &= (\|A\|_{\mathcal{F}} + 1)\,\|B\|_{\Psi_A},\end{aligned}$$

that is, taking $c = \|A\|_{\mathcal{F}} + 1$, the proof is enclosed.

Lemma 1 ensures that the usual norm $\|\cdot\|_{\mathcal{F}}$ is superiorly limited by the induced norm $\|\cdot\|_{\Psi_A}$ in $\mathbb{R}_{\mathsf{F}(A)}$. This result will be particularly useful to guarantee the convergence of the numerical methods presented in Section 5 w.r.t. the usual norm $\|\cdot\|_{\mathcal{F}}$.

Example 1. Let $A = \left(-2; -\frac{1}{2}; -\frac{1}{4}\right) \in \mathbb{R}_{\mathcal{F}}$ and $B = 1A + 2 \in \mathbb{R}_{\mathsf{F}(A)}$. Thus,

$$\|B\|_{\Psi_A} = \|1A + 2\|_{\Psi_A} = \max\{1, 2\} = 2 = \|A\|_{\mathcal{F}}$$

and taking $c = 3$, we have

$$\|B\|_{\mathcal{F}} = \left\|\left(0; \frac{3}{2}; \frac{7}{4}\right)\right\|_{\mathcal{F}} = \frac{7}{4} \leq 3 \cdot 2 = 6.$$

It is a well-known fact that the usual arithmetic operations in the class of fuzzy numbers is based on the Zadeh's Extension principle. These operations are also called *non-interactive* [6]. On the other side, the arithmetic operations defined in the space $\mathbb{R}_{\mathsf{F}(A)}$ are said to be *interactive*, once they are associated with a special type of joint possibility distribution defined in the aforementioned space (see [4] for more details). The next example depicts an initial comparison between these two arithmetic operations.

Example 2 (Interactive arithmetic $\times$ non-interactive arithmetic). Consider $B = -4A$ and $C = 2A$, where $A = (0; 1; 3) \in \mathbb{R}_{\mathsf{F}}$.

- The non-interactive sum $B + C$ is given by

  $$[B + C]_\alpha = [-12 + 8\alpha, -4\alpha] + [2\alpha, 6 - 4\alpha] = [-12 + 10\alpha, 6 - 8\alpha],$$

  for all $\alpha \in [0, 1]$. Thus $B + C = (-12; -2; 6)$.
- The interactive sum $B +_{\Psi_A} C$ is given by

  $$B +_{\Psi_A} C = (-4 + 2)A = -2A = (-6; -2; 0).$$

In this case, note that

$$diam(B +_{\Psi_A} C) < diam(B) \text{ and } diam(B +_{\Psi_A} C) = diam(C),$$

which does not occur in the non-interactive sum.

In this work, we study fuzzy number-valued functions of the form $f : [a,b] \longrightarrow \mathbb{R}_{\mathsf{F}(A)} \subset \mathbb{R}_{\mathsf{F}}$, where $A \in \mathbb{R}_{\mathsf{F}}$ is non-symmetric. Note that, by definition of $\mathbb{R}_{\mathsf{F}(A)}$, there exist functions $q, r : [a,b] \to \mathbb{R}$ such that $f(t) = q(t)A + r(t)$, $\forall t \in [a,b]$. Thus, we have that $f = \Psi_A \circ p$ where $p(t) = (q(t), r(t))$ for all $t \in [a,b]$. Functions $f : [a,b] \to \mathbb{R}_{\mathsf{F}(A)}$ are also called A-linearly correlated fuzzy processes.

For each fuzzy function $f : [a,b] \to \mathbb{R}_{\mathsf{F}(A)}$, we have a family of multivalued functions $f_\alpha : [a,b] \to \mathcal{K}$, defined by $f_\alpha(t) = [f(t)]_\alpha$, where $\mathcal{K}$ indicates the family of closed subintervals of $\mathbb{R}$. Levelwise, we can write

$$[f(t)]_\alpha = [f_\alpha^-(t), f_\alpha^+(t)] = q(t)[a_\alpha^-, a_\alpha^+] + r(t),\ \forall \alpha \in [0,1].$$

Lemma 2. *[4,9] Let $A \in \mathbb{R}_{\mathsf{F}(A)}$ be non-symmetric and $f : [a,b] \to \mathbb{R}_{\mathsf{F}(A)}$. There exists a unique function $(q,r) = p : \mathbb{R} \to \mathbb{R}^2$ such that $f = \Psi_A \circ p$.*

Proposition 1. *[4,18] Let $A \in \mathbb{R}_{\mathsf{F}}$ be non-symmetric and $f = \Psi_A \circ p : [a,b] \to \mathbb{R}_{\mathsf{F}(A)}$.*

a. The function f is bounded if, and only if, $p : [a,b] \longrightarrow \mathbb{R}^2$ is bounded.
b. The function f is continuous if, and only if, $p : [a,b] \to \mathbb{R}^2$ is continuous.

Let $A \in \mathbb{R}_{\mathsf{F}}$ be non-symmetric and $f, g : [a,b] \to \mathbb{R}_{\mathsf{F}(A)}$ be such that $f(t) = \Psi_A(q_f(t), r_f(t)) = q_f(t)A + r_f(t)$ and $g(t) = \Psi_A(q_g(t), r_g(t)) = q_g(t)A + r_g(t)$. By Equations (3) and (4), we have that the addition and subtraction between f and g, and the scalar multiplication are defined, respectively, by

$$(f \pm_{\Psi_A} g)(t) = f(t) \pm_{\Psi_A} g(t) = (q_f(t) + q_g(t))A \pm (r_f(t) + r_g(t)),$$
$$(\lambda \cdot_{\Psi_A} f)(t) = \lambda \cdot_{\Psi_A} f(t) = \lambda f(t) = \lambda q_f(t)A + \lambda r_f(t)$$

for all $t \in [a,b]$ and $\lambda \in \mathbb{R}$.

The next example shows that the interactive operations between $A-$linearly correlated fuzzy processes does not coincide with the standard operations between these processes in general.

Example 3. Consider $A = \left(-\frac{1}{2}; 0; 1\right) \in \mathbb{R}_{\mathsf{F}}$ and $f, g : [-3,3] \to \mathbb{R}_{\mathsf{F}(A)}$ be given by $f(t) = \Psi_A(t, e^t) = tA + e^t$ and $g(t) = \Psi_A(t^2, 1) = t^2 A + 1$, for all $t \in [-3,3]$. Since A is non-symmetric, we have that

$$\begin{aligned} (f +_{\Psi_A} 2 \cdot g)(t) &= \Psi_A\left(q_f(t) + 2q_g(t), r_f(t) + 2r_g(t)\right) \\ &= \Psi_A\left(t + 2t^2, e^t + 2\right) = \left(t + 2t^2\right)A + \left(e^t + 2\right). \end{aligned} \tag{5}$$

Since $t + 2t^2 \geq 0$ if, and only if, $t \in \left[-3, -\frac{1}{2}\right] \cup [0,3]$, Equation (5) can be rewritten as

$$(f +_{\Psi_A} 2 \cdot g)(t) = \begin{cases} \left(-\dfrac{t}{2} - t^2; 0; t + 2t^2\right) + \left(e^t + 2\right), \text{ if } t \in \left[-3, -\dfrac{1}{2}\right] \cup [0,3] \\ \left(t + 2t^2; 0; -\dfrac{t}{2} - t^2\right) + \left(e^t + 2\right), \text{ if } t \in \left(-\dfrac{1}{2}, 0\right). \end{cases}$$

On the other hand, under the standart arithmetic operations, we have

$$(f + 2 \cdot g)(t) = f(t) + 2 \cdot g(t) = \left(tA + e^t\right) + \left(2t^2 A + 2\right)$$

where "+" and "·" denote the usual sum and scalar multiplication in $\mathbb{R}_{\mathsf{F}}$. Since $(\mathbb{R}_{\mathsf{F}}, +, \cdot)$ is semilinear, we have that $tA + 2t^2 A = \left(t + 2t^2\right) A$ if, and only if, $t \geq 0$. Therefore,

$$(f + 2 \cdot g)(t) = \begin{cases} \left(t - t^2; 0; -\dfrac{t}{2} + 2t^2\right) + \left(e^t + 2\right), \text{ if } t \in [-3, 0) \\ \left(-\dfrac{t}{2} - t^2; 0; t + 2t^2\right) + \left(e^t + 2\right), \text{ if } t \in [0, 3], \end{cases}$$

that is, $(f +_{\Psi_A} 2 \cdot g)(t) = (f + 2 \cdot g)(t)$ if, and only if, $t \in [0, 3]$.

2.1 Integrals in $\mathbb{R}_{\mathsf{F}}$

In this subsection we will point out differences between non-interactive and the interactive Riemann integral.

We will begin with the notion of fuzzy Aumann integral introduced by Puri and Ralescu [17]. Fundamental theorems of calculus considering this notion of integrability and the H-differentiability was presented in [12].

Let $f : [a, b] \to \mathbb{R}_{\mathsf{F}}$ be a fuzzy function. The concept of *Aumann integral* is given in terms of set-valued functions. In fact, the Aumann integral of $[f(t)]_\alpha$ is defined by

$$(A) \int_a^b [f(t)]_\alpha dt = \left\{ \int_a^b g(t)dt \,\middle|\, g : [a, b] \to \mathbb{R} \text{ is a measurable selection for } [f(t)]_\alpha \right\}$$

for all $\alpha \in [0, 1]$ [12]. The notion of *fuzzy Aumann integral* extends this notion to fuzzy-number valued functions. According to [12,17], the fuzzy Aumann integral of f over $[a, b]$, denoted by $(FA) \int_a^b f(t)dt$, is defined by α-levels:

$$\left[(FA) \int_a^b f(t)dt\right]_\alpha = (A) \int_a^b [f(t)]_\alpha \, dt = (A) \int_a^b \left[f_\alpha^-(t), f_\alpha^+(t)\right] dt, \tag{6}$$

assuming that the multifunction family $[f(t)]_\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$ is Aumann integrable and satisfies the Stacking Theorem [16]. For a complete study of the existence of the Aumann integral see [12,17].

By Remark 4.2 of [12], if the fuzzy number-valued function f is fuzzy Aumann integrable over $[a, b]$, then the integrals of the endpoint functions f_α^- and f_α^+ over $[a, b]$ exist and the following equality holds:

$$\left[(FA) \int_a^b f(t)dt\right]_\alpha = \left[\int_a^b f_\alpha^-(t)dt, \int_a^b f_\alpha^+(t)dt\right], \tag{7}$$

for all $\alpha \in [0, 1]$.

According to [11,15], a function $f : [a,b] \longrightarrow \mathbb{R}_\mathsf{F}$ is said to be fuzzy Riemann integrable if there exists $S \in \mathbb{R}_\mathsf{F}$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every partition $a = t_0 < t_1 < \ldots < t_n = b$ with $t_i - t_{i-1} < \delta$, $i = 1, \ldots, n$, we have

$$d_\infty \left(\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),\ S \right) < \epsilon, \tag{8}$$

where $\xi_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$, the summation is given in terms of the standard addition of fuzzy numbers, and d_∞ denotes the Hausdorff distance in $\mathbb{R}_\mathsf{F}$ given by Equation (1). In this case, S is said to be the fuzzy Riemann integral of f over $[a,b]$ and it is denoted by $(FR)\int_a^b f(t)dt$.

Proposition 2. *[5] If $f : [a,b] \longrightarrow \mathbb{R}_\mathsf{F}$ is a continuous function w.r.t. the metric d_∞, then the fuzzy Aumann and Riemann integrals of f exist and are equal, that is,*

$$(FA)\int_a^b f(s)ds = (FR)\int_a^b f(s)ds.$$

Example 4. Let $A \in \mathbb{R}_\mathsf{F}$ be non-symmetric and $f(t) = t^3 A$, for all $t \in [-1,1]$. We have that, for all $\alpha \in [0,1]$

$$\begin{aligned}\left[(FA)\int_{-1}^{1} f(t)dt\right]_\alpha &= \left[\int_{-1}^{1} f_\alpha^-(t)dt, \int_{-1}^{1} f_\alpha^+(t)dt\right] \\ &= \left[\int_{-1}^{0} t^3 a_\alpha^+ dt + \int_{0}^{1} t^3 a_\alpha^- dt, \int_{-1}^{0} t^3 a_\alpha^- dt + \int_{0}^{1} t^3 a_\alpha^+ dt\right] \\ &= \frac{1}{4}[a_\alpha^- - a_\alpha^+, a_\alpha^+ - a_\alpha^-].\end{aligned}$$

Thus, $(FA)\int_{-1}^{1} f(t)dt = (FR)\int_{-1}^{1} f(t)dt = \frac{1}{4}(A - A) \neq 0$, where $(-)$ is the standard subtraction.

Example 5. Let $A \in \mathbb{R}_\mathsf{F}$ be non-symmetric and $f(t) = \sin(t)A$, for all $t \in [-\pi/2, \pi/2]$. We have that, for all $\alpha \in [0,1]$,

$$\left[(FA)\int_{-\pi/2}^{t} f(s)ds\right]_\alpha = \left[\int_{-\pi/2}^{t} f_\alpha^-(s)ds, \int_{-\pi/2}^{t} f_\alpha^+(s)ds\right].$$

Now, we have to split into two cases:

i. $t \in [-\pi/2, 0)$

$$\begin{aligned}\left[(FA)\int_{-\pi/2}^{t} f(s)ds\right]_\alpha &= \left[\int_{-\pi/2}^{0} \sin(s)a_\alpha^+ ds, \int_{-\pi/2}^{t} \sin(s)a_\alpha^- ds\right] \\ &= [-\cos(t)a_\alpha^+, -\cos(t)a_\alpha^-] = -\cos(t)[a_\alpha^-, a_\alpha^+].\end{aligned}$$

So, $(FR)\int_{-\pi/2}^{t} f(s)ds = (FA)\int_{-\pi/2}^{t} f(s)ds = -\cos(t)A.$

ii. $t \in [0, \pi/2]$

$$\begin{aligned}\left[(FA)\int_{-\pi/2}^{t} f(s)ds\right]_{\alpha} &= \left[\int_{-\pi/2}^{0} \sin(s)a_{\alpha}^{+}ds + \int_{0}^{t} \sin(s)a_{\alpha}^{-}ds,\right.\\ &\qquad \left.\int_{-\pi/2}^{0} \sin(s)a_{\alpha}^{-}ds + \int_{0}^{t} \sin(s)a_{\alpha}^{+}ds\right]\\ &= [-a_{\alpha}^{+} + (1-\cos(t))a_{\alpha}^{-}, -a_{\alpha}^{-} + (1-\cos(t))a_{\alpha}^{+}]\\ &= -[a_{\alpha}^{-}, a_{\alpha}^{+}] + (1-\cos(t))[a_{\alpha}^{-}, a_{\alpha}^{+}].\end{aligned}$$

So, $(FR)\int_{-\pi/2}^{t} f(s)ds = (FA)\int_{-\pi/2}^{t} f(s)ds = -A + (1-\cos(t))A.$

Therefore,

$$(FA)\int_{-\pi/2}^{t} f(s)ds = \begin{cases} -\cos(t)A & \text{if } t \in [-\pi/2, 0) \\ -A + (1-\cos(t))A & \text{if } t \in [0, \pi/2] \end{cases}. \tag{9}$$

Note that the diameter of (9) is an increasing function (see Figure 1a) and it is given by

$$\operatorname{diam}\left((FA)\int_{-\pi/2}^{t} f(s)ds\right) = \begin{cases} \cos(t)\operatorname{diam}(A) & \text{if } t \in [-\pi/2, 0) \\ (2-\cos(t))\operatorname{diam}(A) & \text{if } t \in [0, \pi/2] \end{cases}. \tag{10}$$

3 Interactive Riemann Integral (Ψ-Riemann Integral)

Let $A \in \mathbb{R}_{\mathsf{F}}$ be non symmetric. A function $f : [a, b] \longrightarrow \mathbb{R}_{\mathsf{F}(A)}$ is said to be fuzzy interactive Riemann integrable over $[a, b]$ (Ψ-Riemann integrable, for short) if there exists $S \in \mathbb{R}_{\mathsf{F}(A)}$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every partition $a = t_0 < t_1 < \ldots < t_n = b$ with $t_i - t_{i-1} < \delta$, $i = 1, \ldots, n$, we have

$$\left\| (\Psi_A)\sum_{i=1}^{n} f(\xi_i)\,(t_i - t_{i-1}) -_{\Psi_A} S \right\|_{\Psi_A} < \epsilon, \tag{11}$$

where $\xi_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$, and the summation is given in terms of the interactive addition ($+_{\Psi_A}$). In this case, S is said to be the fuzzy interactive Riemann integral of f over $[a, b]$ and it is denoted by $(\Psi_A)\int_a^b f(t)dt$.

Remark 1. The summation that appears in (11) is called the **Ψ-Riemann sum** of f over $[a, b]$ with partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. This summation is relative to the space $\mathbb{R}_{\mathsf{F}(A)}$, that is, if $f(t) = \Psi_A(q(t), r(t)) = q(t)A + r(t)$, $\forall t \in [a, b]$, then

$$\begin{aligned}(\Psi_A)\sum_{i=1}^{n} f(\xi_i)\,(t_i - t_{i-1}) &= \Psi_A\left(\sum_{i=1}^{n} q(\xi_i)(t_i - t_{i-1}), \sum_{i=1}^{n} r(\xi_i)(t_i - t_{i-1})\right) \qquad (12)\\ &= \left(\sum_{i=1}^{n} q(\xi_i)(t_i - t_{i-1})\right)A + \left(\sum_{i=1}^{n} r(\xi_i)(t_i - t_{i-1})\right),\end{aligned}$$

holds, where $\sum_{i=1}^{n} q(\xi_i)(t_i - t_{i-1})$ and $\sum_{i=1}^{n} r(\xi_i)(t_i - t_{i-1})$ are, respectively, the Riemann sums of the real-valued functions q and r with respect to P. So, we highligth that it is possible that the diameter of (12) is smaller than the diameter of some terms $(f(\xi_i)(t_i - t_{i-1}))$, for $i = 1, \ldots, n$, as we shall see in the following. This does not occur with (8) where the sum is given from standard arithmetic between fuzzy numbers.

Example 6. Consider the function $f : [0,1] \to \mathbb{R}_{\mathsf{F}(A)}$, defined by $f(t) = \Psi_A(t^2, t^2) = t^2 A + t^2$, and let the length of each interval be $\Delta_n = \frac{b-a}{n} = \frac{1}{n}$, so that $\xi_i = a + i\Delta_n = \frac{i}{n}$. Thus, the Ψ-Riemann sum of f over $[0,1]$ is given by

$$\begin{aligned}(\Psi_A)\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) &= \Psi_A\left(\sum_{i=1}^{n} q(\xi_i)(t_i - t_{i-1}), \sum_{i=1}^{n} r(\xi_i)(t_i - t_{i-1})\right)\\ &= \Psi_A\left(\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \frac{1}{n}, \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \frac{1}{n}\right)\\ &= \Psi_A\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}, \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\\ &= \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)A + \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right).\end{aligned}$$

Example 7. Consider the function $f : [0,2] \to \mathbb{R}_{\mathsf{F}(A)}$, defined by $f(t) = \Psi_A(e^{-t^2}, e^{-t^2}) = e^{-t^2}A + e^{-t^2}$. We would like to approximate the integral $(\Psi_A)\int_0^2 f(t)dt$ by Ψ-Riemann sum by considering $n = 4$.

We have $\Delta_4 = \frac{2-0}{4} = \frac{1}{2}$, so that $\xi_i = 0 + i\frac{1}{2} = \frac{i}{2}$. Thus, the Ψ-Riemann sum of f over $[0,2]$ is given by

$$\begin{aligned}(\Psi_A)\sum_{i=1}^{4} f\left(\frac{i}{2}\right)\frac{1}{2} &= \Psi_A\left(\sum_{i=1}^{4} q\left(\frac{i}{2}\right)\frac{1}{2}, \sum_{i=1}^{4} r\left(\frac{i}{2}\right)\frac{1}{2}\right)\\ &= \Psi_A\left(\frac{e^{-\frac{1}{2}} + e^{-1} + e^{-\frac{3}{2}} + e^{-2}}{2}, \frac{e^{-\frac{1}{2}} + e^{-1} + e^{-\frac{3}{2}} + e^{-2}}{2}\right)\\ &= \Psi_A(0.6352, 0.6352) = (0.6352)A + (0.6352).\end{aligned}$$

Theorem 1. *Let $A \in \mathbb{R}_{\mathsf{F}}$ be non-symmetric. The function $f : [a,b] \to \mathbb{R}_{\mathsf{F}(A)}$ is Ψ-Riemann integrable, if, only if, the real-valued functions $q, r : [a,b] \to \mathbb{R}$ are Riemann integrable.*

Proof. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be an arbitrary partition of $[a,b]$. By definition, if f is Ψ-Riemann integrable, then there exists $S = q_S A + r_S \in$

$\mathbb{R}_{\mathsf{F}(A)}$ such that $\forall\epsilon > 0$, $\exists\delta > 0$, where for $|P| < \delta$, we have

$$\left\|(\Psi_A)\sum_{i=1}^{n}(t_i - t_{i-1})f(\xi_i) -_{\Psi_A} S\right\|_{\Psi_A} < \epsilon \Leftrightarrow$$

$$\left\|\Psi_A^{-1}\left(\Psi_A\left(\sum_{i=1}^{n}(t_i - t_{i-1})(q(\xi_i), r(\xi_i)) - (q_S, r_S)\right)\right)\right\|_\infty < \epsilon \Leftrightarrow$$

$$\left\|\left(\sum_{i=1}^{n}(t_i - t_{i-1})q(\xi_i) - q_S, \sum_{i=1}^{n}(t_i - t_{i-1})r(\xi_i) - r_S\right)\right\|_\infty < \epsilon \Leftrightarrow$$

$$\max\left\{\left|\sum_{i=1}^{n}(t_i - t_{i-1})q(\xi_i) - q_S\right|, \left|\sum_{i=1}^{n}(t_i - t_{i-1})r(\xi_i) - r_S\right|\right\} < \epsilon$$

where $\xi_i \in [t_{i-1}, t_i]$, for $i = 1, \ldots, n$. Therefore, q_S and r_S are the Riemann integrals of the functions q and r, respectively.

The converse implication follows similarly.

Theorem 1 shows that computing the Ψ-Riemann integral of f is equivalent to computing the Riemann integrals of the functions q, r, that is,

$$(\Psi_A)\int_a^b f(t)dt = \Psi_A\left(\int_a^b q(t)dt, \int_a^b r(t)dt\right) = \left(\int_a^b q(t)dt\right)A + \left(\int_a^b r(t)dt\right). \tag{13}$$

The next proposition ensures that the Ψ-Riemann integral of f does not depend on the choice of A.

Proposition 3. *[4,18] Let A_1 and A_2 be non-symmetric fuzzy numbers and let $q_i, r_i : [a, b] \to \mathbb{R}$, $i = 1, 2$, be integrable functions such that*

$$\Psi_{A_1}(q_1(s), r_1(s)) = \Psi_{A_2}(q_2(s), r_2(s))$$

for all $s \in [a, b]$. The following equality holds true:

$$\Psi_{A_1}\left(\int_a^b q_1(s)ds, \int_a^b r_1(s)ds\right) = \Psi_{A_2}\left(\int_a^b q_2(s)ds, \int_a^b r_2(s)ds\right)$$

or equivalently,

$$\left(\int_a^b q_1(s)ds\right)A_1 + \left(\int_a^b r_1(s)ds\right) = \left(\int_a^b q_2(s)ds\right)A_2 + \left(\int_a^b r_2(s)ds\right).$$

Example 8. Let $A \in \mathbb{R}_{\mathsf{F}}$ be non-symmetric and $f(t) = \Psi_A(t^3, 0) = t^3 A$, for all $t \in [-1, 1]$. We have that,

$$(\Psi_A)\int_{-1}^1 f(t)dt = \Psi_A\left(\int_{-1}^1 t^3 dt, \int_{-1}^1 0\,dt\right) = \Psi_A(0,0) = 0A + 0 = 0. \tag{14}$$

Note that, in this case, $(FA)\int_{-1}^1 f(t)dt \neq (\Psi_A)\int_{-1}^1 f(t)dt$ (see Example 4).

Example 9. Let $A \in \mathbb{R}_\mathsf{F}$ be non-symmetric and $f(t) = \Psi_A(\sin(t), 0) = \sin(t)A$, for all $t \in [-\pi/2, \pi/2]$. We have that,

$$\begin{aligned}(\Psi_A)\int_{-\pi/2}^{t} f(s)ds &= \Psi_A\left(\int_{-\pi/2}^{t} \sin(s)ds, \int_{-\pi/2}^{t} 0ds\right) \\ &= \Psi_A(-\cos(t), 0) = -\cos(t)A.\end{aligned} \tag{15}$$

Note that the diameter of (15) is given by

$$\text{diam}\left((\Psi_A)\int_{-\pi/2}^{t} f(s)ds\right) = \cos(t)\text{diam}(A) \tag{16}$$

which is strictly decreasing in $[0, \pi/2]$ (see Figure 1b). Such a property is not possible when we consider the fuzzy Aumann integral as is was shown in Example 5.

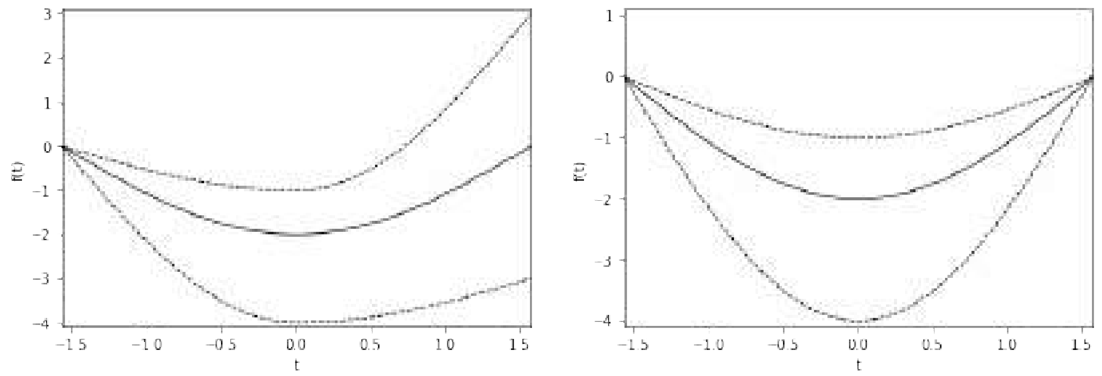

Fig. 1: From left to right, the dashed curves are the 0-levels and the solid curves are the 1-levels of the fuzzy integral functions (9) and (15), respectively, where $A = (1; 2; 4)$.

3.1 Cross product and the method of integration by parts

In this subsection we will present the product rule and the method of integration by parts in $\mathbb{R}_{\mathsf{F}(A)}$. To this end, we will recall the definitions of Ψ-derivative and Ψ-cross product.

Let $A \in \mathbb{R}_\mathsf{F}$ be non-symmetric. According to [10,19], the function $f : [a, b] \to \mathbb{R}_{\mathsf{F}(A)}$ is said to be Ψ-differentiable at $t \in [a, b]$ if there exists a fuzzy number $f'(t) \in \mathbb{R}_{\mathsf{F}(A)}$ satisfying

$$\lim_{h\to 0} \frac{1}{h}(f(t+h) -_{\Psi_A} f(t)) = f'(t).$$

Proposition 4. *[10,19] Let $A \in \mathbb{R}_{\mathsf{F}}$ be non-symmetric. The function f is Ψ-differentiable at $t \in [a,b]$ if, and only if, $q, r : [a,b] \to \mathbb{R}$ is differentiable at t. Additionally,*

$$f'(t) = q'(t)A + r'(t). \tag{17}$$

Let $A \in \mathbb{R}_{\mathsf{F}}$ be non symmetric and $[A]_1 = \{a\}$ be a unitary set. According to [13,14], the Ψ_A-cross product between $B \in \mathbb{R}_{\mathsf{F}(A)}$ and $C \in \mathbb{R}_{\mathsf{F}(A)}$ is defined by

$$P = B \odot_{\Psi_A} C = c_1 B +_{\Psi_A} b_1 C - b_1 c_1, \tag{18}$$

where $[B]_1 = \{b_1 = q_B a + r_B\}$, $[C]_1 = \{c_1 = q_C a + r_C\}$ and $P \in \mathbb{R}_{\mathsf{F}(A)}$.

Rewriting (18) as

$$B \odot_{\Psi_A} C = (2q_B q_C a + q_B r_C + q_C r_B)A + (r_B r_C - a^2 q_B q_C),$$

it is easy to see that if $B, C \in \mathbb{R}$, then $B \odot_{\Psi_A} C = BC$. Additionally, if the A-linearly correlated fuzzy processes $f, g : [a,b] \to \mathbb{R}_{\mathsf{F}(A)}$ are given by $f(t) = q_f(t)A + r_f(t)$ and $g(t) = q_g(t)A + r_g(t)$ for all $t \in [a,b]$, the Ψ_A-cross product between f and g is given by

$$\begin{aligned} f(t) \odot_{\Psi_A} g(t) = {} & (2q_f(t)q_g(t)a + q_f(t)r_g(t) + q_g(t)r_f(t))\, A \\ & + \left(r_f(t)r_g(t) - a^2 q_f(t)q_g(t)\right), \ \forall t \in [a,b]. \end{aligned} \tag{19}$$

The next proposition states some properties concerning the Ψ_A-cross product of A-linearly correlated fuzzy processes:

Proposition 5. *Let $A \in \mathbb{R}_{\mathsf{F}}$ be non-symmetric with $[A]_1 = \{a\}$, and $f, g : [a,b] \to \mathbb{R}_{\mathsf{F}(A)}$ be A-linearly correlated fuzzy processes. The following properties hold true:*

i) If f and g are continuous on $[a,b]$, then $f \odot_{\Psi_A} g$ is continuous on $[a,b]$, where the continuity is given w.r.t. d_{Ψ_A};
ii) If f and g are Ψ_A-differentiable at $t \in [a,b]$, then $f \odot_{\Psi_A} g$ is $\Psi-$differentiable at t and

$$(f(t) \odot_{\Psi_A} g(t))' = f'(t) \odot_{\Psi_A} g(t) +_{\Psi_A} f(t) \odot_{\Psi_A} g'(t);$$

iii) If f and g are Ψ_A-Riemann integrable on $[a,b]$, then $f \odot_{\Psi_A} g$ is Ψ_A-Riemann integrable on $[a,b]$ and

$$(\Psi_A) \int_a^b f'(t) \odot_{\Psi_A} g(t)dt = [f(t) \odot_{\Psi_A} g(t)]_a^b -_{\Psi_A} (\Psi_A) \int_a^b f(t) \odot_{\Psi_A} g'(t)dt.$$

Proof. Consider $f(t) = q_f(t)A + r_f(t)$ and $g(t) = q_g(t)A + r_g(t)$.

i) If f and g are continuous on $[a,b]$ w.r.t. d_{Ψ_A}, then (q_f, r_f) and (q_g, r_g) are continuous w.r.t. the max norm in $\mathbb{R}^2$. Then, it follows immediately from Equation (19) that $f \odot_{\Psi_A} g$ is continuous w.r.t. d_{Ψ_A} on $[a,b]$.

ii) *If f and g are Ψ_A-differentiable at $t \in [a,b]$, then q_f, r_f, q_g and r_g are differentiable at $t \in [a,b]$. Thus, immediately from Equation* (19)*, we have that $f \odot_{\Psi_A} g$ is Ψ_A-differentiable at $t \in [a,b]$. Moreover,*

$$\begin{aligned}
&(f(t) \odot_{\Psi_A} g(t))' = \\
&= (2q_f(t)q_g(t)a + q_f(t)r_g(t) + q_g(t)r_f(t))' A + \left(r_f(t)r_g(t) - a^2 q_f(t)q_g(t)\right)' \\
&= [(2aq_f'(t)q_g(t) + q_f'(t)r_g(t) + q_f(t)r_g'(t))A + (r_f'(t)r_g(t) - a^2 q_f'(t)q_g(t))] \\
&+_{\Psi_A} [(2aq_f(t)q_g'(t) + q_f(t)r_g'(t) + q_g'(t)r_f(t))A + (r_f(t)r_g'(t) - a^2 q_f(t)q_g'(t))] \\
&= f'(t) \odot_{\Psi_A} g(t) +_{\Psi_A} f(t) \odot_{\Psi_A} g'(t).
\end{aligned}$$

iii) *If f and g are Ψ_A-Riemann integrable on $[a,b]$, then q_f, r_f, q_g and r_g are Riemann integrable on $[a,b]$. Thus, immediately from Equation* (19)*, we have that $f \odot_{\Psi_A} g$ is Riemann integrable on $[a,b]$. From item ii) and the linearity of the $\Psi-$Riemann integral, we have that*

$$(\Psi_A)\int_a^b f'(t) \odot_{\Psi_A} g(t)dt = [f(t) \odot_{\Psi_A} g(t)]_a^b -_{\Psi_A} (\Psi_A)\int_a^b f(t) \odot_{\Psi_A} g'(t)dt,$$

since $(\Psi_A)\int_a^b G'(t)dt = G(b) - G(a) = G(t)\,\big|_a^b$ [18].

Example 10. Compute $(\Psi_A)\int_0^\pi [\cos tA \odot_{\Psi_A} tA]\,dt$, where $[A]_1 = \{a\} > 0$. Consider $g(t) = tA$ and $f'(t) = \cos tA$. By Theorem 13 of [10], we have that $f(t) = \sin tA$. Thus,

$$\begin{aligned}
&(\Psi_A)\int_0^\pi [\cos tA \odot_{\Psi_A} tA]dt \\
&= \left[\sin tA \odot_{\Psi_A} tA\right]_0^\pi -_{\Psi_A} (\Psi_A)\int_0^\pi [\sin tA \odot_{\Psi_A} 1A]dt \\
&= \left[(2at\sin t)A + (-a^2 t\sin t)\right]_0^\pi \\
&-_{\Psi_A} (\Psi_A)\int_0^\pi (2a\sin t)A + (-a^2\sin t)dt \\
&= [0A + 0] -_{\Psi_A} \left[(-2a\cos t)A + (-a^2\cos t)\right]_0^\pi \\
&= [0A + 0] -_{\Psi_A} [(4a)A + (-2a^2)] = (4a)A + (-2a^2).
\end{aligned}$$

In the next section, we will study the Malthusian decay model and the logistic growth model.

4 Growth and decay models

In this section, we present the malthusian decay model and the logistic growth model for $A-$linearly correlated fuzzy processes under the $\psi-$derivative and the $\psi-$cross product.

4.1 Malthusian decay model

Consider the Malthusian decay model given by

$$\begin{cases} X'(t) = -\Lambda \odot_{\Psi_A} X(t) \\ X(0) = \frac{1}{2}A + 2 \in \mathbb{R}_{\mathsf{F}(A)} \end{cases}, \tag{20}$$

where $A \in \mathbb{R}_{\mathsf{F}}$ is a non-symmetric fuzzy number given with $[A]_1 = \{a > 0\}$, and $\Lambda = \lambda_1 A + \lambda_2 \in \mathbb{R}_{\mathsf{F}(A)}$, with $\lambda_1, \lambda_2 > 0$. We are assuming that the solution of (20) is an A-linearly correlated fuzzy process $X : [0, \infty) \to \mathbb{R}_{\mathsf{F}(A)}$ given by $X(t) = q(t)A + r(t)$ for all $t \geq 0$. By Equation (18), we have that

$$-\Lambda \odot_{\Psi_A} X(t) = \left(-2a\lambda_1 q(t) - \lambda_1 r(t) - q(t)\lambda_2\right)A + \left(-\lambda_2 r(t) + a^2\lambda_1 q(t)\right). \tag{21}$$

The injectivess of Ψ_A ensures that the fuzzy initial value problem (FIVP) (20) is equivalent to the IVP

$$\begin{cases} q'(t) = -2a\lambda_1 q(t) - \lambda_1 r(t) - q(t)\lambda_2 \\ r'(t) = -\lambda_2 r(t) + a^2\lambda_1 q(t) \end{cases}, \tag{22}$$

with initial conditions given by $q(0) = \frac{1}{2}$ and $r(0) = 2$, whose matrix representation is given by

$$\begin{cases} \begin{pmatrix} q(t) \\ r(t) \end{pmatrix}' = \begin{pmatrix} -2a\lambda_1 - \lambda_2 & -\lambda_1 \\ a^2\lambda_1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} q(t) \\ r(t) \end{pmatrix} \\ \begin{pmatrix} q(0) \\ r(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \end{cases}. \tag{23}$$

Let us denote $M = \begin{pmatrix} -2a\lambda_1 - \lambda_2 & -\lambda_1 \\ a^2\lambda_1 & -\lambda_2 \end{pmatrix}$ the matrix associated to the IVP (22). Note that $\det(M) = 2a\lambda_1\lambda_2 + \lambda_2^2 + a^2\lambda_1^2 = (a\lambda_1 + \lambda_2)^2 > 0$ whenever $a \neq -\frac{\lambda_2}{\lambda_1}$. Since we assumed that $a, \lambda_1, \lambda_2 > 0$, the IVP (22) always have a solution. Moreover, since the characteristic polinomial of M is given by

$$p(\mu) = (\mu + (a\lambda_1 + \lambda_2))^2,$$

the eigenvalues are given by $\mu_1 = \mu_2 = -a\lambda_1 - \lambda_2$. Thus, the solutions of (22) satisfies $q(t), r(t) \to 0$ whenever $t \to \infty$. Note that since $\text{diam}((X)(t)) = q(t)\text{diam}(A)$, then $\text{diam}(X(t)) \to 0$ when $t \to \infty$. This means that the fuzziness of $X = X(t)$ is decreasing and vanishes when $t > 0$ is sufficiently large. In addition, the solution of the FIVP (20), which is depicted in Figure 2, always tends to $0 \in \mathbb{R}_{\mathsf{F}(A)}$ (w.r.t. d_{Ψ_A}) with time.

4.2 Logistic growth model

Consider the logistic growth model with fuzzy carrying capacity $K \in \mathbb{R}_{\mathsf{F}(A)}$

$$\begin{cases} X'(t) = X(t) \odot_{\Psi_A} (K -_{\Psi_A} X(t)) \\ X(0) = \frac{1}{2}A + 1 \in \mathbb{R}_{\mathsf{F}(A)} \end{cases}, \tag{24}$$

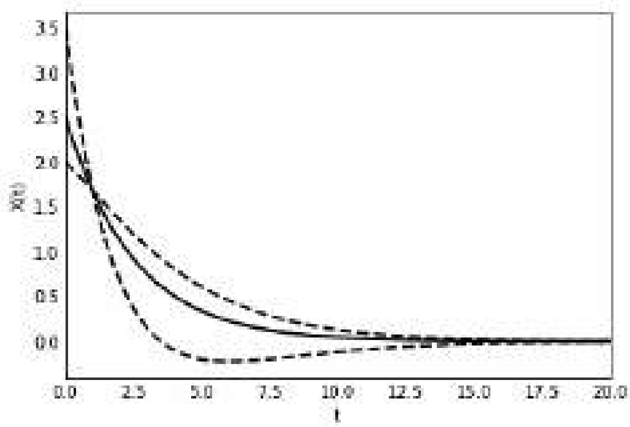

Fig. 2: The dashed curve represents the 0-level and the solid curve represents the 1-level solution $X(t)$ of the FIVP (20) with $\Lambda = 0.2A + 0.2 = (0.2; 0.4; 0.8)$, $A = (0; 1; 3)$ and initial condition given by $X_0 = (2; 2.5; 3.5)$.

where $K = k_1 A + k_2$ and $k_1, k_2 \geq 0$. By Equation (18), we have that

$$\begin{aligned} X(t) \odot_{\Psi_A} (K -_{\Psi_A} X(t)) = &\left(2ak_1 q(t) - 2aq^2(t) + k_2 q(t) - 2q(t)r(t) + k_1 r(t)\right) A \\ &+ \left(k_2 r(t) - r^2(t) - a^2 k_1 q(t) + a^2 q^2(t)\right), \end{aligned} \tag{25}$$

where $[A]_1 = \{a\}$ with $a > 0$. By the injectiveness of Ψ_A, FIVP (24) is equivalent to the following IVP

$$\begin{cases} q'(t) = 2ak_1 q(t) - 2aq^2(t) + k_2 q(t) - 2q(t)r(t) + k_1 r(t) \\ r'(t) = k_2 r(t) - r^2(t) - a^2 k_1 q(t) + a^2 q^2(t) \end{cases}, \tag{26}$$

with initial conditions given by $q(0) = \frac{1}{2}$ and $r(0) = 1$, and whose matrix representation is

$$\begin{cases} \begin{pmatrix} q(t) \\ r(t) \end{pmatrix}' = \begin{pmatrix} 2ak_1 + k_2 - 2aq(t) & -2q(t) + k_1 \\ -a^2 k_1 + a^2 q(t) & k_2 - r(t) \end{pmatrix} \begin{pmatrix} q(t) \\ r(t) \end{pmatrix} \\ \begin{pmatrix} q(0) \\ r(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \end{cases}. \tag{27}$$

The solution of the PVIF (24) is found by numerically solving (27). The solution can be seen in Figure 3, and note that like the classical logistic model, the fuzzy logistic curve tends to fuzzy carrying capacity K.

For the case where the carrying capacity is deterministic, i.e. $k \in \mathbb{R}$, the system (24) becomes

$$\begin{cases} X'(t) = X(t) \odot_{\Psi_A} (k -_{\Psi_A} X(t)) \\ X(0) = \frac{1}{2}A + 1 \in \mathbb{R}_{\mathsf{F}(A)} \end{cases}, \tag{28}$$

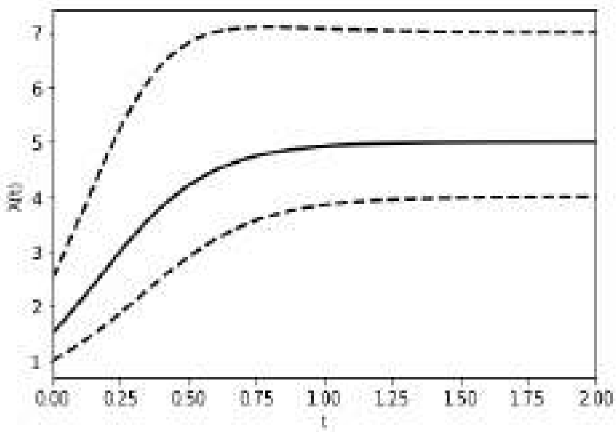

Fig. 3: The dashed curves represent the 0-level and the solid curve represents the 1-level solution $X(t)$ of the system (24) with $K = 1A + 4 = (4; 5; 7)$, $A = (0; 1; 3)$ and initial condition $X_0 = (1; 1.5; 2.5)$.

where $k \geq 0$. By Equation (18), we have that

$$X(t) \odot_{\Psi_A} (k -_{\Psi_A} X(t)) = \left(-2aq^2(t) + kq(t) - 2q(t)r(t)\right) A + \left(kr(t) - r^2(t) + a^2q^2(t)\right), \tag{29}$$

where $[A]_1 = \{a\}$ with $a > 0$. By the injectiveness of Ψ_A, FIVP (28) is equivalent to the following IVP

$$\begin{cases} q^{'}(t) = -2aq^2(t) + kq(t) - 2q(t)r(t) \\ r^{'}(t) = kr(t) - r^2(t) + a^2q^2(t) \end{cases}, \tag{30}$$

with initial conditions given by $q(0) = \frac{1}{2}$ and $r(0) = 1$, and whose matrix representation is

$$\begin{cases} \begin{pmatrix} q(t) \\ r(t) \end{pmatrix}' = \begin{pmatrix} -2aq(t) + k & -2q(t) \\ a^2q(t) & k - r(t) \end{pmatrix} \begin{pmatrix} q(t) \\ r(t) \end{pmatrix} \\ \begin{pmatrix} q(0) \\ r(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \end{cases}. \tag{31}$$

The solution of the FIVP (28) is found by numerically solving (31). The solution can be seen in Figure 4, and note that as in the previous logistic model, the fuzzy logistic curve tends to the carrying capacity k, which is now deterministic.

5 Numerical methods for interactive integration in $\mathbb{R}_{\mathsf{F}(A)}$

In this section, we introduce some numerical methods for the approximated calculation of the interactive Riemann integral

$$I(f) = (\Psi_A) \int_a^b f(t)dt,$$

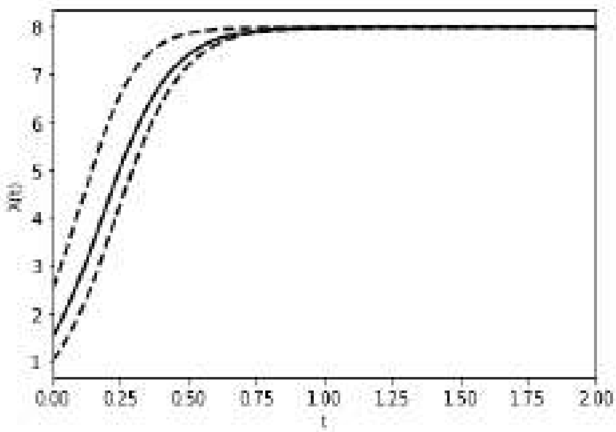

Fig. 4: The dashed curves represent the 0-level and the solid curve represents the 1-level solution $X(t)$ of the system (28) with $k = 8$, $A = (0; 1; 3)$ and initial condition $X_0 = (1; 1.5; 2.5)$.

where $f : [a, b] \to \mathbb{R}_{\mathsf{F}(A)}$, defined by $f(t) = \Psi_A(q(t), r(t)) = q(t)A + r(t)$, is a continuous fuzzy function on $[a, b]$ and $q, r : [a, b] \to \mathbb{R}$. Hereafter, we will denote $C^n([a, b])$ the class of all real-valued functions with derivative of order n continuous.

5.1 Trapezoidal rule

Let us begin with a very simple method of approximating the Ψ-integral, called the trapezoidal rule. The Ψ-integral is approximated by

$$(\Psi_A) \int_a^b f(t)dt \approx \frac{h}{2}[f(a) +_{\Psi_A} f(b)] \equiv Q_T[f] \tag{32}$$

where $b - a = h$. From Equation 32 and definition $+_{\Psi_A}$, we can see that

$$\begin{aligned} Q_T[f] &= \Psi_A\left(\frac{h}{2}[q(a) + q(b)], \frac{h}{2}[r(a) + r(b)]\right) \\ &= \Psi_A\left(Q_T[q], Q_T[r]\right) = Q_T[q]A + Q_T[r], \end{aligned} \tag{33}$$

where $Q_T[q]$ and $Q_T[r]$ are the trapezoidal rule for the functions q and r over $[a, b]$, respectively.

The approximation error in (32) is given by

$$\begin{aligned} E_T[f] &= \left((\Psi_A) \int_a^b f(t)dt\right) -_{\Psi_A} \left(\frac{h}{2}[f(a) +_{\Psi_A} f(b)]\right) \qquad (34) \\ &= \Psi_A\left(\int_a^b q(t)dt - Q_T[q], \int_a^b r(t)dt - Q_T[r]\right) \\ &= \Psi_A(E_T[q], E_T[r]) = E_T[q]A + E_T[r], \end{aligned}$$

where $E_T[q]$ and $E_T[r]$ are the approximation error of the real-valued functions q and r over $[a,b]$. Note that, if $q,r \in C^2[a,b]$, $E_T[q] = -\frac{h^3}{12}q''(t_q)$ and $E_T[r] = -\frac{h^3}{12}r''(t_r)$ for some $t_q, t_r \in [a,b]$ [1].

Thus, for some $t_q, t_r \in [a,b]$

$$E_T[f] = -\frac{h^3}{12}\Psi_A\left(q''(t_q), r''(t_r)\right) = -\frac{h^3}{12}[q''(t_q)A + r''(t_r)]. \tag{35}$$

Thus, the approximation error for the trapezoidal rule applied to $A-$linearly correlated fuzzy processes is given by a fuzzy number. In particular, it is given by the $A-$linearly correlated fuzzy number (35). In order to improve this approximation, we will follow the idea of defining an integral and partition the interval $[a,b]$ into n subintervals, $[t_{i-1}, t_i]$, for $i = 1,\ldots,n$, with $t_0 = a$ and $t_n = b$. For simplicity we will consider all the intervals with the same size $h = \frac{b-a}{n}$. Thus, using the idea above in each subinterval, we get

$$\begin{aligned}(\Psi_A)\int_a^b f(t)dt &\approx h\left(\frac{(f(t_0) +_{\Psi_A} f(t_1))}{2} +_{\Psi_A} \ldots +_{\Psi_A} \frac{(f(t_n) +_{\Psi_A} f(t_{n-1}))}{2}\right) \qquad (36)\\ &= \frac{h}{2}\left(f(t_0) +_{\Psi_A} 2f(t_1) +_{\Psi_A} \ldots +_{\Psi_A} 2f(t_{n-1}) +_{\Psi_A} f(t_n)\right) \equiv Q_{CT}[f].\end{aligned}$$

Equation 36 is called the composite trapezoidal rule and we can rewrite it by

$$\begin{aligned}Q_{CT}[f] &= \Psi_A\left(\frac{h}{2}\left[q(t_0) + 2q(t_1) + \ldots + 2q(t_{n-1}) + q(t_n)\right],\right.\\ &\qquad \left.\frac{h}{2}\left[r(t_0) + 2r(t_1) + \ldots + 2r(t_{n-1}) + r(t_n)\right]\right)\\ &= \Psi_A(Q_{CT}[q], Q_{CT}[r]) = Q_{CT}[q]A + Q_{CT}[r],\end{aligned} \tag{37}$$

where $Q_{CT}[q]$ and $Q_{CT}[r]$ are the composite trapezoid rule for the functions q and r over $[a,b]$, respectively.

In this case, the approximation error of (36) is given by

$$E_{CT}[f] = -\frac{h^3}{12n^2}\Psi_A\left(q''(t_q), r''(t_r)\right) = -\frac{h^3}{12n^2}[q''(t_q)A + r''(t_r)], \tag{38}$$

for some $t_q, t_r \in [a,b]$, and $\|E_{CT}\|_{\Psi_A} = \frac{h^3}{12n^2}\max\{|q''(t_q)|, |r''(t_r)|\}$.

Example 11. Let $A \in \mathbb{R}_{\mathsf{F}}$ be non-symmetric and $f(t) = \Psi_A(\sin(t), 0) = \sin(t)A$, for all $t \in [0, \pi/2]$. Compute numerically the Ψ-integral $(\Psi_A)\int_0^{\pi/2}\sin(t)Adt$.

Using the trapezoid rule, we obtain

$$Q_T[f] = \Psi_A\left(Q_T[q], Q_T[r]\right) = \Psi_A\left(0.785, 0\right) = (0.785)A \tag{39}$$

and using the composite trapezoid rule with $n = 5$, we obtain

$$Q_{CT}[f] = \Psi_A\left(Q_{CT}[q], Q_{CT}[r]\right) = \Psi_A\left(0.992, 0\right) = (0.992)A. \tag{40}$$

We know that the analytical solution of the Ψ-integral $(\Psi_A)\int_0^{\pi/2}\sin(t)Adt$ is $1A$. Thus, we obtain $E_T[f] = 0.215A$ and $E_{CT}[f] = 0.008A$.

5.2 Simpson rule

In Simpson rule, the integration interval $[a, b]$ is divided into two parts by the intermediate point $\frac{a+b}{2}$. Thus, the three interpolation points t_0, t_1 and t_3 are given by $t_0 = a$, $t_1 = a + h$ and $t_2 = a + 2h = b$, where $h = \frac{b-a}{2}$ is the separation between consecutive points. Thus Simpson rule for A-linearly correlated functions is given by

$$(\Psi_A) \int_a^b f(t)dt \approx \frac{h}{3} \left(f(t_0) +_{\Psi_A} 4f(t_1) +_{\Psi_A} f(t_2) \right) \equiv Q_S[f] \tag{41}$$

Equation 41 can rewrite it by

$$\begin{aligned} Q_S[f] &= \Psi_A \left(\frac{h}{3} \left(q(t_0) + 4q(t_1) + q(t_2) \right), \frac{h}{3} \left(r(t_0) + 4r(t_1) + r(t_2) \right) \right) \\ &= \Psi_A(Q_S[q], Q_S[r]) = Q_S[q]A + Q_S[r], \end{aligned} \tag{42}$$

where $Q_S[q]$ and $Q_S[r]$ are the Simpson rule for the functions q and r over $[a, b]$, respectively.

The approximation error in (41) is given by

$$\begin{aligned} E_S[f] &= \left((\Psi_A) \int_a^b f(t)dt \right) -_{\Psi_A} \left(\frac{h}{3} \left[f(t_0) +_{\Psi_A} 4 \cdot_{\Psi_A} f(t_1) +_{\Psi_A} f(t_2) \right] \right) \\ &= \Psi_A \left(\int_a^b q(t)dt - Q_S[q], \int_a^b r(t)dt - Q_S[r] \right) \\ &= \Psi_A \left(E_S[q], E_S[r] \right) = E_S[q]A + E_S[r], \end{aligned} \tag{43}$$

where $E_S[q]$ and $E_S[r]$ are the approximation error of the real-valued functions q and r over $[a, b]$. Note that if $q, r \in C^4([a, b])$, then $E_S[q] = -\frac{h^5}{90} q^{(4)}(t_q)$ and $E_T[r] = -\frac{h^5}{90} r^{(4)}(t_r)$ for some $t_q, t_r \in [a, b]$ [1].

Thus, for some $t_q, t_r \in [a, b]$

$$E_S[f] = -\frac{h^5}{90} \Psi_A \left(q^{(4)}(t_q), r^{(4)}(t_r) \right) = -\frac{h^5}{90} \left[q^{(4)}(t_q)A + r^{(4)}(t_r) \right]. \tag{44}$$

Now let us divide the interval $[a, b]$ into $n \geq 2$ small subintervals and apply Simpson rule to each subinterval. The sum of all approximations of each subinterval produce an approximation for the integral over the entire range. This method is called the composite Simpson rule. Let the interval $[a, b]$ be splitted into n subintervals, with n an even number such that the intervals have the same size $h = \frac{b-a}{n}$. Thus,

$$(\Psi_A) \int_a^b f(t)dt \approx \sum_{k=1}^{n/2} \frac{h}{3} \cdot_{\Psi_A} \left[f(t_{2k-2}) +_{\Psi_A} 4f(t_{2k-1}) +_{\Psi_A} f(t_{2k}) \right] \equiv Q_{CS}[f]. \tag{45}$$

We can rewrite Equation 45 by

$$\begin{aligned} Q_{CS}[f] = \Psi_A \Bigg(& \sum_{k=1}^{n/2} \frac{h}{3} \left[q(t_{2k-2}) + 4q(t_{2k-1}) + q(t_{2k}) \right], \\ & \sum_{k=1}^{n/2} \frac{h}{3} \left[r(t_{2k-2}) + 4r(t_{2k-1}) + r(t_{2k}) \right] \Bigg) \\ = {} & \Psi_A \left(Q_{CS}[q], Q_{CS}[r] \right) = Q_{CS}[q]A + Q_{CS}[r], \end{aligned} \tag{46}$$

where $Q_{CS}[q]$ and $Q_{CS}[r]$ are the composite Simpson rule for the functions q and r over $[a, b]$, respectively.

In this case, the approximation error of (45) is given by

$$E_{CS}[f] = -\frac{h^5}{90n^4} \Psi_A \left(q^{(4)}(t_q), r^{(4)}(t_r) \right) = -\frac{h^5}{90n^4} \left[q^{(4)}(t_q)A + r^{(4)}(t_r) \right] \tag{47}$$

for some $t_q, t_r \in [a, b]$, and $\|E_{CS}\|_{\Psi_A} = \frac{h^3}{90n^4} \max \left\{ \left| q^{(4)}(t_q) \right|, \left| r^{(4)}(t_r) \right| \right\}$.

Example 12. Let $A \in \mathbb{R}_{\mathsf{F}}$ be non-symmetric and $f(t) = \Psi_A(\sin(t), 0) = \sin(t)A$, for all $t \in [0, \pi/2]$. Compute numerically the Ψ-integral $(\Psi_A) \int_0^{\pi/2} \sin(t)A \, dt$.

Using the Simpson rule, we obtain

$$Q_S[f] = \Psi_A \left(Q_S[q], Q_S[r] \right) = \Psi_A (1.002, 0) = (1.002)A \tag{48}$$

and using the composite Simpson rule with $n = 6$, we obtain

$$Q_{CS}[f] = \Psi_A \left(Q_{CS}[q], Q_{CS}[r] \right) = \Psi_A (1.000026, 0) = (1.000026)A. \tag{49}$$

We know that the analytical solution of the Ψ-integral $(\Psi_A) \int_0^{\pi/2} \sin(t)A \, dt$ is $1A$. Thus, we get $E_S[f] = 0.002A$ and $E_{CS}[f] = 0.000026A$.

Remark 2. Note that, by Lemma 1, the numerical convergence of trapezoidal rule and Simpson rule for the interactive integral, given w.r.t. the induced norm $\|\cdot\|_{\Psi_A}$ implies the convergence w.r.t. the usual norm $\|\cdot\|_{\mathcal{F}}$ in the class of fuzzy numbers.

6 Final remarks

In this work, we studied the interactive Riemann integral via the Ψ-Riemann sum of an $A-$linearly correlated fuzzy process when $A \in \mathbb{R}_{\mathsf{F}}$ is a non-symmetric fuzzy number. We demonstrated that computing the $\Psi-$Riemann integral of f is equivalent to compute the Riemann integral of its coordinates in the space $\mathbb{R}_{\mathsf{F}(A)}$. We compared it with non-interactive fuzzy integrals, and introduced the product rule and method of integration by parts using the concept of $\psi-$cross product [14,13].

We also studied the malthusian decay and the logistic growth models with decay rate and carrying capacity given by $A-$linearly correlated fuzzy numbers, respectively. We showed that the mathusian decay always have a solution under some weak conditions, which always has a vanishing fuzziness with time. In addition, the solution always converges to $0 \in \mathbb{R}_{\mathsf{F}(A)}$, similiar to what happens in the classical modeling.

Additionally, we presented numerical approximations for the interactive Riemann integral of $A-$linearly correlated fuzzy processes. More specifically, we presented the trapezoidal and the Simpson rules for the fuzzy cases. The interesting thing is that computing these methods for $f : [a, b] \to \mathbb{R}_{\mathsf{F}(A)}$ is equivalent to computing them for the corresponding real-valued functions q and r. Moreover, Lemma 1 ensured the numerical convergence w.r.t. the usual norm $\| \cdot \|_{\mathcal{F}}$ in the class of fuzzy numbers.

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Fuzzy Kalman Filter Modeling Based on Sensor Triangulation Applied to Suborbital Rocket Position Estimation

Arthur Medeiros Macedo[1] [0000−0002−7801−4707] and Ginalber Luiz de Oliveira Serra[2] [0000−0002−4424−618X]

[1] Universidade Federal do Maranhão, Av. dos Portugueses, 1966 - Vila Bacanga, 65080-805, São Luís - MA, Brasil
arthurmacedo01@gmail.com
[2] Instituto Federal do Maranhão, Av. Getúlio Vargas, 04, Monte Castelo, 65030-005, São Luís - MA, Brasil
ginalber@ifma.edu.br

Abstract. In order to track a suborbital vehicle, some tracking systems are used. Tracking radar can be used for this purpose. This paper presents a method using triangulation of antennas other than radars, for estimating the position of the vehicle. The goal is to improve the capability of keeping tracking the vehicle, even in loss of signal scenarios. Also, a dynamic filter is proposed for improving the estimation accuracy, based on a fuzzy system identification. Simulations were performed to demonstrate the proposed methods in estimating a suborbital vehicle position.

Keywords: Input and excitation design · Estimation and Filtering · Fuzzy Clustering · Observer Kalman Filter Identification · Position Tracking.

1 Introduction

A sounding rocket is a space vehicle designed for carrying a scientific payload that mostly takes measurements and performs some scientific experiments when it is in a sub-orbital environment, during its flight [1]. In addition, the development of systems for sounding rockets also contributes to the development of orbital vehicles. One can divide a sounding flight in three phases [5]. The first phase is called boost phase. In this phase, the vehicle takes off, the thrust forces increase the velocity of the rocket. The second phase, called burnout, occurs after the propellent is burn, and its characterized for the decrease of the vehicle velocity due the resistive aerodynamic forces. During the third phase, called recovery, after the apogee, the rocket velocity increases again and it returns to the ground.

In Brazil, Alcantara Launch Center, located at State of Maranhão, among others activities, launches sounding rocket from Brazilian territory [4]. One of the responsibilities of a launch site is to guarantee the safety of the rocket flight in regard to the safe of people, environment and facilities. In order to guarantee the flight safety, the launch site have its flight termination system, which have the capability of destructing the vehicle during its flight [6]. The flight termination system uses remote control to send the termination command. The moment for commanding the flight termination depends on the flight safety plan [17]. Generally, one of the criteria for terminating a flight is loss of tracking data, for controlled vehicles. This procedure prevents the possibility of the vehicle to enter a protected zone. In Brazil, Alcantara Launch Center has tracking radars [12], in order to keep tracking the vehicle during its flight. The tracking radars are systems dedicated for this activity during the flight.

This paper presents a methodology for tracking a sounding rocket using signal other than tracking radar signal. The proposed method uses some data from telemetry antennas instead of tracking radars for determining the position of the rocket during its flight by triangulation.

Some works have been published in the area of rocket position estimation however from different approaches of this work. [3] addresses the combination of using satellite navigation receivers combined with inertial measurement unit (IMU). [11] studied the use of bi-static radars for the detection and estimation of the position of supersonic rockets. In the scope of orbital rockets, [7] presents a methodology for estimating the position and orientation of the vehicle in relation to a spacecraft using camera sensors.

The method proposed in this work intends to address, in academic level, the challenge of keeping tracking the vehicle in case of loss of data from radar, with some precision close to the radar precision. The ultimate goal is to increase the availability of tracking data and then to increase the flight safety. The outcome of this work should to assist the Safety Officer for decision taking in the moment of loss of radar signal. This paper uses data from telemetry antennas, initially using triangulations of signals from the antennas in pairs. The outcome of these triangulations, should be used as input of a dynamic filter in order to obtain a signal close to that one delivered from the radar. This dynamic filter design should consider non-linearities present in the trajectory [13]. To handle those non-linearities, a fuzzy clustering method is used for separating the trajectory in clusters and treat the filter dynamic as a fuzzy system using those clusters. A system identification technique is applied as part of the filter design in order to transform data based on the triangulation to data close to the signal from a tracking radar. Finally, some simulations were performed and the results were presented demonstrating the performance of the method using real data from a sounding rocket launch. The results were compared and discussed.

2 Antennas Triangulation

In addition to the tracking radar, Alcantara Launch Center also has some telemetry stations. Each telemetry ground system has its own antenna with azimuth and elevation angles that point on the direction of the rocket position. The challenge is to calculate the vehicle position based only on those antennas' angles.

In order to do that, one can use more than one antenna data to perform a triangulation and then to determine the position of the vehicle with some level of accuracy [10]. Since the Launch Center has a few telemetry station, one can perform more than one triangulation, resulting in multiple position estimations based on triangulations only. Those results can be used all together in the filter design for increasing the accuracy. The filter design will be explained in other section of this paper.

Lets first consider only one pair of antennas. One can define the relation of the position of the vehicle and the position of the antennas with the equation that follows:

$$\overrightarrow{Da_1a_2} = \overrightarrow{P_1} - \overrightarrow{P_2} \quad (1)$$

$\overrightarrow{P_1}$ and $\overrightarrow{P_2}$ are the position of the vehicle in relation to antenna 1 and 2 respectively. $\overrightarrow{Da_1a_2}$ is a distance vector between antenna 1 and antenna 2, as shown in Fig. 1.

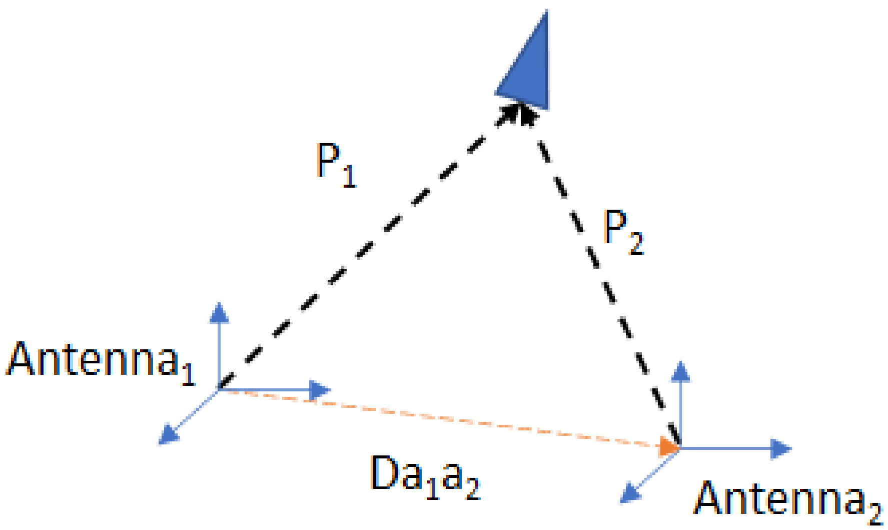

Fig. 1. Triangulation considering vectorial sum

$\overrightarrow{P_1} \in \mathbb{R}^{(1\times3)}$ and $\overrightarrow{P_2} \in \mathbb{R}^{(1\times3)}$ are row vectors as follow:

$$\overrightarrow{P_1} = D_1[cos(\beta_1)cos(\alpha_1), cos(\beta_1)sen(\alpha_1), sen(\beta_1)] \quad (2)$$

$$\overrightarrow{P_2} = D_2[cos(\beta_2)cos(\alpha_2), cos(\beta_2)sen(\alpha_2), sen(\beta_2)] \quad (3)$$

where α_i and $\beta_i \in \mathbb{R}$ are the azimuth and elevation angles of each antenna respectively.

Therefore, $D_1(t)$ and $D_2(t)$ are the only unknown values. Since the position is a three dimensional vector, the equation (1) can be express in each axis as follows:

$$T = M(t)D(t) \tag{4}$$

$$T = \begin{bmatrix} Da_1a_{2x} \\ Da_1a_{2y} \\ Da_1a_{2z} \end{bmatrix}, D = \begin{bmatrix} D_1(t) \\ D_2(t) \end{bmatrix} \tag{5}$$

$$M(t) = \begin{bmatrix} cos(\beta_1(t))cos(\alpha_1(t)) & -cos(\beta_2(t))cos(\alpha_2(t)) \\ cos(\beta_1(t))sen(\alpha_1(t)) & -cos(\beta_2(t))sen(\alpha_2(t)) \\ sen(\beta_1(t)) & -sen(\beta_2(t)) \end{bmatrix} \tag{6}$$

Then,

$$D(t) = (M(t)^T M(t))^{-1} M(t)^T T \tag{7}$$

Therefore, one can compute $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$ from $D_1(t)$ and $D_2(t)$. For simulation purpose, this paper consider the launch pad as the origin of the coordinate system. Then one should add the distance vector from the radar to the launch pad in order to have all the tracking data in the same coordinate system origin. The coordinate system is then ENU (East, North, Upward) with the origin at the launch pad.

It's important to mention that the position calculated through triangulation gives only an estimation of the real position of the vehicle. Comparing with a tracking radar dedicated to this activity, the triangulation yields a high noisy estimation with errors embedded. In order to reduce those error and noise, one can design a filter which has as its input the triangulation signal, and gives as output an estimation with higher accuracy and precision.

3 Fuzzy Clustering

Considering that the dynamic of the filter that reduces the error and noise from the triangulations output can be a very complex nonlinear system, one can approach to this challenge by partitioning the trajectory of the rocket into subsets [9] and design a simpler dynamic filter for each subset.

Clustering techniques can be applied to a dataset from a previous flight in order to define the fuzzy clusters of triangulations values based on a data driven clustering method [14]. This paper used Gustafson-Kessel algorithm as the clustering method. Gustafson-Kessel algorithm is part of a family of algorithms derived from the basics of fuzzy c-means functional. Its goal is to minimize the following function:

$$J = \sum_{i=1}^{c} \sum_{k=1}^{N} (\mu_{ik})^m D_{ikA_i}^2 \tag{8}$$

where N is the length of the dataset, μ_{ik} is the membership degree of the k'th data of the i'th cluster to be optimized, c is the total number of clusters, $m > 1$

is a parameter related to the sharpness of the clusters, D is the distance vector given by:

$$D^2_{ikA_i} = (z_k - v_i)^T A_i (z_k - v_i) \tag{9}$$

where A is the induced matrix norm to be optimized according to the Gustafson-Kessel algorithm [14], z_k is the k'th data, v_i is the center of the i'th cluster to be optimized. To obtain a feasible solution, A must be constrained. Generally, the determinant of A is constrained in a fixed value.

An advantage of this algorithm over the classical c-meas, is that it can identify clusters of different shapes and orientations [14].

3.1 Membership Degree

After determining the center of each cluster, the membership degree of a certain data in relation to a cluster i can be computed as:

$$\mu_i = \frac{1}{\sum_{j=1}^{c} (D_{iA_i}/D_{jA_i})^{2/(m-1)}} \tag{10}$$

where $\mu_i \in [0, 1]$. Therefore, if $D_{jA_i} = 0$ then $\mu_i = 0$, whereas if $D_{iA_i} = 0$ then $\mu_i = 1$.

3.2 Determining the Number of Clusters

The Gustafson-Kessel algorithm considers the total number of clusters already defined beforehand. A standard approach to determining an appropriate number of clusters is the cluster validity measures. This paper used the Fuzzy Hypervolume measurement for validating the number of clusters.

The fuzzy hypervolume V_h is defined by [14]:

$$V_h = \sum_{i=1}^{c} [det(F_i)]^{1/2} \tag{11}$$

where F_i is the ith' cluster covariance matrix. A small value of V_h indicates an appropriate number of clusters. Then one should analyze the V_h for different numbers of clusters and choose the number that yields a small V_h and also keeps the number of clusters feasibly small.

4 System Identification

After the dataset partition, one should design a filter that takes the values from the triangulation step and filter them to a signal as close to a radar signal as possible. In order to do that, one can use a system identification algorithm. The method here presented aim to obtain a state-space model from the input and output data, its called Observer/Kalman Filter Identification (OKID) [8].

Consider the discrete linear systems with a space state representation after adding and subtracting the term $Ky(k)$ to the right-hand side to yield:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) - Ky(k) + Ky(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{12}$$

where $u(k)$ is the input of the system and $y(k)$ the output. Considering the tracking problem, $u(k)$ is the estimations based on triangulation of a particular cluster, and $y(k)$ is the vehicle position given by the radar.

Rearrange the terms to yield:

$$\begin{aligned} x(k+1) &= \bar{A}x(k) + \bar{B}v(k) \\ \bar{A} &= A - KC \\ \bar{B} &= \left[B - KD,\, K\right] \\ v(k) &= \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \end{aligned} \tag{13}$$

The Observer Markov Parameters can be obtain as:

$$\bar{Y} = \bar{y}\bar{V}^T[\bar{V}\bar{V}^T]^{-1} \tag{14}$$

where

$$\begin{aligned} \bar{y} &= \left[y(p)\; y(p+1)\; \dots\; y(l-1)\right] \\ \bar{V} &= \begin{bmatrix} u(p) & u(p+1) & \dots & u(l-1) \\ v(p-1) & v(p) & \dots & v(l-2) \\ v(p-2) & v(p-1) & \dots & v(l-3) \\ \vdots & \vdots & \vdots & \vdots \\ v(0) & v(1) & \dots & v(l-p-1) \end{bmatrix} \end{aligned} \tag{15}$$

where l is the length of dataset, p is an arbitrary small integer where $\bar{A}^p$ is sufficiently small.

Once the Observer Markov Parameters $\bar{Y}$ are calculated, one can recover the original system Markov Parameters by first partitioning $\bar{Y}$ into:

$$\bar{Y} = \left[\bar{Y}_0\; \bar{Y}_1\; \bar{Y}_2\; \dots\; \bar{Y}_p\right] \tag{16}$$

and

$$\begin{aligned} \hat{D} &= \bar{Y}_0 \\ \bar{Y}_k &= \left[\bar{Y}_k^{(1)}\; -\bar{Y}_k^{(2)}\right]; k = 1, 2, 3, \dots \end{aligned} \tag{17}$$

where $\bar{Y}_k^{(1)}$ and $\bar{Y}_k^{(2)}$ are submatrices of $\bar{Y}_k$ [8] that will be used for calculating the Markov Parameters afterwards.

Then, the system Markov Parameters Y can be obtained by:

$$\begin{aligned} \hat{D} &= Y_0 \\ \left[Y_k \quad Y_k^o\right] &= \left[\bar{Y}_k^{(1)} - \bar{Y}_k^{(2)}D \qquad \bar{Y}_k^{(2)}\right] - \textstyle\sum_{i=1}^{k-1} \bar{Y}_i^{(2)} \left[Y_{k-i} \quad Y_{k-i}^o\right] \end{aligned} \tag{18}$$

where Y_n is the n-th Markov parameter. Y_n^o will used in future for calculating the Kalman gain K in (12).

Once the System Markov Parameters are calculated, one can use an identification method to determine A, B and C matrix in (12). Here, the time domain method Eigensystem Realization Algorithm (ERA) was used [15].

4.1 Eigensystem Realization Algorithm

With the System Markov Parameters Y_n already calculated from the previous step, one can determine the Hankel matrix $\alpha.m \times \beta r$, as follows:

$$H(k-1) = \begin{bmatrix} Y_k & Y_{k+1} & \dots & Y_{k+\beta-1} \\ Y_{k+1} & Y_{k+2} & \dots & Y_{k+\beta} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{k+\alpha-1} & Y_{k+\alpha} & \dots & Y_{k+\alpha+\beta-2} \end{bmatrix} \tag{19}$$

where n is the order of the system, $\alpha \geq n$ and $\beta \geq n$. m is the number of outputs and r the number of inputs.

According to ERA, the matrix A can be estimated as

$$\hat{A} = \Sigma_n^{-1/2} R_n^T H(1) S_n \Sigma_n^{-1/2} \tag{20}$$

where Σ_n, R_n and S_n are calculated through the singular value decomposition of $H(0)$.

The singular value decomposition of $H(0)$ is defined by:

$$H(0) = R \Sigma S^T \tag{21}$$

where the columns of matrices R and S are orthonormal and Σ is a rectangular matrix that can be approximated as:

$$\Sigma = \begin{bmatrix} \Sigma_n & 0 \\ 0 & 0 \end{bmatrix} \tag{22}$$

Then, R_n and S_n are the first n columns of R and S matrices, where n is the dimension of the system.

The matrix B and C can be estimated as:

$$\begin{aligned} \hat{B} &= \Sigma_n^{1/2} S_n^T E_r \\ \hat{C} &= E_m^T R_n \Sigma_n^{1/2} \end{aligned} \tag{23}$$

defining O_i as a null matrix of order i and I_i as an identity matrix of order i, $E_m^T = [I_m \; O_m \; \dots \; O_m]$ and $E_r^T = [I_r \; O_r \; \dots \; O_r]$.

4.2 Kalman Gain

Given that the estimation of A, B, C and D has already been calculated in the steps above, the Kalman Gain K in (12) is:

$$K = -(P^T P)^{-1} P^T Y^o \tag{24}$$

where Y^o was calculated in (18), and P is the system observability matrix:

$$P = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^k \end{bmatrix} \tag{25}$$

The process of the system identification presented so far considering the fuzzy clustering step is shown in Fig. 2.

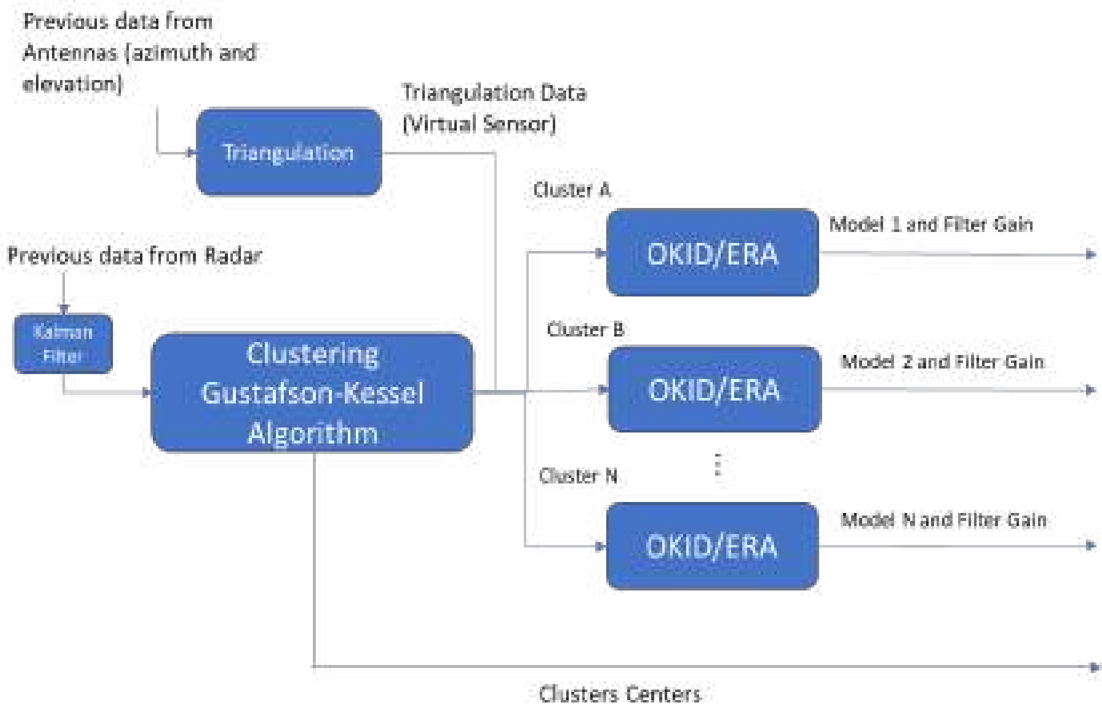

Fig. 2. System Identification Diagram

5 Fuzzy Kalman Filter

As mentioned before, each pair of antennas can give one estimation of the position of a sounding rocket based on the triangulation method presented in Chapter

2. If we consider more than one pair of antennas, one can have many estimations. Those values can be used as input of a filter that converts those data from the triangulations to data similar the ones from the tracking radar sensor. In order to find the dynamic of the filter that performs this conversion, the OKID algorithm [16] can be run taking as inputs all the data from different triangulations and as output the data from a radar sensor. After separating the trajectory in clusters as explained before, instead of using all the tracking data in the OKID algorithm, one should use only those data from the same cluster (highest membership degree) for each OKID algorithm turn. At the end of this process, one should have c number of dynamic filters (A, B, C, D and K), where c is the number of clusters.

Up to here, all the procedures was performed offline, in order to define the c filters that can be used during a flight. Therefore, during a flight, one can have the filtered estimated position using the equations:

$$\hat{x}(k+1) = [\hat{A}_i - K_i\hat{C}_i]\hat{x}_i(k) + [\hat{B}_i - K_i\hat{D}_i]u(k) + K_i y(k) \tag{26}$$

$$\hat{y}_i(k) = \hat{C}_i\hat{x}_i(k) + \hat{D}_i u(k) \tag{27}$$

The signals $u(k)$ and $y(k)$ are the signals from the triangulations procedure and the tracking radar sensor respectively. $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$, $\hat{D}_i$ and K_i are the matrices from the i-th filter estimated from OKID algorithm. The total number of filters are the total number of clusters. Hence, the final estimation considering all those filters can be computed as:

$$\hat{y}(k) = \mu_1 \times \hat{y}_1(k) + \mu_2 \times \hat{y}_2(k) + \ldots + \mu_c \times \hat{y}_c(k) \tag{28}$$

where μ_i is the membership degree of $\hat{y}(k-1)$ related to the cluster i as presented in (10).

Notice that (26) uses data from the tracking radar in $y(k)$ in order to update the state $\hat{x}(k)$. But, once the radar signal is available, one can use itself as the final position estimation, giving no advantage of using (27) and (28) for estimating the position of the rocket. However, if the data from the tracking radar $y(k)$ is not available, one can use the dynamic filters already designed for estimating the state using:

$$\hat{x}(k+1) = \hat{A}_i\hat{x}(k) + \hat{B}_i u(k) \tag{29}$$

and then using (27) and (28) for estimating the position of the rocket.

6 Blind Interval

For study purpose, one can define a method for the case where neither the data from the tracking radar is available nor the data from the triangulations at some period during the flight. In this case, one should estimate the position of the rocket based only in previous data from the sensors. In the case where there is no position data from any source or sensor, the Flight Safety Officer of the Launch Site may have the decision for terminating the flight if the Flight Safety Plan says to do so. However, this paper presents a method for study propose that propagates the dynamic of the rocket up to the moment of loss of data. This propagation of dynamics can be used to estimate the position of the rocket during this blind interval. The radar system can then use this estimation to adjust its servo mechanisms for trying to track the vehicle again.

The Standard Kalman Filter [18] can be used as the method for propagating the dynamics of the vehicle. Considering the state vector as position, velocity and acceleration on all three axis, one can represent the dynamic of the rocket as the following state space representation:

$$x_{k+1} = Ax_k + w_k \tag{30}$$

$$y_k = Cx_k + n_k \tag{31}$$

where w_k is the process noise, an independent random variable with mean equal zero and covariance Q. n_k is the measurement noise, an independent random variable with mean equal zero and covariance R.

The state vector can be written as:

$$x = \begin{bmatrix} P_x \; \dot{P}_x \; \ddot{P}_x \; P_y \; \dot{P}_y \; \ddot{P}_y \; P_z \; \dot{P}_z \; \ddot{P}_z \end{bmatrix}^T \tag{32}$$

where P_x, P_y and P_z is the position in the x axis, y axis and z axis respectively. $\dot{P}$ is the first derivative of position with respect to time. $\ddot{P}$ is the second derivative of position with respect to time.

$$A = \begin{bmatrix} 1 & \delta_T & \delta_T^2/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \delta_T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \delta_T & \delta_T^2/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \delta_T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \delta_T & \delta_T^2/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \delta_T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

where δ_T is the signal sampling time.

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{34}$$

The Standard Kalman Filter is a procedure with two steps. The prediction step and the update step [2]. First, the prediction step can be calculated as:

$$\begin{aligned} \hat{x}_{k/k-1} &= A\hat{x}_{k-1/k-1} \\ P_{k/k-1} &= AP_{k-1/k-1}A^T + Q \\ \hat{y}_k &= C\hat{x}_{k/k-1} \end{aligned} \tag{35}$$

and the update step as:

$$\begin{aligned} K_k &= P_{k/k-1}C^T[CP_{k/k-1}C^T + R]^{-1} \\ \hat{x}_{k/k} &= \hat{x}_{k/k-1} + K_k[y_k - \hat{y}_k] \\ P_{k/k} &= P_{k/k-1} - K_k C P_{k/k-1} \\ \hat{y}_k &= C\hat{x}_{k/k} \end{aligned} \tag{36}$$

where $k/k-1$ indicates the value at the instant k before the sensor data is known. On the other hand, k/k indicates the value at the instant k after the sensor data is known.

Therefore, during the flight, in the case where there is at least one source of position data available (either the radar and/or triangulation), both the prediction step and the update step can be performed. However, during the period when there is no position data available whatsoever, only the prediction step is performed, then the position estimation $\hat{y}_k$ is calculated based on the previous data only, as desired. A diagram that covers the filter process explained before is shown in Fig. 3.

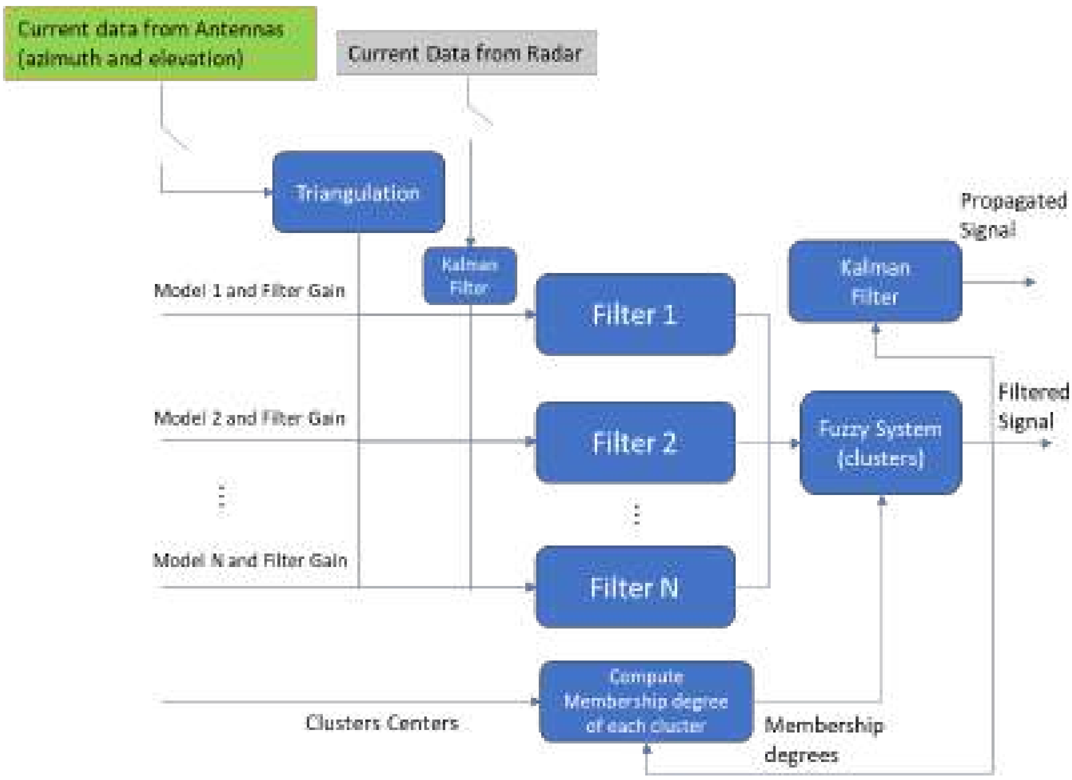

Fig. 3. Fuzzy Kalman Filter for Triangulation

7 Results

A simulation was performed using the methodology presented in this paper. Data from a real flight of a sounding rocket were used. The data from the tracking radar was used for determining the clusters that will be used in the next steps. Following the methodology presented before, the hyper volume for different numbers of clusters was calculated by the Gustafson-Kessel algorithm, with $m = 1.5$.

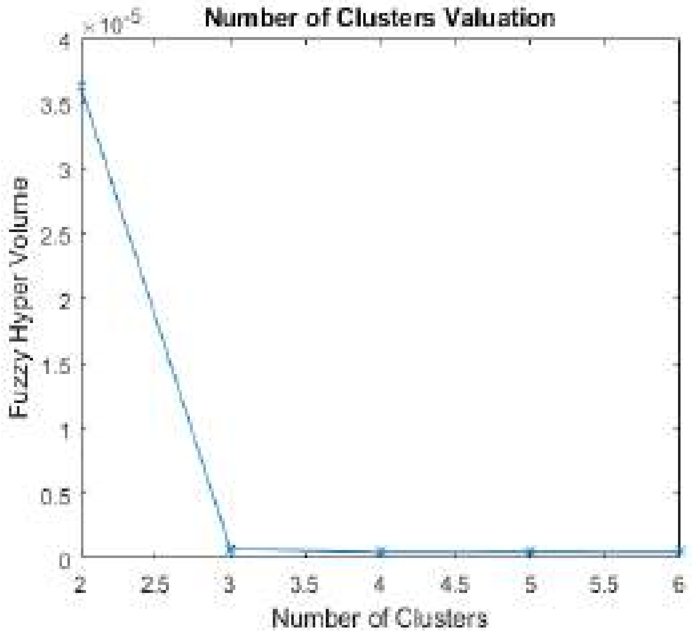

Fig. 4. Number of Clusters Valuation

The curve representing the hyper volume for different choice of number of clusters is shown in Fig. 4. According to the Fig. 4, three clusters is a reasonable number, for it yields a low hyper volume as well as keep the number of clusters low. Therefore, the Gustafson-Kessel algorithm for three clusters was performed. Fig. 5 shows how the dataset was divided in clusters.

For each cluster, the OKID algorithm was performed for determining the Markov Parameters. Two triangulations was performed as the input of OKID algorithm. Each triangulation is a result of calculations from data of a different pair of telemetry antennas. In this step the OKID was performed for each axis, and binded together afterwards. All the OKID turns used the parameter $n = 2$, $p = 2$ and $\alpha = \beta = 2$ for each axis. After this, the ERA algorithm determined the space state matrices for each cluster and each axis. Therefore, the final models have order 6 for each cluster. For simulating the final results of the filter, two different scenarios was simulated. In the first scenario, the filter was used together with the radar signal during the flight except during a certain period,

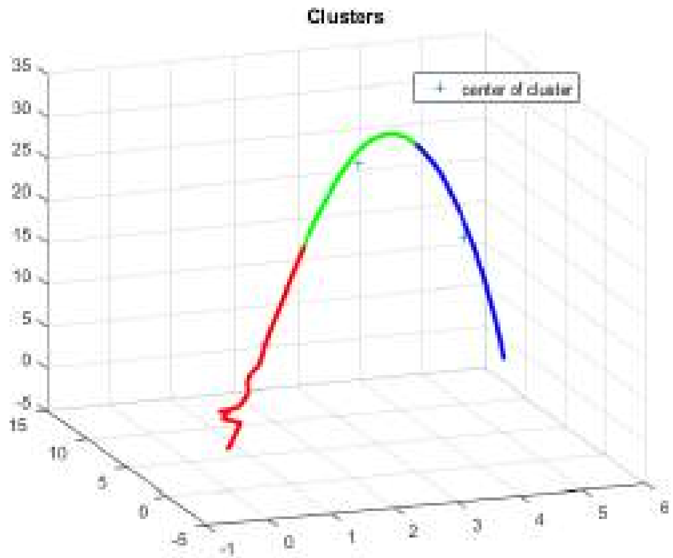

Fig. 5. Dataset divided in Clusters

that simulates a loss of signal from the tracking radar. During this period, only the triangulation signals were available. The simulation of loss of radar signal in the interval from 5 to 25 seconds, from 60 to 80 seconds and from 110 to 130 seconds, is shown in Fig. 6. The total flight took 166 seconds.

Note that the estimated position follows the real trajectory showing that the filter have succeeded in taking the triangulations signal as input, and yielding an estimated position close to the real trajectory.

For comparison purposes, Fig. 7 shows the absolute error during the simulated loss of signal interval. Note that the level of error in each interval in lower for the output of the designed filter if compared to each one of the triangulation signals.

Also, Table 1 shows some performance metrics for the position estimation during the interval that the loss of radar signal was simulated. Note that for all metrics, Root Mean Square Error (RMSE), Mean Absolute Error (MAE) and Mean Absolute Percentage Error (MAPE), the estimation using the designed filter has been performing better than a triangulation solely.

Table 1. Performance Metrics.

Tracking Method	RMSE	MAE	MAPE
Designed Filter	0.09	0.07	0.01
Triangulation 1	0.52	0.40	0.04
Triangulation 2	0.51	0.40	0.04

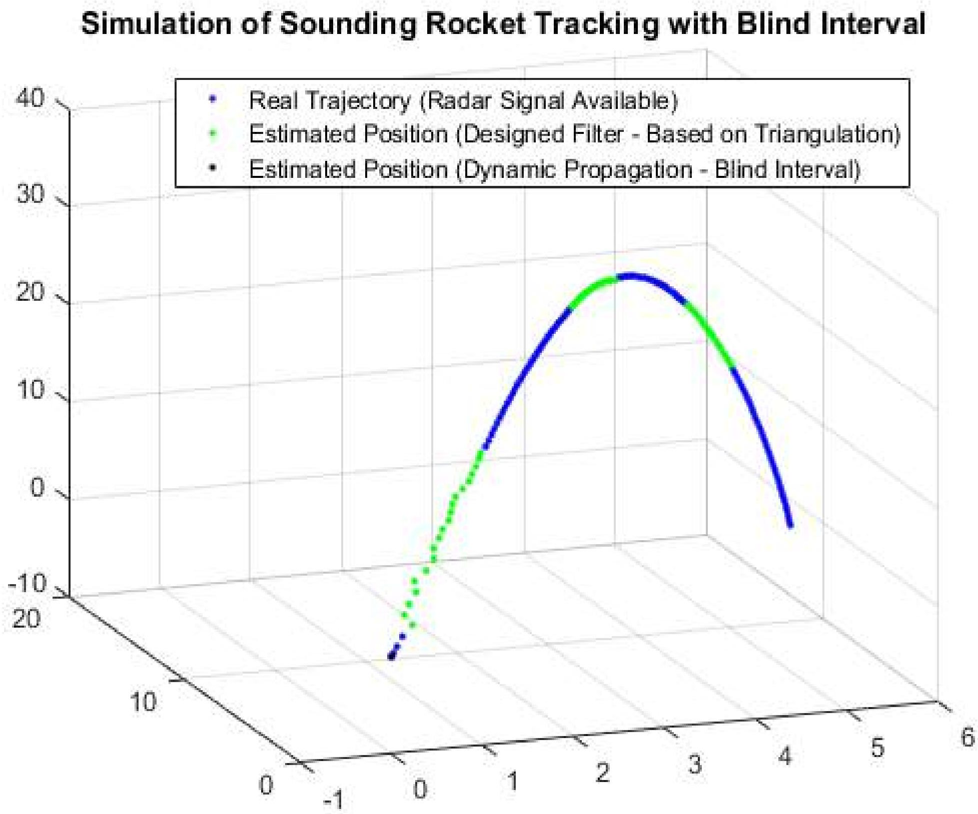

Fig. 6. Simulation of Sounding Rocket Tracking with Loss of Radar Signal

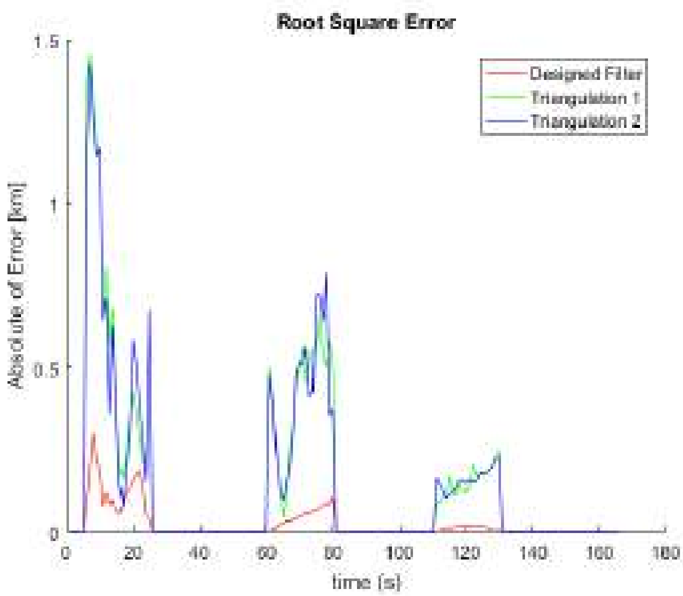

Fig. 7. Simulation of Sounding Rocket Tracking with Loss of Radar Signal

It is important to mention that the performance metrics presented here considered the simulation of three intervals lasting 20 seconds each, as said before. During the rest of the flight, the radar signal also served as input for the observer of the designed filter. The radar signal is important for it helps the filter improving its accuracy through its observer. If we consider the complete flight with no radar signal whatsoever, the designed filter won't have the same performance, and the triangulation solely, can eventually perform better.

The second simulated scenario includes intervals where neither the signals from radar nor signal from triangulation were available, then the propagation based on Kalman filter only was used during this complete blind interval. The Kalman filter used $Q = 10^{-3}I$, $R = 10^{-1}I$ and $P_0 = 10^3 I$. A loss of radar signal was simulated in the interval from 5 to 30 seconds, from 60 to 85 seconds and from 110 to 135 seconds, and the loss os triangulation signal was simulated in the interval from 20 to 25 seconds, from 80 to 85 seconds and from 125 to 130 seconds. Note that even with the scenario simulating the complete loss of signal, the filter could propagate the dynamic of the rocket based on historical data in order to estimate its position, as shown in Fig. 8.

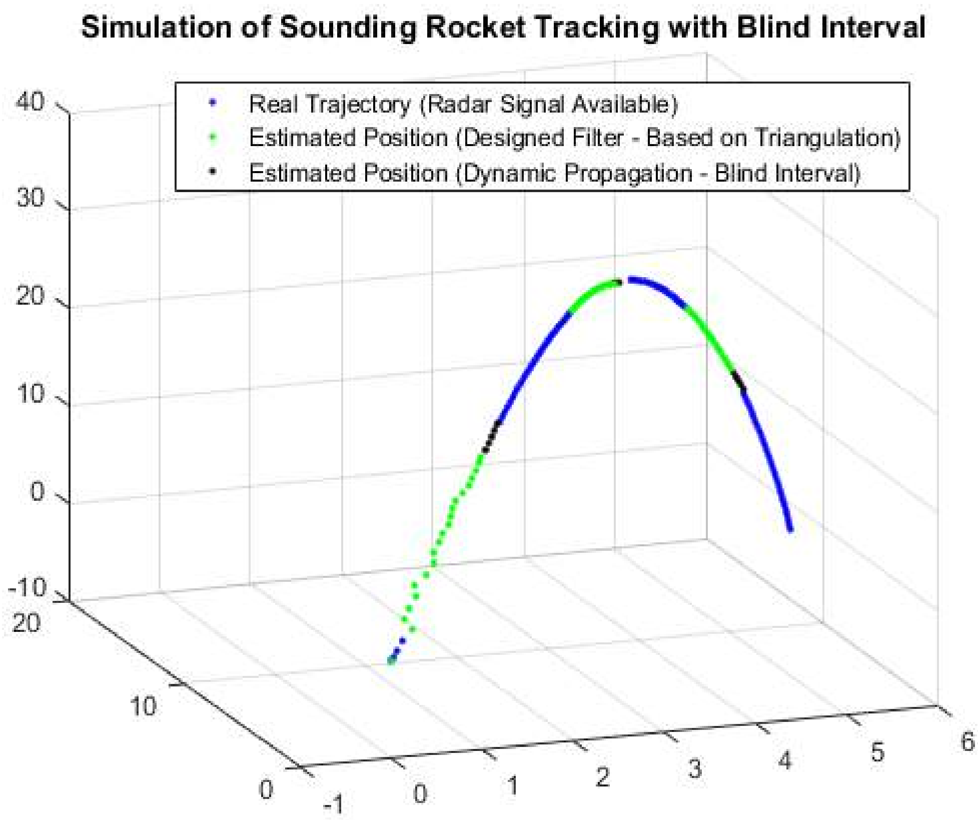

Fig. 8. Simulation of Sounding Rocket Tracking with Blind Interval

8 Final Considerations

This paper presented a method for estimating the position of a suborbital vehicle using triangulation of antennas. It aims to increase the launch site capability of keeping tracking the vehicle even if the tracking radar is not available at some interval. Also, this paper presented a method for designing a dynamic filter with observer, based on a fuzzy system, that improves the accuracy for the position estimation. Some simulations were performed considering different scenarios of loss of radar signal during a suborbital flight.

The results show that the triangulation method yields a rocket position estimation, however it is necessary to filter its output in order to increase the estimation accuracy. The triangulation alone carryout high estimation error level that can be related to sensors dynamics, to the antennas servomechanisms limitations or even to signal trajectory distortion during its propagation. A method for filtering the signal from the triangulation step was presented to decrease the error level. A fuzzy approach was described, where the rocket trajectory was divided into fuzzy subsets in order to deal with non-linearity of the error dynamics. The proposed filter could reduce the error estimation increasing the estimation accuracy. The filter design includes a observer that takes advantage of the tracking radar signal, during the period it is available, for updating the estimation.

In the scenario with a radar signal loss, the designed filter don't uses the observer, however still deliver a position estimation based on the triangulation step only. One could notice that the filter decreased the estimation error compared to the triangulation without the filter. Nonetheless, staying in this scenario can cause the estimation error rise, since the observer wouldn't update the estimation. Moreover the scenario of all signal loss was simulated, considering no signal from tracking radar neither from triangulation. For this design, a approach of the rocket dynamics propagation was considered. However if the vehicle is controlled, then it is important to be aware that the dynamic of the vehicle could change abruptly. To sum up, this paper has proposed a method for rocket position estimation that can collaborate for the launch site flight safety.

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Interactive Fuzzy Mathematics: a space with vector structure for calculation with uncertainties

Laécio C. Barros⋆1 and Estevão Esmi ⋆⋆1

Universidade Estadual de Campinas
Instituto de Matemática, Estatística e Computação Científica
Campinas, São Paulo 13081-970, Brasil
{laeciocb@ime.unicamp.br,eelaureano@ime.unicamp.br}

Abstract. One of the main virtues of mathematical structures that deal with uncertainties is their great modeling potential, in contrast to the deterministic theory. However, in such structures, there are difficulties to develop a theory of differential and integral calculus. Conversely, deterministic mathematics has a rich foundation for differential and integral calculus. This article is a survey that highlights a space formed by possibilistic variables, which includes both uncertainties (since the distributions of these variables can be interpreted as fuzzy numbers) and the Banach space structure, a very appropriate space for the development of differential and integral calculus, differential equations and mathematical analysis in general.

Keywords: Banach Spaces, Fuzzy Arithmetic, Joint Possibility Distribution, Possibilistic Variables, Interactive Fuzzy Numbers.

1 Introduction

One of the great areas in mathematics is the one that refers to Differential and Integral Calculus. It is quite developed in the classical (or deterministic) mathematics, as in the case of the real calculus or $\mathbb{R}^n$. The concepts of derivative and integral are well established in the Banach space, which is a complete normed vector space. In such spaces, the definitions of the Fréchet derivative and the Riemann integral are highlighted [1,2].

If in the dynamic modeling of a phenomenon there is some aspect of uncertainty, for example in a parameter, when we opt for deterministic mathematics it is necessary that this uncertainty is disregarded at the beginning of the process, so that a precise value (a real number) is adopted for the respective parameter. This is usually done by adopting an average value (either the mode or another statistical measure) estimated from data that are often inaccurate and/or incomplete. A strong criticism (among others) is that this procedure prevents possible

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states (responses) of the model, due to uncertainties that are not considered. On the other hand, if one can adopt a mathematical tool that makes possible the uncertainty “flow” along with the dynamics, there may be an enlarging of the answers. Furthermore, if the adopted tools which allow “to exclude” the uncertainty at an appropriate moment, there is certainly an enlarging of the quality of the solution obtained, as uncertainty about the past state of a process can eventually affect its evolution as a whole. Therefore, “excluding” the uncertainty at the beginning of the process can produce very different answers, for example, from those in which the uncertainty is excluded at the end.

On the one hand, if the theory of calculus is well established in Banach spaces, as is the case with $\mathbb{R}^n$, on the other hand, there is a limitation in the use of this theory to deal with phenomena in which aspects of uncertainty are relevant in its study. In general, most spaces that arise from theories that deal with uncertainties - namely, stochastic theory and fuzzy set theory - do not fulfill all the requirements of a Banach space, which makes it difficult to adequately or satisfactorily “transpose” the concepts of derivation and integration in this context.

In stochastic calculus, it is assumed that the state of the process at each instant is a random variable, in such a way that it can produce an answer of the form: “the variable takes on value 2 with probability 0.7”, which is wider than simply answering that the variable is equal to 2, as occurs when we use deterministic theory.

This increasing of information in the answer has a difficulty in the manipulation (arithmetic) of random variables since it involves the so-called joint distributions. Intuitively, such a concept is what makes it possible to mathematically “operate” the variables involved. That is, from joint distributions, two random variables are added, subtracted, multiplied, or divided. Thus, the result of an arithmetic operation on two random variables depends on the respective joint probability distribution. Now, due to a particularity of the arithmetic (subtraction) on random variables, it is not possible to define derivative through the limit of quotients of differences, as it is done in real calculus theory. This fact is corroborated by the so-called *random walk* or *Brownian motion*, in which a particle can move in any direction from its current position, clearly hindering the notion of a tangent line at any moment. So, roughly speaking, the theory of stochastic calculus boils down to the theory of integration of stochastic processes [3], excluding even the notion of Fréchet derivative.

In order to compare probability theory with fuzzy set theory, from a mathematical point of view in the treatment of uncertainties, we can say that the first one (probability theory) has its origin in the study of the frequency (repetition) with which an event occurs, and the second one (fuzzy set theory) studies the recognition of a certain event. On the other hand, the first evaluates the risk through probability and the second does it through the degree of membership, which makes the fuzzy approach a set theory.

The first article dealing with fuzzy in mathematics, entitled “Fuzzy sets” [4], appeared with Zadeh (1965). Initially, an arithmetic structure on fuzzy sets was developed extending the interval arithmetic via the Representation Theorem (see Lemma 2.1 in [5]). As it is well known, such operations present some "inconveniences", such as the increase of the diameter (that is, in the uncertainty) of the result in comparison to the diameters of the operands. These arithmetic operations are often called "standard operations' on fuzzy numbers (see Subsection 2.1), whose result depends only on the operands, without needing additional information about elements of the operands. Without wishing to deepen debates on terminology, we inform that these operations are typical of the so-called **conjunctive sets**, which are associated with **ontic variables**, that is, variables whose values are sets, in contrast to **epistemic variables**, which assume a single value within a set of possibilities, called **disjunctive set** [6], [7]. Examples of epistemic variables are the famous **random variables**, and as we present in Section 3, also the **possibilistic variables**.

Similar to the arithmetic on random variables, the study of arithmetic on possibilistic variables also uses the concept of joint possibilistic (membership) distribution (see [8],[9],[10]). In this case, some inconveniences pointed out in arithmetic on ontic variables disappear (See Subsection 2.1). One of the first articles dealing with fuzzy derivative for "conjunctive" fuzzy set-valued function is [5] and for "disjunctive" fuzzy set-valued function is [10].

We end this introduction saying that one of the motivations for writing this text is to highlight a class known as **linearly correlated fuzzy numbers** or **linearly interactive fuzzy numbers** (whose precise term should be **linearly interactive fuzzy variables**, as we argued in Section 3). This class includes both aspects of uncertainties (since their distributions can be interpreted as membership functions of fuzzy numbers) and the Banach space structure, which is seen as an ideal framework for development of differential and integral calculus, as it is known in classical literature.

2 Preliminary

A fuzzy (sub)set A of a universe X is characterized by a membership function $\varphi_A : X \to [0,1]$, where $\varphi_A(x)$ represents the membership degree of the element x in A. The class of fuzzy sets over X is denoted by $\mathcal{F}(X)$.

Definition 1 (Fuzzy number) *[11] A fuzzy subset A is called* fuzzy number *when the universe, in which its membership function φ_A is defined, is the set of real numbers $\mathbb{R}$ and satisfies the conditions:*

(i) all the α-levels of A are closed and nonempty intervals;
(ii) $suppA = \{x \in \mathbb{R} : \varphi_A(x) > 0\}$ is bounded.

Thus, the α-levels of the fuzzy number A, given by $[A]_\alpha = \{x \in \mathbb{R} : \varphi_A(x) \geq \alpha\}$, with $0 < \alpha \leq 1$ and $[A]_0 = cl(suppA)$, where $cl(Y)$ is the closure of the set Y, are closed intervals that we denote by

$$[A]_\alpha = [a_\alpha^-, a_\alpha^+].$$

The diameter of A is given by $diamA := a_0^+ - a_0^-$. We denote by $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers. From definition, a real number r can be regarded as a degenerated interval $[r, r]$ and as a fuzzy number with the membership function given by its characteristic function and, in this case, for the sake of simplicity, we denote the corresponding degenerated interval and the fuzzy number by r.

We take this opportunity to clarify that the fuzzy number A is said to be conjunctive when $[A]_\alpha$ are conjunctive, and A is said to be disjunctive when $[A]_\alpha$ is disjunctive, for all $\alpha \in [0, 1]$.

The result below identifies a fuzzy number as a family of intervals (see [5], [12]).

Lemma 1 *Let A_α, $\alpha \in [0, 1]$, be a family of closed and nonempty intervals of $\mathbb{R}$. If*

- $A_{\alpha_2} \subset A_{\alpha_1}$ *whenever* $\alpha_1 \leq \alpha_2$
- $\bigcup A_\alpha \subset A_0$
- $A_\alpha = \bigcap\limits_{k \geq 0} A_{\alpha_k}$ *with* $\alpha_k \to \alpha$,

then there is a unique fuzzy number so that $[A]_\alpha = A_\alpha$.

Let $f : X \to Y$ and $A \in \mathcal{F}(X)$. The **Zadeh extension** of f in A is the fuzzy set $\hat{f}(A) \in \mathcal{F}(Y)$ whose membership function is given by

$$\varphi_{\hat{f}(A)}(y) = \sup_{x : f(x) = y} \varphi_A(x). \tag{2.1}$$

For the case where f is a function with multiple arguments, the Zadeh extension is defined as follows. Let $f : X_1 \times \ldots \times X_n \to Y$ and $A_i \in \mathcal{F}(X_i)$, $i = 1, \ldots, n$. The Zadeh extension of f in $(A_1, \ldots, A_n)$ is the fuzzy set $\hat{f}(A_1, \ldots, A_n) \in \mathcal{F}(Y)$ whose membership function is given by

$$\varphi_{\hat{f}(A_1, \ldots, A_n)}(y) = \sup_{(x_1, \ldots, x_n) : f(x_1, \ldots, x_n) = y} \left[\varphi_{A_1}(x_1) \wedge \ldots \wedge \varphi_{A_n}(x_n)\right], \tag{2.2}$$

where the symbol $\wedge$ stands for the minimum operator.

Theorem 1 (On the Continuity of The Zadeh's Extension [13], [14]) *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \in \mathbb{R}_{\mathcal{F}}$, then $\hat{f}(A) \in \mathbb{R}_{\mathcal{F}}$ and $[\hat{f}(A)]_\alpha = f([A]_\alpha)$ for all $\alpha \in [0, 1]$.*

Similarly, if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $A_i \in \mathbb{R}_{\mathcal{F}}$, $i = 1, \ldots, n$, then $\hat{f}(A_1, \ldots, A_n) \in \mathbb{R}_{\mathcal{F}}$ and $[\hat{f}(A_1, \ldots, A_n)]_\alpha = f([A_1]_\alpha, \ldots, [A_n]_\alpha)$.

2.1 Standard arithmetic on fuzzy numbers

The operations given by **standard arithmetic** on fuzzy numbers are obtained from standard interval arithmetic on the α-levels of the operands combined with Lemma 1. More precisely, let $A, B \in \mathbb{R}_{\mathcal{F}}$ and $\star \in \{+, -, \cdot, \div\}$. The fuzzy number $A \star B$ is defined through the α-levels as follows

1. $[A+B]_\alpha = [a^-_\alpha + b^-_\alpha, a^+_\alpha + b^+_\alpha]$;
2. $[A-B]_\alpha = [a^-_\alpha - b^+_\alpha, a^+_\alpha - b^-_\alpha]$;
3. $[A \cdot B]_\alpha = [\min Y_\alpha, \max Y_\alpha]$, where $Y_\alpha = \{a^-_\alpha b^-_\alpha, a^-_\alpha b^+_\alpha, a^+_\alpha b^-_\alpha, a^+_\alpha b^+_\alpha\}$;
4. $[A \div B]_\alpha = [\min Z_\alpha, \max Z_\alpha]$, where $Z_\alpha = \left\{\frac{a^-_\alpha}{b^-_\alpha}, \frac{a^-_\alpha}{b^+_\alpha}, \frac{a^+_\alpha}{b^-_\alpha}, \frac{a^+_\alpha}{b^+_\alpha}\right\}$ if $0 \notin [B]_0$,

for all $\alpha \in [0,1]$.

It is possible to prove (see [15],[12]) that each item above satisfies Lemma 1 so that each defines a fuzzy number.

Note that the operations above have the "inconveniences" $A - A \neq 0$ and $A \div A \neq 1$ if $A \notin \mathbb{R}$. The Hukuhara difference, given by $C = A -_H B \Leftrightarrow A = B + C$, eliminates the defect $A - A \neq 0$ if $A \notin \mathbb{R}$. This fact elected it as a candidate for the notion of derivative of an interval function (later extended to fuzzy set-valued functions) via limit of quotients of differences [5]. However, $C = A -_H B$ does not exist when $diamB > diamA$.

As we already mentioned, the above arithmetic operations on α-levels coincide with the operations on intervals (which are classical sets) via Minkowski arithmetic. Recalling that the result of Minkowski operation $*$ on intervals, say X and Y, is given by the set of values $x * y$ for ALL possible ordered pairs, $\{(x,y) \in X \times Y\}$. Such a procedure suggests a kind of "independence" between X and Y. If it is necessary to adopt some rule (restriction) in choosing the pairs (x,y) involved in the operation, the result may be different from that given by standard arithmetic, which depends only on the operands involved. For example, for $X = [1,2]$ and $Y = [2,4]$, we have $X + Y = [1,2] + [2,4] = [3,6]$. On the other hand, if the pairs that are part of the sum are restricted to $R = \{(x,y) : y = -2x+6, x \in [1,2]\}$, then we define $X +_R Y = \{x + (-2x+6) : x \in [1,2]\} = \{6 - x : 1 \leq x \leq 2\} = [4,5]$. It is worth noting that all elements $x \in X$ and $y \in Y$ that participate of the sum "$X +_R Y$" are such that for each $x \in X$ the element $y \in Y$ is already fixed and is given by $y = -2x + 6$. This restriction on the pairs (x,y) that participate of the operation is a kind of "dependency" between the sets X and Y. This simple example is a typical case of what Zadeh called **interactivity** between fuzzy sets [16], which also takes into account the membership functions of the fuzzy sets involved.

3 Interactivity

First of all, we want to note that the term "interactive fuzzy numbers" is common in the literature [8],[10]. However, we believe that the appropriate terminology is **possibilistic (or fuzzy) variables interactive** (see [7]). In fuzzy analysis, the concept of **interactivity** is directly related to the concept of **dependence** in stochastic analysis, whose mathematical objects involved are random variables. In this case, two (or more) random variables may be dependent or independent. Note that the concept of (in)dependence concerns random variables and not their probability distributions which are "fixed" and can be normal, Poisson, binomial, beta, etc. Similarly, possibilistic numerical variables whose distributions of possibilities can be identified with membership functions of fuzzy numbers may be interactive or non-interactive. Thus, the notion of interactivity is related to possibilistic variables and not to their corresponding possibility distributions which are fixed. However, due to the vast literature with the terminology **interactive fuzzy numbers** [8],[9],[10],[17],[18], here we also commit this abuse and we use this terminology instead of **interactive possibilistic variables**.

In the case of numerical variables, it is interesting to note that both random variables and possibilistic variables fall into the so-called **epistemic variables**, which have a single value, that is, the values assumed by them are real numbers within a specified range, so, only one numerical value must be assumed, although there are vast possibilities of values for each of them [7].

Let A and B be fuzzy numbers (classical subsets, random variables, or possibilistic variables). The aim here is to analyze A and B together (i.e., bivariate analysis). In a language from the set theory, φ_A and φ_B can be seen as "quantifiers" (weights) of the qualities of the elements of A and B.

The fuzzy subset J of $\mathbb{R}^2$ is a joint distribution (of membership, or of possibilities) for A and B if their projections on the axes coincide with A and B, i.e., if $P_1(J) = A$ and $P_2(J) = B$. Figure 1 illustrates a possible joint distribution for two fuzzy numbers A and B.

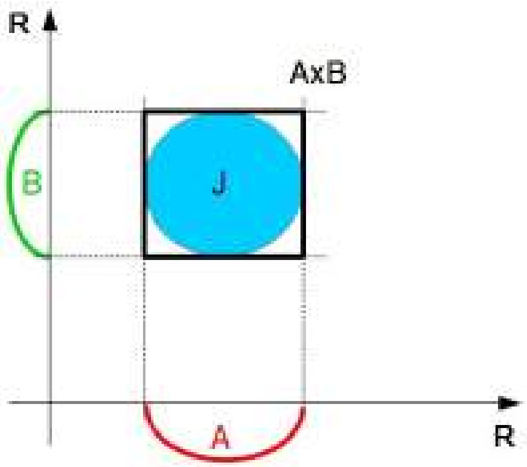

Fig. 1. J represents a joint probability distribution for A and B.

Remark 1 *:*

- *The requirement that $P_1(J) = A$ and $P_2(J) = B$ is due to the fact that every element of A and every element of B must be part of some pair in J.*

- *If A and B are random variables and J is a joint probability distribution, then $\varphi_J(x, y)$ is such that the marginal distributions are given by $\varphi_A(x)= \int \varphi_J(x, y)dy$ and $\varphi_B(y)=\int \varphi_J(x, y)dx$. In this case, A and B are independent if and only if $\varphi_J(x, y) = \varphi_A(x)\varphi_B(y)$ for all (x, y). If this condition is not satisfied, then A and B are dependent.*

– *If A and B are fuzzy sets (i.e., possibilistic variables) and J is a joint possibility (or membership) distribution, then the membership degree of the pair (x, y) to J is related to the membership degrees from x to A and from y to B through marginal membership distributions: $\varphi_A(x) = sup_y \varphi_J(x, y)$ and $\varphi_B(y) = sup_x \varphi_J(x, y)$. We say that A and B are non-interactive if, and only if, $\varphi_J(x, y) = \varphi_A(x) \wedge \varphi_B(y)$ for all (x, y). Otherwise, A and B are said to be interactive.*

– *If A and B are classical sets or intervals (that is, possibilistic variables with uniform distributions), then they are non-interactive if, and only if, $J = suppJ = A \times B$, since $\varphi_J(x, y) = \varphi_A(x) \wedge \varphi_B(y) = \varphi_{A\times B}(x, y)$ for all (x, y).*

Next, we focus on the fuzzy case (or possibilistic variables).

Definition 2 *A fuzzy subset $J \in \mathcal{F}(\mathbb{R}^n)$ is a joint possibility (or membership) distribution for $A_1, \ldots, A_n \in \mathbb{R}_{\mathcal{F}}$ if $P_i(J) = A_i, \forall i = 1, \ldots, n$, that is,*

$$\varphi_{P_i(J)}(y) = sup_{(x_1,\ldots,x_n)\in\mathbb{R}^n : x_i = y} \varphi_J(x_1, ..., x_n), \quad \forall y \in \mathbb{R}. \tag{3.3}$$

If $\varphi_J(x_1, ..., x_n) = \varphi_{A_1}(x_1) \wedge \ldots \wedge \varphi_{A_n}(x_n)$ (or equivalently $J = A_1 \times \ldots \times A_n$), then $A_1, \ldots, A_n$ are said to be non-interactive. Otherwise, $A_1, \ldots, A_n$ are called J-interactive, or simply interactive.

It is worth noting that the concept of interactivity is defined on an ordered n-tuple of fuzzy numbers since it is characterized by a given joint possibility distribution such that its projection onto ith axis corresponds exactly to the ith element of the n-tuple.

Definition 3 (linearly correlated [9]) *We say that the fuzzy numbers A and B are linearly correlated (or completely correlated or linearly interactive) if there are $q, r \in \mathbb{R}$ such that their joint distribution of possibilities $J_{q,r}$ is given by*

$$\varphi_{J_{q,r}}(x, y) = \chi_L(x, y)\varphi_A(x) \qquad \text{and} \qquad \varphi_{J_{q,r}}(x, y) = \chi_L(x, y)\varphi_B(y)$$

for all $x, y \in \mathbb{R}$, where χ_L is the characteristic function of the line $L = \{(u, qu + r) \mid u \in \mathbb{R}\}$. If $q > 0$ ($q < 0$), we speak of a positive (negative) correlation.

We would like to highlight that Definition 3 extends slightly the original definition provided in [9] by allowing $q = 0$. From Definition 3, assuming that A and B are linearly correlated is equivalent to assuming that A and B are $J_{q,r}$-interactivity, for some $q, r \in \mathbb{R}$, where $P_1(J_{q,r}) = A$ and $P_2(J_{q,r}) = B$. In this case, we can prove that $B = \hat{f}(A) = qA + r$ since $J_{q,r}$ is defined in terms of L which is the graph of the linear function $f(x) = qx + r$. Furthermore, by Theorem 1, we have $[B]_\alpha = q[A]_\alpha + r$. It is worth noting that saying "A and B are $J_{q,r}$-interactivity" is not the same as saying "B and A are $J_{q,r}$-interactivity". In the last case, we have $P_1(J_{q,r}) = B$, $P_2(J_{q,r}) = A$, and $[A]_\alpha = q[B]_\alpha + r$ instead.

3.1 Interactive Arithmetic

As we commented above, what characterizes the interactivity between A and B are what we call joint possibility distributions. The degree of membership of each real number z to the result $(A * B)$ of the arithmetic operation "$*$" is calculated through A, B, and the joint distribution of A and B. Again, this procedure is analogous to that used in arithmetic operations on random variables, where the joint probability distribution is needed to obtain the distribution of the arithmetic operation "$*$" in question, which is a random variable.

The arithmetic operations on J-interactive fuzzy numbers are obtained via the sup-J extension principle applied to the classical arithmetic operations $\star \in \{+, -, \cdot, \div\}$. More precisely, let $A, B \in \mathbb{R}_{\mathcal{F}}$, we define the fuzzy number $A \star_J B$ with membership function

$$\varphi_{A\star_J B}(z) = \sup_{x*y=z} \varphi_J(x,y). \tag{3.4}$$

When the fuzzy numbers A and B are non-interactive, their joint probability distribution J has membership function $\varphi_J(x,y) = \varphi_A(x) \wedge \varphi_B(y)$, so that the arithmetic operations are given by

$$\varphi_{A\star_J B}(z) = \sup_{x*y=z} \varphi_A(x) \wedge \varphi_B(y), \tag{3.5}$$

which is often called arithmetic via the Zadeh extension Principle.

Remark 2 *: One can note the similarity of the formula* (3.4) *with that of the stochastic case: if A and B were random variables with joint probability distribution J, then*

$$\varphi_{A\star_J B}(z) = \int_{x*y=z} \varphi_J(x,y)dxdy,$$

where $\varphi_J(x,y)$ is the joint probability distribution function of A and B.

As in the case of standard arithmetic, we also have two ways to handle interactive arithmetic. One explicitly involves the joint distribution (3.4) which is similar to arithmetic for random variables. The other way is from its α-levels (which refers to interval arithmetic) obtained by applying Theorem 1 since each arithmetic operation $\star : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

If in Equation (3.4) every $[J]_\alpha$ is compact, then we have that [19,20,21]:

$$[A \star_J B]_\alpha = \left[\min_{(x,y)\in[J]_\alpha} (x \star y), \max_{(x,y)\in[J]_\alpha} (x \star y) \right], \tag{3.6}$$

for all $\alpha \in [0,1]$. For the non-interactive case, i.e., $[J]_\alpha = [A]_\alpha \times [B]_\alpha$, it becomes

$$[A \star B]_\alpha = \left[\min_{(x,y)\in[A]_\alpha\times[B]_\alpha} (x \star y), \max_{(x,y)\in[A]_\alpha\times[B]_\alpha} (x \star y) \right], \tag{3.7}$$

for all $\alpha \in [0,1]$.

Proposition 1 (Addition and subtraction on linearly correlated fuzzy numbers[10]) *Let A and B be $J_{q,r}$-linearly correlated fuzzy numbers. For all $\alpha \in [0,1]$, we have*

$$[A +_{J_{q,r}} B]_\alpha = \begin{cases} [a_\alpha^- + b_\alpha^-, a_\alpha^+ + b_\alpha^+] \ , \ \mathit{if}\, 0 \leq q \\ [a_\alpha^- + b_\alpha^+, a_\alpha^+ + b_\alpha^-] \ , \ \mathit{if}\, -1 \leq q < 0 \\ [a_\alpha^+ + b_\alpha^-, a_\alpha^- + b_\alpha^+] \ , \ \mathit{if}\, q < -1 \end{cases} , \tag{3.8}$$

and

$$[A -_{J_{q,r}} B]_\alpha = \begin{cases} [a_\alpha^+ - b_\alpha^+, a_\alpha^- - b_\alpha^-] \ , \ \mathit{if}\, 1 \leq q \\ [a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+] \ , \ \mathit{if}\, 0 \leq q < 1 \\ [a_\alpha^- - b_\alpha^+, a_\alpha^+ - b_\alpha^-] \ , \ \mathit{if}\, q < 0 \end{cases} . \tag{3.9}$$

Although the formulas above are for disjunctive intervals, they may coincide with formulas for conjunctive arithmetic. If $q \geq 0$, then (3.8) coincides with the standard addition whereas (3.9) coincides with the generalized Hukuhara difference or, for short, the gH-difference.

We end this section by emphasizing some fundamentals already highlighted throughout the exposition above, because we consider them to be important.

Although the author does not use the terminology “interactivity” in [22], the reader can find an interesting study on the topic discussed here. In this article, the author focuses on arithmetic operations whose results depend on additional information about the operands. He considers also epistemic variables. Although he does not use this term, he makes it clear in his study that each variable must assume only one value within a possible range. However, he gives little emphasis to variables and works explicitly with the sets of values they can assume. Thus, in our view, the author opts for a language from the set theory to deal with fuzzy arithmetic operations with additional information (considered to be requisite constraints). Such an information is formalized from bivariate fuzzy relations, that is, a fuzzy subset of $\mathbb{R} \times \mathbb{R}$, a typical term from the set theory.

In our approach, the operands and the response are “possibilistic” epistemic variables, just like in the stochastic case. Furthermore, the response is a variable whose distribution can be changed according to the type of interactivity between the operands. Interactivity (dependence, in the stochastic case) is formally given by a joint possibility distribution, whose counterpart in the approach of [22] is the fuzzy relation that models the additional information. Since the stochastic approach is already well established, it seems to us that the approach discussed here is a good option for the formalization of interactivity. What connects our approach with that from the set theory, presented by Klir [22], is the fact that the possibility distributions of possibilistic variables are real functions whose codomain is the range $[0,1]$, which can be identified as membership functions of fuzzy sets. Thus, there is a clear connection between the possibilistic variable (which takes values on the real line) and the fuzzy number, identified by a membership function. However, there are strong criticisms of using the terminology: “interactivity between fuzzy numbers”. For example, this may give the idea that the result of an operation on (fuzzy) numbers only depends on the operands and nothing else. In fact, in our opinion, this is not a conceptual problem but a terminology problem. Thus, the terminology "interactivity between fuzzy numbers" is a abuse of language which refers to the existence of an interactivity between possibilistic variables with possibility distributions functions given in terms of membership functions of fuzzy numbers. The assumption of the existence of certain "interactivity" between the operands is in agreement with ideas exposed in [23]. We believe that the option for the terminology "interactive possibilistic variables" or "interactive fuzzy numbers" depends on the most adequate interpretation for the phenomenon at hand: possibility or membership. In this work, we also use the term "interactive fuzzy numbers" instead of "interactive possibilistic variables", which despite being an abuse of language is often found in the literature.

As seen in Subsection 2.1, the standard arithmetic is proper for ontic variables (or set-valued function), whose values assumed by them are conjunctive (classical or fuzzy) sets. Objectively, it is given from the interval arithmetic on the α-levels of the operands, without the need of any additional information in contrast to the study of [23]. Intuitively, this fact tells us that there is a kind of "independence" in standard arithmetic. However, the notion of dependence (or interactivity) is specific to epistemic variables, whose sets (classical or fuzzy) of possible values assumed by them are disjunctive. Thus, even though the α-levels in standard arithmetic coincide with the α-levels in NON-interactive arithmetic (where the joint distribution is given by the minimum t-norm, i.e., the arithmetic given by the Zadeh extension principle), there is no reason to perform comparisons between standard arithmetic and interactive arithmetic. In the first case, the fuzzy numbers are conjunctive and in the second one they are disjunctive. Furthermore, mathematical concepts such as derivative and integral are distinct from each other. One is for multi-valued functions and the other is for single-valued functions. This makes all the difference in using one approach or another to solve a problem at hand.

4 The space $\mathbb{R}_{\mathcal{F}(A)}$

Although the theory of differential and integral calculus for fuzzy number spaces has evolved with the notion of interactivity, it still encounters many problems due to the lack of adequate mathematical structure. However, as we will see, the class of non-symmetric linearly correlated fuzzy numbers (or possibilistic variables) forms a Banach space, which is the ideal space to develop a theory of differential and integral calculus.

A deeper study of what we present here can be found in [17,24,25]. For each $A \in \mathbb{R}_{\mathcal{F}}$ consider the operator $\Psi_A : \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ that associates each vector $(q, r) \in \mathbb{R}^2$ to the fuzzy number $\Psi_A(q, r) = qA + r,\ \forall q, r \in \mathbb{R}$. The range Ψ_A is denoted by $\mathbb{R}_{\mathcal{F}(A)} = \{\Psi_A(q, r) \,|\, (q, r) \in \mathbb{R}^2\}$.

Remark 3 *:*

– *$\mathbb{R}$ can be immersed in $\mathbb{R}_{\mathcal{F}(A)}$ since $r \in \mathbb{R}$ can be identified as $\Psi_A(0,r) \in \mathbb{R}_{\mathcal{F}(A)}$, that is, $\mathbb{R} \subseteq \mathbb{R}_{\mathcal{F}(A)}$.*
– *$\mathbb{R}_{\mathcal{F}(A)}$ is the class of all fuzzy numbers linearly correlated with each other and also with A.*
– *If $B \in \mathbb{R}_{\mathcal{F}(A)} \setminus \mathbb{R}$, then $\mathbb{R}_{\mathcal{F}(A)} = \mathbb{R}_{\mathcal{F}(B)}$.*

In order not to make the reader anxious, what follows is to show that if A is non-symmetric, then $\mathbb{R}_{\mathcal{F}(A)}$ is a Banach space. See Theorem 4 below.

Theorem 2 *[17] Let $(q,r),(u,v) \in \mathbb{R}^2$ and $A \in \mathbb{R}_{\mathcal{F}}$. The operator Ψ_A satisfies the properties:*

1. *$\Psi_A(q+u, r+v) \subseteq \Psi_A(q,r) + \Psi_A(u,v)$;*
2. *$\Psi_A(q+u, r+v) = \Psi_A(q,r) + \Psi_A(u,v)$ if $\frac{u}{q} \geq 0$ for $q \neq 0$.*

Theorem 3 *[17] Given $A \in \mathbb{R}_{\mathcal{F}}$, the operator Ψ_A is injective if and only if A is non-symmetric, i.e., if there is no $x \in \mathbb{R}$ such that $\varphi_A(x-y) = \varphi_A(x+y)$, for all $y \in \mathbb{R}$.*

Theorem 4 *[17] Let $A \in \mathbb{R}_{\mathcal{F}}$ be non-symmetric. The set $\mathbb{R}_{\mathcal{F}(A)}$ is a normed vector space over $\mathbb{R}$ with addition, scalar product, and norm defined as follows:*

i) $B +_A C = \Psi_A(\Psi_A^{-1}(B) + \Psi_A^{-1}(C)) = (q_B + q_C)A + (r_B + r_C)$, $\forall B, C \in \mathbb{R}_{\mathcal{F}(A)}$;
ii) $\eta \cdot_A B = \Psi_A(\eta \Psi_A^{-1}(B)) = (\eta \cdot q_B)A + (\eta \cdot r_B)$, $\forall B \in \mathbb{R}_{\mathcal{F}(A)}$ *and* $\forall \eta \in \mathbb{R}$;
iii) $B -_A C = \Psi_A(\Psi_A^{-1}(B) - \Psi_A^{-1}(C)) = (q_B - q_C)A + (r_B - r_C)$, $\forall B, C \in \mathbb{R}_{\mathcal{F}(A)}$;
iv) $\| B \|_A = \| \Psi^{-1}(B) \|_\infty$, *for all* $B \in \mathbb{R}_{\mathcal{F}(A)}$.

Furthermore, $\mathbb{R}_{\mathcal{F}(A)}$ is isomorphic to $\mathbb{R}^2$ via the linear isomorphism Ψ_A.

From Theorem 4, the set $\mathbb{R}_{\mathcal{F}(A)}$ forms a Banach space. Note that if $r, \eta \in \mathbb{R}$, then $B +_A r = B + r$ and $\eta \cdot_A B = \eta \cdot B$. Furthemore, the set of non-symmetric fuzzy numbers is open and dense in $\mathbb{R}_{\mathcal{F}}$, with the metric D_∞ [18]. This indicates that every fuzzy number can be approximated as closely as desired by an element of some space $\mathbb{R}_{\mathcal{F}(A)}$.

5 Differential and integral calculus in $\mathbb{R}_{\mathcal{F}(A)}$ with non-symmetric A

Consider the Banach space $(\mathbb{R}_{\mathcal{F}(A)}, +_A, \cdot, \| \cdot \|_A)$. Here we summarize some results for the derivation and integration of functions $f : D \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}(A)}$. First, note that there are unique functions $q, r : D \to \mathbb{R}$ such that

$$f(t) = \Psi_A(q(t), r(t)) = q(t)A + r(t), \qquad \forall t \in D.$$

Moreover, if q and r are bounded (continuous), then f is also bounded (continuous).

Definition 4 (Derivation) *[17,25] Let* $f(t) = \Psi_A(q(t), r(t)) = q(t)A + r(t)$ *defined in the open set* $D \subset \mathbb{R}$*. We say that* f *is* Ψ*-differentiable in* $t \in D$ *if there is* $f'(t) \in \mathbb{R}_{\mathcal{F}(A)}$ *such that*

$$\lim_{h \to 0} \frac{f(t+h) -_A f(t)}{h} = f'(t). \tag{5.10}$$

The function f *is* Ψ*-differentiable in* D *if* f *is* Ψ*-differentiable at every* $t \in D$*.*

Theorem 5 *Function* $f : D \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}(A)}$ *is* Ψ*-differentiable in* t *if and only if functions* q *and* r *are differentiable in* t*. In this case, we have*

$$f'(t) = \Psi_A(q'(t), r'(t)) = q'(t)A + r'(t).$$

The first derivative concept for this type of function was given in [17], in the sense of Fréchet derivative. However, the concepts of Fréchet derivative and Ψ-derivative are equivalent in $\mathbb{R}_{\mathcal{F}(A)}$ [18].

Definition 5 (Riemann integral) *[26] The function* $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}(A)}$ *is said to be Riemann integrable if there is* $S \in \mathbb{R}_{\mathcal{F}(A)}$ *such that for every* $\epsilon > 0$ *there is* $\delta > 0$ *such that for every partition* $a = x_0 < x_1 < \ldots < x_n = b$ *with* $x_k - x_{k-1} < \delta$, $k = 1, \ldots, n$, *we have*

$$\| (A) \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) -_A S \|_A < \epsilon$$

where $t_k \in [x_{k-1}, x_k]$, $k = 1, \ldots, n$.

Theorem 6 *[24] The function* $f : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}(A)}$ *given by* $f(t) = q(t)A + r(t)$ *is Riemann integrable on* $[a, b]$ *if and only if the functions* $q, r : [a, b] \to \mathbb{R}$ *are Riemann integrable on* $[a, b]$*. In this case, we have*

$$\int_a^b f(s)ds = \left(\int_a^b q(t)dt \right) A + \int_a^b r(t)dt. \tag{5.11}$$

Theorem 7 (Fundamental theorems of calculus) *[24,25] Let* $f : [a, b] \to \mathbb{R}_{\mathcal{F}(A)}$*.*

i. If f *is continuous, then the function* $G(t) = \int_a^t f(s)ds$ *is* Ψ*-differentiable and* $G'(t) = f(t), \forall t \in [a, b]$*.*
ii. If f *is* Ψ*-differentiable and* f' *Riemann integrable, then*

$$\int_a^b f'(s)ds = f(b) -_A f(a).$$

Example 1 *Consider the function* $f : [0, 1] \to \mathbb{R}_{\mathcal{F}}$ *given by* $f(t) = \cos(\pi t)A + e^t$*, with* A *non-symmetric. Then, the* Ψ*-derivative of* f *is*

$$f'(t) = -\pi \sin(\pi t)A + e^t, \qquad \forall t \in [0, 1],$$

and the Riemann integral is

$$\int_0^1 f(t)dt = \left(\int_0^1 \cos(\pi t)dt \right) A + \left(\int_0^1 e^t dt \right) = e - 1 \in \mathbb{R}.$$

Remark 4 *:*

- *If the function* $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}(A)}$ *is* Ψ*-differentiable in* t*, then* f *is gH-differentiable in* t *and both derivatives coincide [27].*
- *As the Riemann Integral seen above considers interactivity, the notion of Riemann integral for functions of* $[a, b]$ *to* $\mathbb{R}_{\mathcal{F}(A)}$ *does not coincide with the usual notions of Aumann, Riemann, and Henstock integrals in the fuzzy sense [15], as we can see in Example 1 above.*

Because $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}(A)}$ is fully characterized by the functions $q(\cdot)$ and $r(\cdot)$ (see Theorems 3 and 4). Thus, theories in $\mathbb{R}^2$ such as differential equations, numerical methods of integration, etc., are fully transferred to f, via classical functions q and r. In what follows, we present an example of Partial Differential Equations, an almost intractable topic in fuzzy function theory via Hukuhara derivative and its generalizations, since the candidate solutions must satisfy Lemma 1 [28]. The fuzzy solutions obtained through our methods do not require the verification of the Representation Theorem.

5.1 Fuzzy linear partial differential equations in $\mathbb{R}_{\mathcal{F}(A)}$

We present here a study started in [26] on Fuzzy Linear Partial Differential Equation of the form

$$a\frac{\partial^2 U}{\partial x^2} +_A b\frac{\partial^2 U}{\partial x \partial t} +_A c\frac{\partial^2 U}{\partial t^2} +_A d\frac{\partial U}{\partial x} +_A e\frac{\partial U}{\partial t} +_A fU +_A G = 0, \tag{5.12}$$

where the coefficients a, b, c, d, e, f are real functions of (x, t) and $G(x, t) \in \mathbb{R}_{\mathcal{F}(A)}$ is given by $G(x, t) = g_1(x, t)A + g_2(x, t)$, with $A \in \mathbb{R}_{\mathcal{F}}$ non-symmetric, so that the solution of (5.12) is of the form $U(x, t) = q(x, t)A + r(x, t)$. Thus, we obtain the decoupled system for real functions $q(x, t)$ and $r(x, t)$:

$$a\frac{\partial^2 q}{\partial x^2} + b\frac{\partial^2 q}{\partial x \partial t} + c\frac{\partial^2 q}{\partial t^2} + d\frac{\partial q}{\partial x} + e\frac{\partial q}{\partial t} + fq + g_1 = 0 \tag{5.13}$$

and

$$a\frac{\partial^2 r}{\partial x^2} + b\frac{\partial^2 r}{\partial x \partial t} + c\frac{\partial^2 r}{\partial t^2} + d\frac{\partial r}{\partial x} + e\frac{\partial r}{\partial t} + fr + g_2 = 0, \tag{5.14}$$

so that the fuzzy solution of (5.12) is given by $U(x, t) = q(x, t)A + r(x, t)$, where $q(x, t)$ and $r(x, t)$ are real solutions of (5.13) and (5.14), respectively. Thus, the methods adopted to obtain the fuzzy solution $U(x, t)$ are those known from the classical theory that provide $q(x, t)$ and $r(x, t)$. This is illustrated in the study of the heat equation, where we use the methods of the Laplace transform and separation of variables.

It is worth noting that (5.12) only makes sense if the addition is interactive, since the standard (or non-interactive) addition could not result in $Zero$. In the following example, we adopt two solving methods to obtain the heat solution. First via Laplace Transform and then via variable separation.

Example 2 (Fuzzy heat equation) *Consider the following equation*

$$\frac{\partial U}{\partial t} -_A c^2 \frac{\partial^2 U}{\partial x^2} = 0 \Leftrightarrow \frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}, \quad c \neq 0, \tag{5.15}$$

where $A \in \mathbb{R}_{\mathcal{F}}$ is non-symmetric.

Solutions

– *Via Laplace Transform*

Here we study the Problem (5.15) *subject to the conditions*

$$\begin{gathered} U(x,0) = q(x,0)A + r(x,0) \\ U(0,t) = q(0,t)A + r(0,t) \\ \| U(x,t) \|_A = \| q(x,t)A + r(x,t) \|_A \text{ is bounded.} \end{gathered}$$

Considering the Theorem 5 and the injectivity of Ψ_A, the fuzzy problem (5.15) *is equivalent to the classic decoupled systems:*

$$\begin{cases} \dfrac{\partial q}{\partial t} = c^2 \dfrac{\partial^2 q}{\partial x^2} \\ q(x,0) \\ q(0,t) \\ |q(x,t)| \text{ bounded,} \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial r}{\partial t} = c^2 \dfrac{\partial^2 r}{\partial x^2} \\ r(x,0) \\ r(0,t) \\ |r(x,t)| \text{ bounded.} \end{cases} \tag{5.16}$$

As it can be seen in [26], the Laplace transform is used to obtain the solutions of (5.16).

Additionally, if we take $q(x,0) = q_0 \in \mathbb{R}$ and $q(0,t) = q_1 \in \mathbb{R}$, the classical solutions for $q(x,t)$ and $r(x,t)$ are given by

$$q(x,t) = \left(q_1 + (q_0 - q_1) erf\left(\frac{x}{2c\sqrt{t}}\right)\right)$$

and

$$\left(r_1 + (r_0 - r_1) erf\left(\frac{x}{2c\sqrt{t}}\right)\right),$$

so that the fuzzy solution of (5.15) *is given by*

$$U(x,t) = \left(q_1 + (q_0 - q_1) erf\left(\frac{x}{2c\sqrt{t}}\right)\right) A + \left(r_1 + (r_0 - r_1) erf\left(\frac{x}{2c\sqrt{t}}\right)\right).$$

The fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ can be interpreted as a temperature linearly correlated with the initial temperature of the object and erf is the Gauss error function [29].

– *Via Separations of variables*

Here we study the Problem (5.15) *on the interval* $[0, L]$ *under the homogeneous boundary condition and regular initial condition, that is,*

$$\begin{cases} (B.C.) & U(0,t) = q(0,t)A + r(0,t) = q(L,t)A + r(L,t) = U(L,t) \\ (I.C.) & U(x,0) = q(x,0)A + r(x,0) \end{cases},$$

with $q(x,0) = q(x)$ *and* $r(x,0) = r(x)$ *regular functions.*

In this case, the focus is to investigate a general solution for (5.15) *which takes the form*

$$U(x,t) = U_T(x,t) +_A \bar{U}(x) = (q_T(x,t) + \bar{q}(x))A + (r_T(x,t) + \bar{r}(x)),$$

where $U_T(x,t)$ $(q_T(x,t),\, r_T(x,t))$, *is a transient space and time-dependent function that decays to zero and* $\bar{U}(x)$ $(\bar{q}(x),\, \bar{r}(x))$ *is a stationary state solution. The separation of variables is to find* $U_T(x,t)$, *which means to write*

$$U_T(x,t) = S(x)T(t) = (q_S(x)q_T(t))A + r_S(x)r_T(t),$$

where S (q_S, r_S) *depends only on* x *and* T (q_T, r_T) *only on* t.

Again, considering the Theorem 5 and the injectivity of Ψ_A, *the fuzzy problem* (5.15) *with homogeneous boundary conditions and initial condition as above is equivalent to decoupled classical systems:*

$$\begin{cases} \dfrac{\partial q}{\partial t} = c^2 \dfrac{\partial^2 q}{\partial x^2} \\ (BC) : q(0,t) = q(L,t) = 0 \\ (IC) : q(x,0) = q(x), \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial r}{\partial t} = c^2 \dfrac{\partial^2 r}{\partial x^2} \\ (BC) : r(0,t) = r(L,t) = 0 \\ (IC) : r(x,0) = r(x). \end{cases} \tag{5.17}$$

To find q_S, q_T $(\neq 0)$ *we use the first equation of* (5.17), *so that*

$$q_S.q_T' = c^2 q_S''.q_T \Rightarrow \frac{q_T'}{q_T} = \frac{c^2 q_S''}{q_S}.$$

As the first member of the equality only depends on t *and the second member depends only on* x, *we can say that this equality only holds for all* t *and all* x *if*

$$\frac{q_T'}{q_T} = \frac{c^2 q_S''}{q_S} = k \Rightarrow \begin{cases} q_T' = kq_T \\ c^2 q_S'' = kq_S \end{cases},$$

for some constant k. *In this case,* q_T *and* q_S *are solutions of first and second order linear ordinary differential equations, respectively. Furthermore, these solutions are of the form*

$$q_T(t) = e^{kt} \text{ and } q_S(x) = e^{\pm\sqrt{\gamma}x}, \text{ where } \gamma = \sqrt{\frac{k}{c^2}}.$$

Now, due to the boundary condition, it is easy to see that $\bar{q}(x) = 0$ *and that*

$$q_S(0).q_T(t) = q_S(L).q_T(t) = 0.$$

Thus, as we are assuming $q_T \neq 0$, since $sin(0) = 0$, then, necessarily $k < 0$ and the second-order ODE above implies

$$q_S(x) = B \sin\left(\frac{n\pi x}{L}\right), \text{ with } n \in \mathbb{N}.$$

Therefore, the transient solution is given by

$$q_T(x,t) = Be^{-\frac{c^2n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right), \text{ with } n \in \mathbb{N}.$$

The stationary solution ($\bar{q}(x)$) must satisfy the equation $\frac{\partial^2 C}{\partial x^2} = 0$ and the boundary conditions. So it is easy to see that $\bar{q}(x) = 0$ for all x. Thus, the general solution is $q(x,t) = q_T(x,t)$. Finally, from a deeper analysis (see [29]) involving the initial condition $q(x,0) = q(x)$, we have the following general solution

$$q(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{c^2n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L}\int_0^L q(x) \sin\left(\frac{n\pi x}{L}\right) dx, \text{ with } n \in \mathbb{N}.$$

Analogously, we find the general solution for $r(x,t)$, which is given by

$$r(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{c^2n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L r(x) \sin\left(\frac{n\pi x}{L}\right) dx, \text{ with } n \in \mathbb{N}.$$

Thus, the solution to the fuzzy heat equation (5.15) is given by

$$\begin{aligned} U(x,t) &= q(x,t)A + r(x,t) \\ &= \left(\sum_{n=1}^{\infty} a_n e^{-\frac{c^2n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)\right)A + \left(\sum_{n=1}^{\infty} b_n e^{-\frac{c^2n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)\right), \end{aligned}$$

where

$$a_n = \frac{2}{L}\int_0^L q(x) \sin\left(\frac{n\pi x}{L}\right) dx \text{ and } b_n = \frac{2}{L}\int_0^L r(x) \sin\left(\frac{n\pi x}{L}\right) dx, \text{ with } n \in \mathbb{N}.$$

In this example, we highlight that the expression $\dfrac{\partial U}{\partial t} -_A c^2 \dfrac{\partial^2 U}{\partial x^2} = 0$ only makes sense if we consider interactive arithmetic, because, physically, the difference between the fuzzy quantities involved must result in *Zero*. In [30] the reader can find a study of the equation (5.15) in which the solution is obtained from the Zadeh extension Principle.

The last topic in this section is an introduction to fractional calculus in $\mathbb{R}_{\mathcal{F}(A)}$. More specifically, we just define Caputo derivative in this space and interpret its result through statistical expectation. The reader interested in more results can consult [31].

6 Fractional Calculus: Caputo derivative under Fréchet fuzzy derivative

Definition 1. *[31] Let $f \in L_{loc}([a,\infty),\mathbb{R}_{\mathcal{F}(A)})$ and $\alpha \in (0,1)$. The Caputo fractional derivative of f is defined by*

$$({}^{C}D^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(1-\alpha)}[\int_a^t (t-s)^{-\alpha}(f(s) -_A f(a))ds]', \ \ \forall t \in (a,\infty), \quad (6.18)$$

where $'$ represents Fréchet derivative on $\mathbb{R}_{\mathcal{F}(A)}$ (see [17],[25]).

Now, if $f(t) = q(t)A + r(t)$, from Theorems 4iv), 5 and 6, the Equation (6.18) can be rewrite by

$$\frac{1}{\Gamma(1-\alpha)}[\int_a^t (t-s)^{-\alpha}(q(s)-q(a))ds]'A + \frac{1}{\Gamma(1-\alpha)}[\int_a^t (t-s)^{-\alpha}(r(s)-r(a))ds]'.$$

That is, for all $t \in (a,\infty)$, we have

$$^{\mathrm{C}}D^{\alpha}_{a+}f(t) = (^{\mathrm{C}}D^{\alpha}_{a+}q(t))A + {}^{\mathrm{C}}D^{\alpha}_{a+}r(t). \quad (6.19)$$

Besides that, if $q, r \in AC_{loc}([a,\infty),\mathbb{R})$, from [32] and Theorem 5, then

$$\begin{aligned} ^{\mathrm{C}}D^{\alpha}_{a+}f(t) &= (^{\mathrm{C}}D^{\alpha}_{a+}q(t))A + {}^{\mathrm{C}}D^{\alpha}_{a+}r(t) \\ &= (\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}q'(s)ds)A \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}r'(s)ds \\ &= \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}f'(s)ds, \ \ \forall t \in (a,\infty). \end{aligned}$$

Theorem 8 *Consider f in terms of q and r as above and $\alpha \in (0,1)$. In this case,*

$$^{C}D^{\alpha}_{0}f(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\int_0^1 \frac{(1-w)^{(1-\alpha)-1}}{B(1,1-\alpha)}f'(tw)dw = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}E[f'(tW)], \quad (6.20)$$

where E is the mathematical expectation and $W \sim B(1,1-\alpha)$. That is, W is a random variable with probability density function being the distribution beta with parameters 1 and $1-\alpha$.

It is worth noting that there is an abuse in the last equality of Equation (6.20), because in this text we have not rigorously defined the meaning of $g(X)$, where g is a function in $\mathbb{R}_{\mathcal{F}(A)}$ and X is a random variable, and we also have not defined the expectation of $g(X)$. However, the understanding here should be analogous to the classic case: if g is a real function and X is a random variable, then $g(X)$ is also a random variable whose expectation is obtained from the density of X. The proof given below helps to give meaning to the last equality.

Proof. The proof follows the steps taken in [33] for the case of real functions, combined with Theorem 6.

According to Proposition 2 in [33],

$$^{\mathrm{C}}D_0^{\alpha} g(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} E[g'(tW)],$$

for a real function g, with assumptions like those of q and r.

Thus, from (6.19) we have

$$\begin{aligned}
^{\mathrm{C}}D_0^{\alpha} f(t) &= (^{\mathrm{C}}D_0^{\alpha} q(t))A + {}^{\mathrm{C}}D_0^{\alpha} r(t) \\
&= \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} E[q'(tW)]A + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} E[r'(tW)] \\
&= [\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^1 \frac{(1-w)^{(1-\alpha)-1}}{B(1,1-\alpha)} q'(tw)dw]A \\
&+ [\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^1 \frac{(1-w)^{(1-\alpha)-1}}{B(1,1-\alpha)} r'(tw)dw] \\
&= \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^1 \frac{(1-w)^{(1-\alpha)-1}}{B(1,1-\alpha)} f'(tw)dw,
\end{aligned}$$

which proves the theorem.

We end this theorem by commenting that $^{\mathrm{C}}D_0^{\alpha} f(t)$ is proportional to the weighted average of the Fréchet derivative $f'(s)$, considering all prior values s less than t ($w = 0$ to 1), distributed according to a beta distribution. In this sense, since $^{\mathrm{C}}D_0^{\alpha} f(t)$ is a fuzzy number in $\mathbb{R}_{\mathcal{F}(A)}$, it can be seen as an average value of the fuzzy process $f'(s) = q'(s)A + r'(s)$, $0 < s < t$, whose value is given by

$$\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \underbrace{(E[q'(tW)]A + E[r'(tW)])}_{E[f'(tW)]}.$$

Example 3 *Consider $f(t) = t^{\lambda}A + r$, with $\lambda \in (-1, \infty)$ and $\alpha \in (0, 1)$. Provided $^{C}D_{a+}^{\alpha} r = E[r'(tW)] = E[0] = 0$ and $^{C}D_{a+}^{\alpha}(t^{\lambda}) = \frac{t^{\lambda-\alpha}\Gamma(\lambda+1)}{\Gamma(1+\lambda-\alpha)}$ (see [33]), we have*

$$^{C}D_{a+}^{\alpha} f(t) = \frac{t^{\lambda-\alpha}\Gamma(\lambda+1)}{\Gamma(1+\lambda-\alpha)} A.$$

7 Final Comments

We began the text by dealing with mathematics for uncertainties. We expose basic methods of fuzzy set theory focused on the arithmetic of interactive fuzzy numbers (interactive possibilistic variables). The analogies with arithmetic operations on random variables were highlighted from the joint distributions: possibility distribution for the fuzzy case, and probability distribution for the stochastic case.

Interactive arithmetic on the class of linearly correlated fuzzy numbers is given in terms of joint distribution $J_{q,r}$ and provides a particular case of interactive calculus with good properties [10], but still encounters some difficulties due to lack of an adequate mathematical structure. The main difficulty is performing the calculus itself and, consequently, classical methods, such as numerical integration, methods to solve ODEs and PDEs, among others, are not immediate defined in the fuzzy domain.

As seen in Section 6, the reported difficulties are fully resolved when we restrict the study to linearly correlated non-symmetric fuzzy numbers, $\mathbb{R}_{\mathcal{F}(A)}$, which is a Banach space. Through an isomorphism with $\mathbb{R}^n$, it is possible to generalize this space to a n-dimensional space with a structure of Banach spaces [18]. The formulation of the space $\mathbb{R}_{\mathcal{F}(A)}$ depends on the non-symmetry property of the fuzzy number A. However, as we pointed out earlier, the set of non-symmetric fuzzy numbers is an open and dense set in the space of fuzzy numbers [18], which indicates that it is sufficient to consider it from a practical point of view. This is intuitive because any perturbing at one of the extremities of the symmetric fuzzy number makes it non-symmetric. Also, for the case where A is symmetric [25],[27], the calculus is extended to $\mathbb{R}_{\mathcal{F}(A)}$ using equivalence class. In this way, a theory of differential and integral calculus, mathematical analysis, etc., can be developed in a space that contemplates uncertainties (since its points are fuzzy numbers) with a structure of Banach spaces [34]. Finally, it is worth mentioning that, as we saw in Section 5 (Theorem 5), the derivative in $\mathbb{R}_{\mathcal{F}(A)}$ coincides with the extension of Zadeh of the classical derivative. More precisely, if $f(t) = q(t)A + r(t)$, then $f'(t) = q'(t)A + r'(t)$, which coincides with the extension of Zadeh of $f'(t) = q'(t)a + r'(t)$, which is the extension of Zadeh of classical derivative of $f(t) = q(t)a + r(t)$, for some $a \in \mathbb{R}$. That is, the derivation in $\mathbb{R}_{\mathcal{F}(A)}$ corresponds to the Zadeh extension of classical function derivation. In [7] the reader can find a study showing that the derivation of fuzzy function interchanges with the extension of the derivative for a class larger than $\mathbb{R}_{\mathcal{F}(A)}$.

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A fractional epidemiological model for COVID-19 using triangular norm

Michele Martins Lopes[1 [0000-0002-9539-4916] ⋆],
Francielle Santo Pedro Simões[2 [0000-0001-5177-563X]],
Laécio Carvalho de Barros[1 [0000-0003-4322-5974] ⋆⋆],
Estevão Esmi[1 [0000-0001-6719-678X] ⋆⋆⋆], and
Daniel Eduardo Sánchez[3 [0000-0003-4660-5884]]

[1] Department of Applied Mathematics, State University of Campinas, Campinas, SP
[2] Multidisciplinary Department, Federal University of São Paulo, Osasco, SP, Brazil
[3] Center of Basic Science Teaching for Engineering, Univ. Austral of Chile, Valdivia, Chile

Abstract. In this manuscript we propose a model to analyze the spread of COVID-19. It is based on the compartmental epidemiological model of the SIR (Susceptible-Infected-Recovered) type, but it contains memory effect through fractional differential equations and uses the Hamacher triangular norm to consider control in the proportion of encounters between susceptible and infected individuals.

Keywords: SIR model · Fractional differential equations · Hamacher triangular norm · COVID-19.

1 Introduction

The COVID-19 disease pandemic appeared in December 2019 in China. There are still some misunderstood questions about this disease, but it is known that it spreads quickly and mainly through respiratory droplets [21]. On the other hand, the current variant of the beginning of the year 2022, called Omicron, was first identified in Botswana and soon after in South Africa in November 2021. It has been identified in many countries due to the sharp increase in the number of infections, indicating that it has greater transmissibility and resistance to neutralizing antibodies [4, 7].

It is also known that the levels of viral RNA are higher in the period when the symptoms start, indicating when the transmission is more likely to occur. Therefore, in the incubation period the spread of the disease is less relevant [25]. Thus, it is reasonable to disregard the incubation period in the propagation dynamics of COVID-19. In addition, what is known about the immunity obtained by those recovered from this disease is not yet conclusive.

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In this way, in this manuscript we use the epidemiological model of the SIR type to analyze the behavior of the new coronavirus [3, 20]:

$$\begin{cases} \frac{dS}{dt} = & -\beta S(t)I(t) \\ \frac{dI}{dt} = & \beta S(t)I(t) - \gamma I(t)\,, \\ \frac{dR}{dt} = & \gamma I(t) \end{cases} \tag{1}$$

where $S(t), I(t)$ and $R(t)$ are functions of susceptible, infected and recovered individuals in time t, respectively, $\beta > 0$ is the contamination rate and $\gamma \in (0, 1)$ is the rate of removal of infectious.

One of the main epidemiological models used in the literature is the classic SIR model whose acronyms denote the health status of susceptible, infected and recovered individuals [3, 6, 20]. Currently, structural variations of the SIR model, stochastic adaptations [10, 16] and prediction or machine learning processes [18] are used to fit the system values and thus estimate the effects and try to predict the spread of the COVID-19 pandemic. However, it is worth noting that many works show significant differences in the approaches of the systems, in the calculation of the parameters and in obtaining the results that can fit the real data [24].

In the SIR model the product SI represents the encounter between infected and susceptible individuals, according to the law of mass action [3]. However, during a pandemic, countries aim to reduce the number of such meetings, through control measures. To consider control in the studied dynamics, we introduce a triangular norm in the model. For that, we consider the population normalized, that is, $S(t), I(t), R(t) \in [0, 1]$ and the entire population is considered equal to 1. Thus, from (1) we have $S(t) + I(t) + R(t) = 1$, for all $t \geq 0$.

For the choice of such control measures, isolation from infected individuals is common [11]. This reaction is taken based on previous experiences of others or the same pandemic [8]. Therefore, these and/or other control measures can be interpreted as human memory effects in front of the spread of a pandemic. So, in this paper we present an epidemiological SIR model that takes into account the memory process and the control measures. We use fractional differential equations to cover the memory effect [19] and Hamacher t-norm [1,13] to consider control measures.

In [2] and [15] the authors considered the traditional SIR model to study the evolution of the COVID-19 pandemic, but in [15] the model does not involve a memory effect and considers the Hamacher t-norm to model the encounter between susceptible and infected, and in [2] the model does not consider triangular norms and there is a memory effect in the dynamics. In this manuscript, both are considered together in the SIR model.

2 Mathematical modeling

In this section we see the path to the formulation of the proposed model.

2.1 Fractional SIR model

Fractional differential equations using the Caputo derivative are a tool to insert memory into a dynamic and allows a better description of the studied phenomenon [2, 17, 19]. As we see in [2], the fractional-order differential equation model fits better to the data than the integer-order differential equation model, that is, considering the formulation without memory.

Moreover, in [2] the memory effect is proven in some fractional operators through statistical expectation of functions dependent on values that occurred in previous times. In particular, it is shown that the Caputo derivative of a function $f(t)$ can be written according to this weighted average of the entire order derivative $f'(s)$ with $0 < s < t$, that is, in present the value is determined by the values that have occurred previously. More precisely, in [2] it is shown that,

$$cD_t^\alpha f(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} E\left[f'(tW)\right], \tag{2}$$

where for $\alpha \in (0,1)$, cD_t and $f'(t)$ are the Caputo derivative and the classical derivative of f at time t, respectively, and W is a random variables with beta distribution, $W \sim B(1, 1-\alpha)$.

Thereby, we adapt the classic SIR model (1) with fractional derivative to introduce the memory effect, generating the following system

$$\begin{cases} cD_t^\alpha S(t) = & -\beta^\alpha S(t)I(t) \\ cD_t^\alpha I(t) = & \beta^\alpha S(t)I(t) - \gamma^\alpha I(t)\,, \\ cD_t^\alpha R(t) = & \gamma^\alpha I(t) \end{cases} \tag{3}$$

where $\alpha \in (0,1]$, such that by varying the parameter α we can obtain different solutions and among them ($\alpha = 1$) is the solution of classic model (1).

Note that, to correct the dimension of the parameters in (3), the parameters β and γ have power α. According to [5], it is indicated that in each equation of (3) a single parameter of each portion of the sum must have power α, since the fractional derivative operator cD_t^α has dimension $(\text{time})^{-\alpha}$ instead of $(\text{time})^{-1}$ as in the classic case. So, in model (3) the dimensions of the parameters β^α and γ^α are, respectively, population$\times(\text{time})^{-\alpha}$ and $(\text{time})^{-\alpha}$.

Another way to do dimension correction in a fractional model is to perform the product between $\frac{1}{\tau^{1-\alpha}}$ and the operators on the left side of (3), where τ is any constant with dimension $(\text{time})^1$. Thus, the term $\frac{1}{\tau^{1-\alpha}} cD_t^\alpha$ has dimension $(\text{time})^{-1}$. In this way, if we take $\bar{\beta} = \tau^{1-\alpha}\beta$ and $\bar{\gamma} = \tau^{1-\alpha}\gamma$, we can rewrite (3) so that the parameters $\bar{\beta}$ and $\bar{\gamma}$ appear in place of β and γ, which are not raised to the power α. It is easy to prove this form of dimension correction through the equation (2), since we can write

$$t^{\alpha-1} cD_t^\alpha f(t) = \frac{E\left[f'(tW)\right]}{\Gamma(2-\alpha)} \tag{4}$$

and $E\left[f'(tW)\right]$ is a weighted average of $f'(s)$ that has dimension: population $\times$ $(\text{time})^{-1}$. Therefore, both sides have dimension population $\times$ $(\text{time})^{-1}$. We chose the first form of dimension correction (generating the model (3)) as this makes it easier to get the parameters of the original model.

In computational tests performed with the model (3) for coronavirus in some countries it was found that the peak value of the solution may be much higher than the peak value of the actual data, resulting in a bad fit, as we can see in Figure 1. In this figure we take as an example the case of Italy. The blue curve represents the real data of the "first wave" while the green curve represents the model solution. Data fitting is done by the least squares method, whose mean square error (MSE) is given by:

$$MSE = \sum_{i=1}^{n} (I(t_i) - data(t_i))^2, \tag{5}$$

where $data$ is the vector with n real data of COVID-19 active cases. Note that because the treated values are proportions, the value of MSE can be very close to 0 even if the model solution did not fit the data very well.

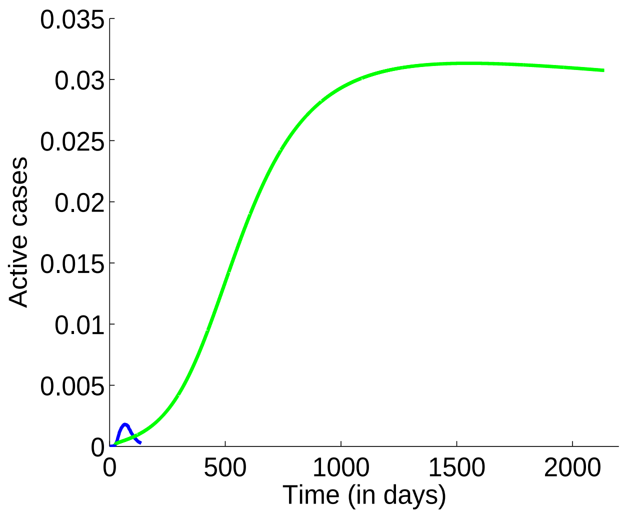


Fig. 1: Solution of the model (3) (green), with peak in $I = 0.03$, and real data of COVID-19 in Italy (blue), with peak in $I = 0.002$. $MSE = 7.189 \cdot 10^{-5}$.

Since $\alpha = 1$ is an option in the data fit performed, this problem occurs not only in the fractional model, but in the classic epidemiological SIR model. To solve this problem, it is necessary to consider a reduction of contagion in the dynamics of the model.

In real life, in fact, each country seeks a way to reduce the number of cases of the disease, adopting specific contingency measures to combat an epidemic. These measures can be, for example, the mandatory use of masks, closing shops, schools and parks, among other measures of social distance [14]. Thus, in model (3), there is still a control measure to add. As we will see in the next section, this control can be considered through triangular norms.

2.2 Triangular norm and SIR model

The concept of triangular norm started in 1942 with the work of Karl Menger, which comes from the probabilistic metric space [13].

Definition 1. *[13] A triangular norm T (also known as t-norm) is a binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ that satisfies:*

1. $T(1, x) = x$;
2. $T(x, y) = T(y, x)$;
3. $T(x, T(y, z)) = T(T(x, y), z)$;
4. *If* $x \leq u$ *and* $y \leq v$, *then* $T(x, y) \leq T(u, v)$.

Some triangular norms are listed below [1, 13]

1. Minimum $t-$norm $T_M(x, y) = \min\{x, y\}$.
2. Product $t-$norm: $T_P(x, y) = xy$.
3. Lukasiewicz $t-$norm: $T_L(x, y) = \max\{0, x + y - 1\}$.
4. Hamacher $t-$norm:

$$T_H(x, y) = \frac{xy}{p + (1 - p)(x + y - xy)}, \quad p \geq 0.$$

In the model (3), direct contact transmission is discribed by the law of mass action. This law assumes that the probability of a susceptible individual finding an infected individual is proportional to the product of their densities. Thus, in system (3), the term βSI coincides with $\beta T_P(S, I)$, that is, using the t-norm of the product.

In a pandemic situation, especially of a disease still little known, the encounter between susceptible and infected individuals is no longer random, since each region looks for ways to reduce them. For this control measures are taken, such as social isolation, the use of a mask and hand sanitizer [9]. One way to insert control into the dynamics would be to change the product t-norm for Hamacher t-norm, that is, in (3) we change $T_P(S, I)$ for $T_H(S, I)$. With that, we get the following new model:

$$\begin{cases} cD_t^\alpha S(t) = & -\dfrac{\beta^\alpha S(t)I(t)}{p + (1 - p)(S(t) + I(t) - S(t)I(t))} \\ cD_t^\alpha I(t) = & \dfrac{\beta^\alpha S(t)I(t)}{p + (1 - p)(S(t) + I(t) - S(t)I(t))} - \gamma^\alpha I(t) \\ cD_t^\alpha R(t) = & \gamma^\alpha I(t) \end{cases}, \tag{6}$$

where $\alpha, \gamma \in (0, 1]$, $\beta > 0$ and $p \geq 1$.

We chose the Hamacher t-norm as it generalizes the product t-norm, which is used in the classical SIR model. That is, we have $T_H(S, I) = T_P(S, I)$ when $p = 1$ and $T_H < T_P$ when $p > 1$. Thus, we have in fact added a control parameter p in relation to the likelihood of contagion when a susceptible individual meets an infected one, or even in relation to the probability of meeting.

In the previous section we saw in Figure 1 that the model (3) presented a problem in fiting data for COVID-19 disease in Italy, the country that we have used as an example. Now with the model (6) the problem is solved and the fit data becomes visually good, as we can see in Figure 2. We can confirm this statement through the mean square error, which changed from $MSE = 7.189 \cdot 10^{-5}$ to $MSE = 6.745 \cdot 10^{-6}$, approximately.

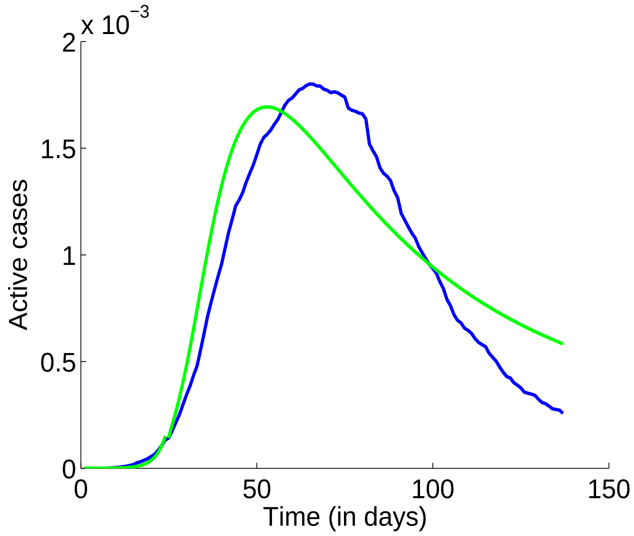


Fig. 2: Solution of the fractional epidemiological SIR model (green) after entering the control parameter p (model (6)) compared to real data of COVID-19 in Italy (blue). $MSE = 6.745 \cdot 10^{-6}$.

Furthermore, in order to show that the use of fractional calculus is also important, if we keep the control parameter p in the model, but we take $\alpha = 1$ (returning to the case without fractional derivative), we can see in the Figure 3 that the data fit is not as good as the one shown in Figure 2. In fact, by limiting $\alpha = 1$ the mean square error increases from $MSE = 6.745 \cdot 10^{-6}$ to $MSE = 5.439 \cdot 10^{-5}$. Therefore, the combination of both (fractional derivative and triangular norm) that allows us to obtain a better result.

2.3 The model

In this paper, we propose the model (6) to analyze the spread of COVID-19, that is, a model that considers the memory effect (with fractional derivatives) and control measures (with the parameter p of Hamacher t-norm).

We first perform the dimensional analysis of the parameter p. Note that the values $S(t), I(t)$ and $R(t)$ are dimensionless. As stated in the section 2.1, β has a dimension population$\times$(time)$^{-\alpha}$, so that $\frac{1}{p+(1-p)(S+I-SI)}$ must be dimensionless. Therefore, p must also be dimensionless.

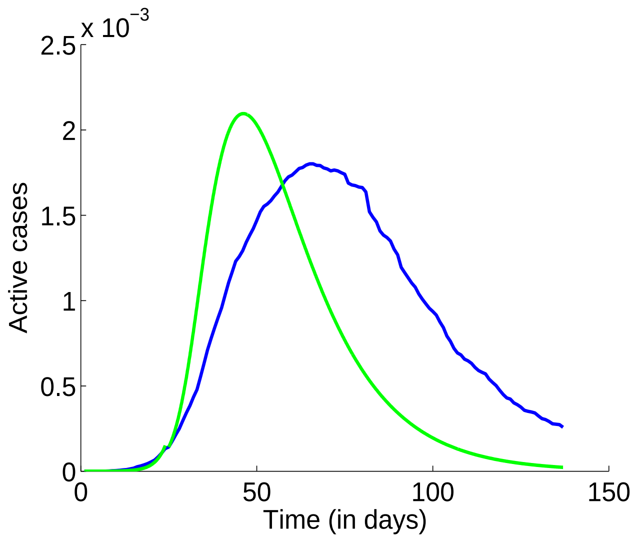


Fig. 3: Solution of the classic epidemiological SIR model (green) with control parameter p (model (6) when $\alpha = 1$) compared to real data of COVID-19 in Italy (blue). $MSE = 5.439 \cdot 10^{-5}$.

The basic reproduction number $R_0^{p,\alpha}$ is the average number of secondary infections produced by a single infectious in a population totally susceptible with size S_0 [6]. It can be analyzed as follows. If we consider $I'(t) > 0$ then we get $R_0^{p,\alpha} > 1$, being $S = S_0$ and $I = I_0$, so that the proportion of infected people is given by only one individual in the entire population. So, for the model (6),

$$\begin{aligned} I'(t) > 0 &\Leftrightarrow \left(\frac{\beta^\alpha S}{p + (1-p)(S + I - SI)} - \gamma^\alpha \right) I > 0 \\ &\Leftrightarrow \frac{\beta^\alpha S}{p + (1-p)(S + I - SI)} - \gamma^\alpha > 0 \\ &\Leftrightarrow \frac{\beta^\alpha S}{\gamma^\alpha p + (1-p)(S + I - SI)} > 1 \\ &\Leftrightarrow \frac{\beta^\alpha S}{\gamma^\alpha} \frac{1}{p + (1-p)(S + I - SI)} > 1. \end{aligned}$$

Thereby, considering the values $S = S_0$ and $I = I_0$ of the $R_0^{\alpha,p}$ analysis,

$$I'(t) > 0 \Leftrightarrow \frac{\beta^\alpha S_0}{\gamma^\alpha} \frac{1}{p + (1-p)(S_0 + I_0 - S_0 I_0)} > 1.$$

In the fractional SIR epidemiological model, that is, when $p = 1$, we know that the basic reproduction number is given by $R_0^\alpha = \left(\frac{\beta}{\gamma}\right)^\alpha S_0$ [3, 12, 20].

So, if $p \geq 1$, since $I_0 = 1 - S_0$, we can obtain the basic reproduction number $R_0^{\alpha,p}$ for model (6) by doing

$$I'(t) > 0 \Leftrightarrow R_0^\alpha \frac{1}{p + (1-p)(S_0 + I_0 - S_0 I_0)} > 1$$

$$\Leftrightarrow R_0^{\alpha} \frac{1}{p + (1-p)(1 - S_0(1 - S_0))} > 1.$$

Therefore, we define

$$R_0^{\alpha,p} = R_0^{\alpha} \frac{1}{p + (1-p)(1 - S_0(1 - S_0))}. \tag{7}$$

Note that $R_0^{\alpha,p} \leq R_0^{\alpha}$ and $R_0^{\alpha,p} \to R_0^{\alpha}$ when $p \to 1^+$. Moreover, $R_0^{\alpha,p} \to 0$ when $p \to \infty$. This fact is in agreement with the interpretation of the parameter p, since it represents control of new infections, that is, the higher its value is, the smaller the number of secondary infections caused by each infected individual ($R_0^{\alpha,p}$).

3 Methodology

In this manuscript, we use the model (6) to analyze the propagation of active cases (infected cases) of COVID-19 in Germany, Italy, Switzerland and United States. This is done by fit data using the least squares method to determine the parameters of model (6) and then their solutions. The real data of COVID-19 are taken from [23] in the periods referring to the first "wave" and the most recent "wave" of each country at this time, early 2022. As commented before, for the use of the triangular norm we consider the population in proportion. Thus, all data values are normalized, that is, they were divided by the total inhabitants of the respective country (N) [20].

Our objective is not only to test the effectiveness of the model, but also to compare the different values of p obtained for each country in order to verify if this parameter is a good indicator of the rigidity of the control adopted by each country.

The data for the so-called first "wave" of each country are classified into two phases, the first representing the period without control measure ($p = 1$) and the second representing the period with control measure ($p > 1$) adopted in accordance with the public policies of each analyzed country. The separation of data to determine each phase is established in the respective time t that indicates the day when quarantine started in the analyzed country, that is, if social isolation in the country started at $t = t^*$, then the first part of the data is defined from the initial time up to t^* (phase 1), while the second part of the data is defined after t^* (phase 2).

We consider the initial value (S_0, I_0, R_0), where I_0 is the proportion of infected on the first day, which is taken as the first value of the normalized data, $R_0 = 0$ and, in turn, $S_0 = 1 - I_0$.

The values of the parameters α, β, γ and p are found by fitting data. In phase 1 (without control) we consider $p = 1$ and also $\alpha = 1$, since in this phase there is no past to be considered for the purpose of memory. So, in this phase we get β and γ. With these values fixed, in phase 2 (with control) we find the values of $\alpha \in (0, 1]$ and $p > 1$.

In this process, two methods are considered to analyze the model (6):

- *Method 1:* β and γ are free parameters, assuming values in the range $(0, 1)$;
- *Method 2:* β is a free parameters, while $\gamma = 0.1$ is fixed.

Setting $\gamma = 0.1$ comes from the fact that in the SIR model the removal rate (γ) is equal to the inverse of the time the individual remains in the infected compartment [3, 20]. That is, this time is determined by $\frac{1}{\gamma}$. It is known that the COVID-19 infection period varies between 7 and 14 days, considering only the period with symptoms, since in the SIR model we do not consider the incubation period [22]. So, we consider 10 days the average time that an individual can be considered as active, that is, who remains in the compartment of those infected. Then, we have adopted $\gamma = \frac{1}{10}$.

4 Results

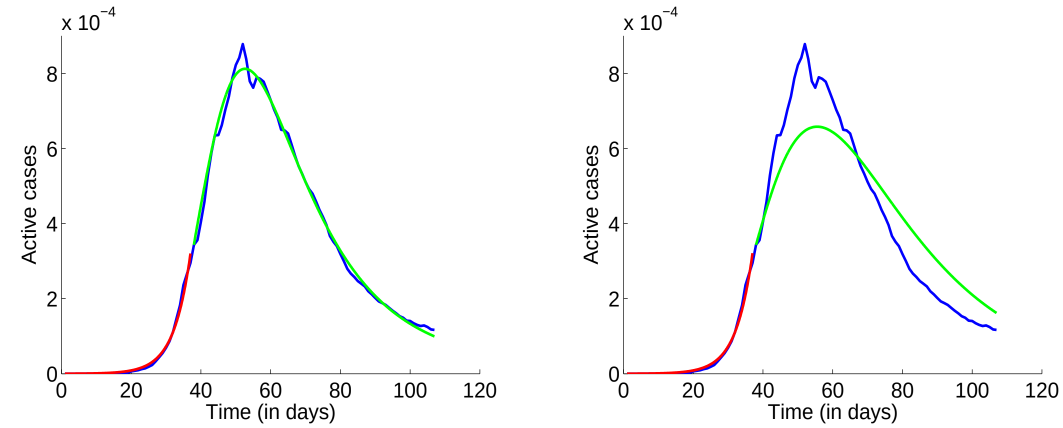
In this section we analyze the results obtained with both methods mentioned in the previous section: method 1, which assumes β and γ with any values in their ranges, and method 2, which assumes β free and $\gamma = 0.1$ fixed. First, we test the model (6) with data from the first "wave" of each country selected for this work, since this is where the first phase is found (when there was still no control measure, just before starting phase 2). Then, we use the model (6) in the most recent "wave" of each country at this time, early 2022. That is, in this second stage, our objective is to project the curve that represents the recent "wave", affected by the Omicron variant, making predictions and comparing the current scenario with that of the first "wave".

4.1 Fisrt "wave"

Figures 4–7 show the results obtained for the first "wave" of each country selected for this work, where the blue curve represents the real data (obtained in [23]), the red curve the model solution in phase 1 (without control) and the green curve the model solution in phase 2 (with control). The graphics on the left side are obtained with method 1 (β and γ free parameters) and the graphics on the right side obtained with method 2 ($\gamma = 0.1$ fixed). We must remember that the parameter values p and α are referring to phase 2, since in phase 1 we consider $p = \alpha = 1$.

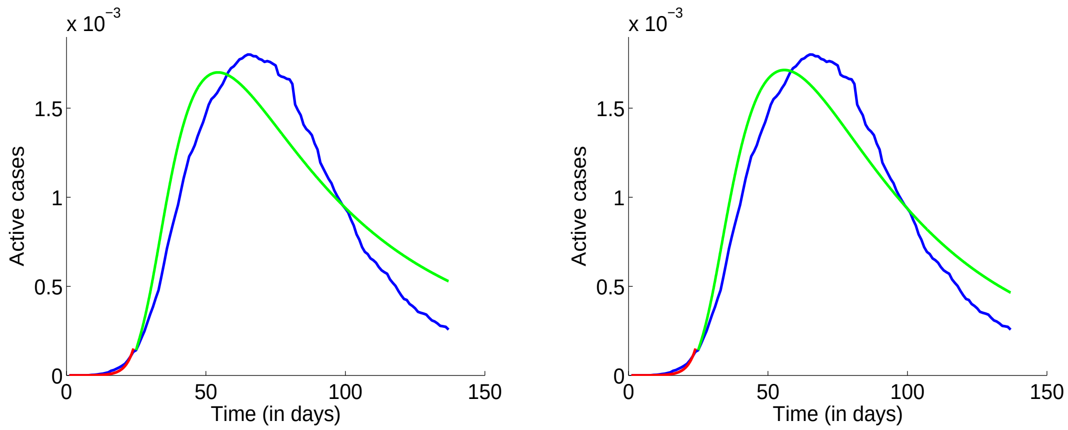
We verified that, to both methods (especially method 1), the model (6) presents a solution $I(t)$ with good ability to describe the data, showing a good approximation of the value of the maximum proportion of active cases (infected cases) and the day on which it occurred.

We recall that, as we saw in the Section 2.2, the parameter p includes control over the number of encounters between susceptible and infected individuals in the model dynamics. The higher the value of $p \geq 1$, the greater the rigidity of the control measures adopted, with value $p = 1$ indicating no control. According to the values obtained for this parameter, through method 1 we conclude that Italy had more rigid control and then Germany, Switzerland and the United



(a) (b)

Fig. 4: Solution curves of the model (6) for Germany in phase 1 (red) and phase 2 (green), compared to real data (blue), with method 1 (a) and method 2 (b). The parameters obtained are (a) $\beta = 0.487, \gamma = 0.276, p = 172.892, \alpha = 0.899$, and (b) $\beta = 0.312, p = 1184.107, \alpha = 0.999$.

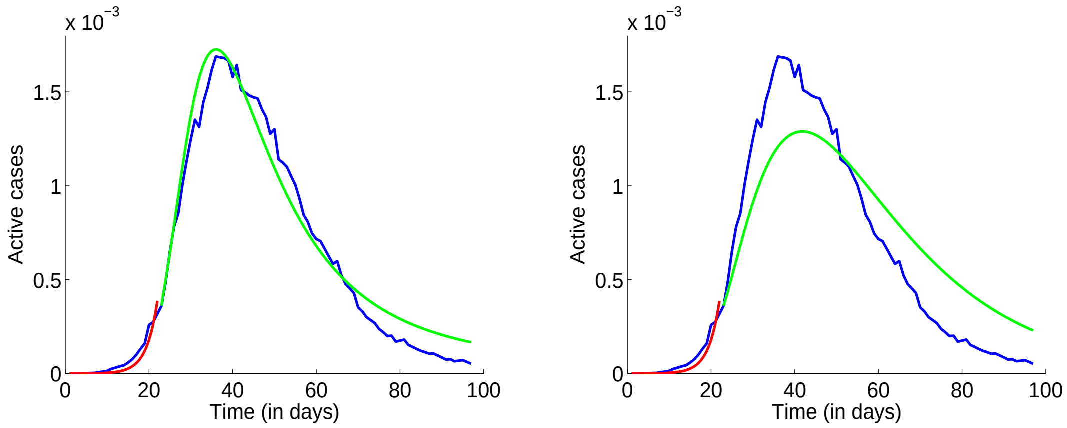


(a) (b)

Fig. 5: Solution curves of the model (6) for Italy in phase 1 (red) and phase 2 (green), compared to real data (blue), with method 1 (a) and method 2 (b). The parameters obtained are (a) $\beta = 0.499, \gamma = 0.151, p = 184.665, \alpha = 0.719$, and (b) $\beta = 0.448, p = 389.618, \alpha = 0.794$.

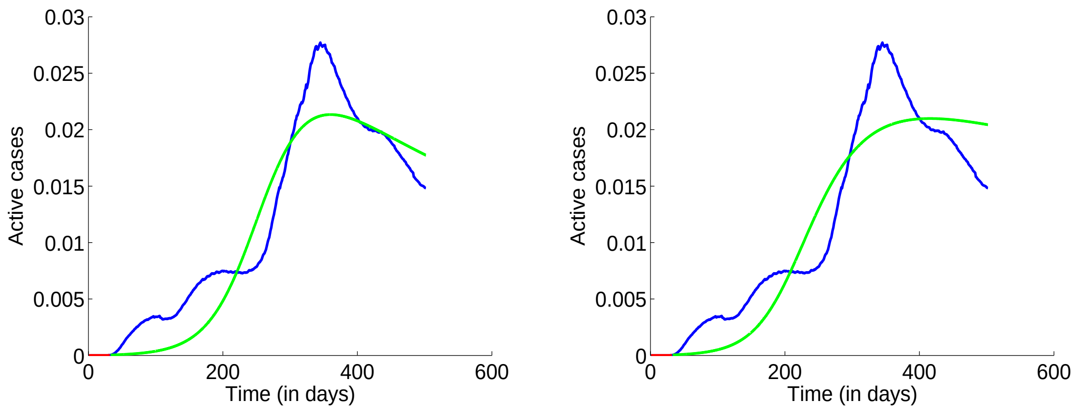
States. By method 2 the country with the strictest control was Germany, then Switzerland, Italy and the United States. Both methods agree that the United States had less control, having almost no control according to method 1.

As we also see in the Section 2.2 (Figures 2 and 3), the width of the curve, in relation to the abscissa axis, is well described from the value of the derivative order (α). The lower the value α, the wider the solution curve of the model (6), that is, the disease spreads more slowly. With this, the so-called "flattening of



(a) (b)

Fig. 6: Solution curves of model (6) for Switzerland in phase 1 (red) and phase 2 (green), compared to real data (blue), with method 1 (a) and method 2 (b). The parameters obtained are (a) $\beta = 0.804, \gamma = 0.418, p = 78.617, \alpha = 0.831$, and (b) $\beta = 0.486, p = 1181.724, \alpha = 0.999$.



(a) (b)

Fig. 7: Solution curves of the model (6) for USA in phase 1 (red) and phase 2 (green), compared to real data (blue), with method 1 (a) and method 2 (b). The parameters obtained are (a) $\beta = 0.469, \gamma = 0.273, p = 1.001, \alpha = 0.504$, and (b) $\beta = 0.297, p = 2.983, \alpha = 0.428$.

the curve" occurs. Therefore, the analysis of the α can influence public health decisions. In addition, low values of α indicate greater memory effect [2].

We conclude from the Table 4 and through both methods that the spread of the disease in the United States occurs more slowly and with more memory effect. In contrast, the other countries have high values of α, indicating a faster spread and with less memory effect. This makes sense since in Germany, Italy and Switzerland it took no more than 150 days to end the "first wave", a period in which little was known about the disease. On the other hand, in the United

States, 500 days have passed and the "first wave" has not yet been completed, indicating in fact a slower spread and greater experience of the population, explaining the greater memory effect.

Comparing the two methods, we see that method 1 presents a better fit data in both phases and this can be seen in the Tables 1 and 2. These tables show the mean squared error obtained in each country, for both phases with the two methods. However, considering what we saw in Section 3, the values of γ should be close to 0.1 and, especially in the case of Switzerland, by method 1 this did not occur. In this same case, by method 1 the value of β also differed considerably from those found in other countries with both methods.

Table 1: Mean squared error values in phase 1 of the first "wave"

	Germany	**Italy**	**Switzerland**	**USA**
Method 1	1.959e-09	2.453e-09	2.198e-08	2.067e-11
Method 2	1.961e-09	2.453e-09	2.202e-08	2.060e-11

Table 2: Mean squared error values in phase 2 of the first "wave"

	Germany	**Italy**	**Switzerland**	**USA**
Method 1	2.784e-08	6.742e-06	7.342e-07	4.030e-03
Method 2	6.304e-07	6.118e-06	4.866e-06	5.328e-03

4.2 Recent "wave"

The most recent "waves" of the countries in early 2022 have a different characteristic: a sharp growth that, consequently, occurs more quickly. This is probably due to the fact that the Omicron variant is more easily transmitted and is more resistant to antibodies [4, 7]. This behavior is not seen at this intensity in previous "waves", with the influence of the other variants. In view of this, the need arises to adapt our methodology to include this new effect.

The rates β and γ are determined in phase 1 of the first wave and they are no longer changed. The rate p includes control in the model, which may decrease the intensity of the infection in the dynamics, but not increase it. Finally, the rate α helps to describe the speed of growth of the curve, but given the values of β, γ and p that denote a scenario of greater control of the spread of the disease, we observe in our experiments that the value of α does not contribute to solving this problem. The solution found is to change the value of N, which is the total population of the country. This represents an increase in the susceptible population in relation to previous "waves", which in fact occurred with the resistance of Omicron to neutralizing antibodies.

Since the values of β and γ are already determined, in the recent "wave" we fit the data again to find new values for p and α. Figures 8–11 show the results obtained for the recent "wave" of each country in early 2022, where the blue curve represents the real data and the green curve the model solution. Data were taken until January 25, 2022, date the paper was written. From January 26, 2022 we present the model solution projection for the next 6 months.

The graphics on the left side are obtained with method 1 (β and γ free parameters) and the graphics on the right side obtained with method 2 ($\gamma = 0.1$ fixed).

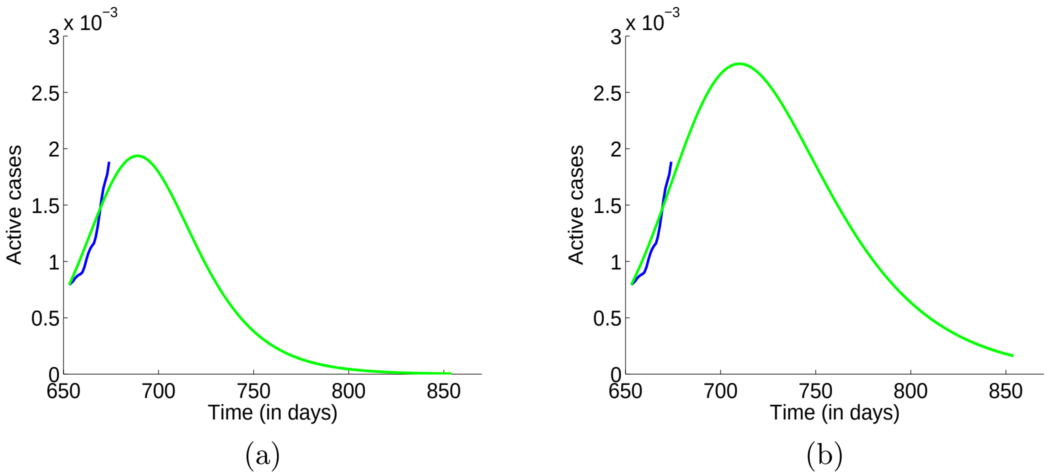


Fig. 8: Solution curve of the model (6) for Germany (green) compared to real data (blue), with method 1 (a) and method 2 (b).
The parameters obtained are (a) $\beta = 0.487, \gamma = 0.276, p = 14.841, \alpha = 0.999$, and (b) $\beta = 0.312, p = 68.585, \alpha = 0.999$.

We can observe that, by both methods, we obtain a good fit of data, especially through method 2. From Table 3 we see that, in fact, method 2 presented lower values of mean square error in all cases, contrary to what we saw in the first "wave".

Table 3: Mean squared error values of recent "waves"

	Germany	**Italy**	**Switzerland**	**USA**
Method 1	3.518e-07	1.073e-05	3.842e-07	1.453e-05
Method 2	2.689e-07	8.085e-06	1.883e-07	8.529e-06

The solutions of the model (6) by method 1 indicate that the curve of active cases of COVID-19 is closer to the peak than by method 2. In Germany, by method 1, there are 15 days left until reaching the maximum number of infected

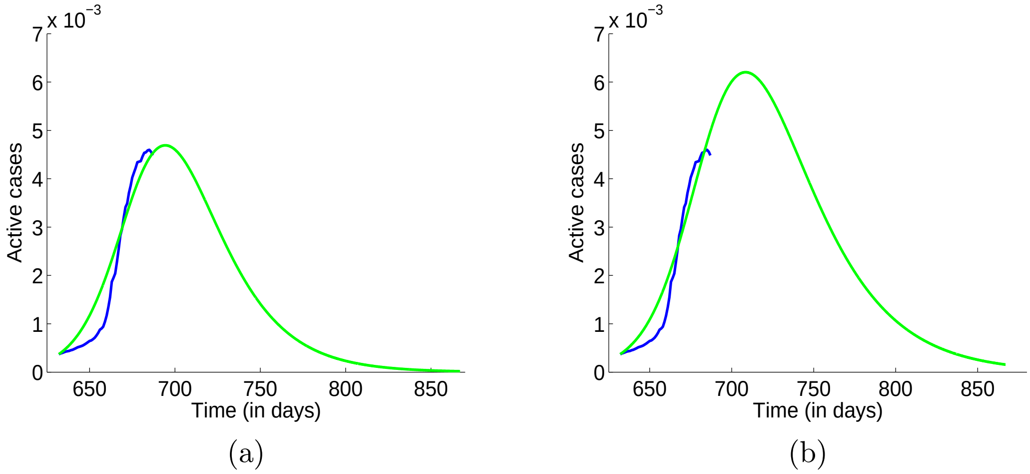


Fig. 9: Solution curve of the model (6) for Italy (green) compared to real data (blue), with method 1 (a) and method 2 (b).
The parameters obtained are (a) $\beta = 0.499, \gamma = 0.151, p = 33.548, \alpha = 0.999$, and (b) $\beta = 0.448, p = 52.658, \alpha = 0.999$.

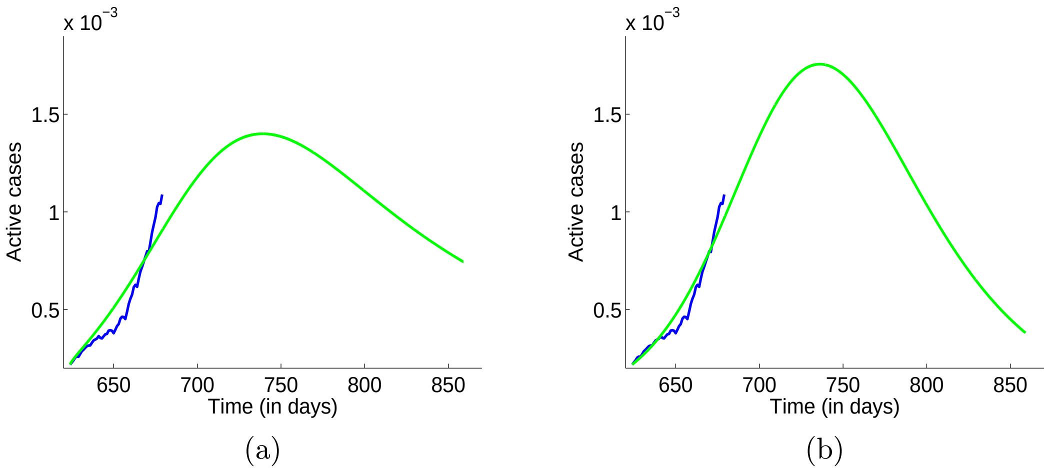


Fig. 10: Solution curve of the model (6) for Switzerland (green) compared to real data (blue), with method 1 (a) and method 2 (b).
The parameters obtained are (a) $\beta = 0.804, \gamma = 0.418, p = 6.033, \alpha = 0.744$, and (b) $\beta = 0.486, p = 84.351, \alpha = 0.999$.

people and this value should not grow much further. By method 2, there are 36 days left until the peak and the number of infected should increase by around 45%.

In Italy, by method 1, there are 9 days left until reaching the maximum number of infected people and this value should not grow much further. By method 2, there are 23 days left until the peak and the number of infected should increase by around 35%.

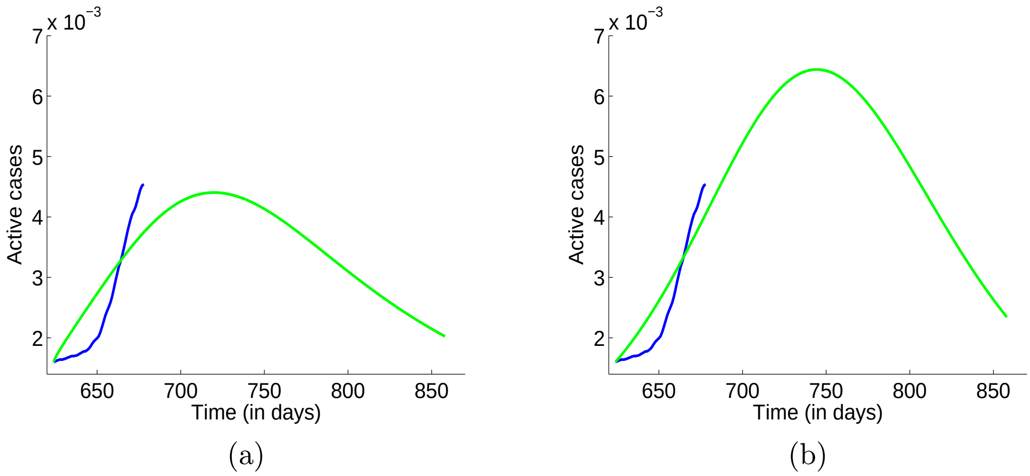


Fig. 11: Solution curve of the model (6) for USA (green) compared to real data (blue), with method 1 (a) and method 2 (b).
The parameters obtained are (a) $\beta = 0.469, \gamma = 0.273, p = 1.285, \alpha = 0.799$, and (b) $\beta = 0.297, p = 4.871, \alpha = 0.999$.

In Switzerland, by method 1, there are 60 days left until reaching the maximum number of infected people and this value should increase by around 28%. By method 2, there are 57 days left until the peak and the number of infected should increase by around 61%.

In USA, by method 1, there are 42 days left until reaching the maximum number of infected people and this value should not grow any more. By method 2, there are 66 days left until the peak and the number of infected should increase by around 42%.

As we saw in the Section 2.2 the parameter p includes control over the number of encounters between susceptible and infected individuals in the model dynamics. The higher the value of $p \geq 1$, the greater the rigidity of the control measures adopted, with value $p = 1$ indicating no control. According to the values obtained for this parameter, through method 1 we conclude that in the most recent "wave" Italy had more rigid control and then Germany, Switzerland and the United States, exactly the same order obtained in the first "wave". By method 2 the country with the strictest control is Switzerland, then Germany, Italy and the United States. Again, both methods agree that the United States had less control, having almost no control according to method 1.

As we also see in the Section 2.2 (Figures 2 and 3), the width of the curve, in relation to the abscissa axis, is well described from the value of the derivative order (α). The higher the value α is, the narrower the solution curve of the model (6) is, that is, the disease spreads faster. Therefore, higher values of this parameter are expected for this "wave". In fact, with method 2 we get $\alpha \sim 1$ for all countries. With method 1 we obtained $\alpha \sim 1$ for Germany and Italy, while for the other countries this do not occur, but still α has high values. Also, high values of α indicate less memory effect [2]. Thus, our model indicates that in the recent "wave" there is little prior knowledge used in current decisions. In fact, there are still many doubts about the behavior of this new variant Omicron.

4.3 What changed from the first scenario to the current one?

Table 4 presents the values of the parameters obtained for each country using both methods. We can see that the two methods agree with the following results. In the recent "wave" the values of p decreased in Germany, Italy and Switzerland, indicating less containment of the spread of the disease. On the other hand, in the USA the value of p has increased a little, but it is still very low, indicating almost no control ($p \sim 1$). Furthermore, the values of α in the recent "wave" either increased or remained the same, with the exception of what is indicated by method 1 for Switzerland. In this way, we concluded that the infected curve started to grow faster, which we can actually observe, and that there is less memory effect nowadays. In fact too, the behavior of the new variant is not well known.

Table 4: Parameter values obtained by both methods

		Method 1		*Method 2*	
		First "wave"	Recent "wave"	First "wave"	Recent "wave"
Germany	β	0.487	0.487	0.312	0.312
	γ	0.276	0.276	0.1	0.1
	p	172.892	14.841	1184.107	68.585
	α	0.899	0.999	0.999	0.999
Italy	β	0.499	0.499	0.448	0.448
	γ	0.151	0.151	0.1	0.1
	p	184.665	33.548	389.618	52.658
	α	0.719	0.999	0.794	0.999
Switzerland	β	0.804	0.804	0.486	0.486
	γ	0.418	0.418	0.1	0.1
	p	78.617	6.033	1181.724	84.351
	α	0.831	0.744	0.999	0.999
USA	β	0.469	0.469	0.297	0.297
	γ	0.273	0.273	0.1	0.1
	p	1.001	1.285	2.983	4.871
	α	0.504	0.799	0.428	0.999

With the model (6) it is also possible to find the values of basic reproduction numbers, according to (7). Table 5 presents these values obtained with each method for the respective countries, considering the first "wave" and the recent "wave" of each one. In agreement with our conclusions above, we see that the values of this parameter have increased in the recent "wave", indicating the faster transmission in the current scenario.

Table 5: Basic reproduction number values ($R_0^{\alpha,p}$)

	Method 1		*Method 2*	
	First "wave"	Recent "wave"	First "wave"	Recent "wave"
Germany	1.686	1.768	3.118	3.119
Italy	2.616	3.301	3.789	4.484
Switzerland	1.729	1.627	4.857	4.858
USA	1.280	1.542	1.665	2.966

5 Conclusion

The classic SIR model is commonly used for the study of pandemics. The memory effect in this model, based on fractional differential equations, allows the description of the phenomenon to be even more detailed, generating good results. Moreover, the use of Hamacher t-norm in the model made it possible to consider in the dynamics the control of encounters between susceptible and infected individuals, making it possible to analyze the rigidity of isolation between the countries analyzed.

We saw that for the presented methodologies the combination of the fractional model with the use of the t-norm is fundamental for good data fit and, with that, better interpretations of the parameters. In addition, we use two different methods to determine the values of the parameters and compare them, identifying the advantages of each one.

We analyze the scenario of the first "wave" and the most recent "waves" from Germany, Italy, Switzerland and the United States, in early 2022. In the latter case, it was possible to make projections of the curve of active cases of COVID-19, estimating the time remaining to reach the peak and how much the increase in the number of cases will be. Furthermore, through the parameters found (Table 4) and the basic reproduction numbers (Table 5), we were able to conclude the higher transmissibility of the Omicron variant.

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Type-1 and Interval Type-2 Fuzzy Sets along with 3D Cellular Automaton: Tools for Qualitative Study of a SIRD Model

Rosana Sueli da Motta Jafelice[1][0000−0001−8489−3974], Jefferson Beethoven Martins[2,3][0000−0001−6804−9802], and Ana Maria Amarillo Bertone[3][0000−0003−4370−9506]

[1] Federal University of Uberlândia, Uberlândia MG, Brasil
rmotta@ufu.br
[2] Federal Institute of Minas Gerais Triangle Mineiro, Uberaba, MG, Brasil
jefferson@iftm.edu.br
[3] Federal University of Uberlândia, Uberlândia MG, Brasil
amabertone@ufu.br

Abstract. A dynamic of the evolution of an epidemic based on the classic SIRD model (Susceptible, Infected, Recovered, Dead) is developed using a three-dimensional cellular automaton in combination with fuzzy rule-based systems of type-1 and interval type-2 that interpret the infection and recovery rates. These systems interact with the automaton rules to determine the next generation behavior in terms of infection and, once infected, determining if the cell is recovered or dead. The aim of working with both types of fuzzy sets is to increase the uncertainty of the phenomenon studied through fuzzy systems, in particular those that originated by the interval type-2 fuzzy sets. The main contribution of this work is the interaction between fuzzy rules of two types with the rules of the cellular automaton itself. In fact, the novelty of this study is the development of its own computer program, with the possibility of implementing other dynamics of epidemiological or population origin. The modeling proposed has been validated through building, for each population of individuals, a range generated by the cellular automaton outputs related to the interval type-2 fuzzy rule-based system. This range contains the deterministic solution of the system of ordinary differential equations that govern the phenomenon and the cellular automaton output corresponding to the type-1 fuzzy rule-based system. As a result, equivalent qualitative aspects are observed.

Keywords: SIRD Model · Cellular Automaton · Type-1 and Interval Type-2 Fuzzy Sets.

1 Introduction

The models denominated "SIRD" to study epidemics, have been widely used [1,2], considering the epidemiological model proposed in 1927 by two Scottish scientists, William Kermack and Anderson McKendrick, as pioneers, becoming known as Kermack–McKendrick models [3]. These models show how an infection evolves when a certain number of infected people are detected in a healthy population. The epidemiological phenomenon can be modeled by a system of ordinary differential equations (ODE), whose parameters are imprecise, however relevant in the significant changes of an approximate solution. More specifically, the Kermack-McKendrick models interpret the dynamics of three populations during an epidemic represented by Susceptible (S), Infected (I) and Removed (R) individuals, with each of these groups having its dynamics described by an ODE that make up the system. In the case of the SIRD [4] models, the R group is divided into two: the one recovery from the disease, continuing to be called R; and the population that dies during the epidemic. For this group, the acronym D is used due to the word "Dead".

The ODE system that governs SIRD-type models is given by:

$$\begin{aligned} \frac{dS}{dt} &= -\frac{\beta IS}{N}, \\ \frac{dI}{dt} &= \frac{\beta IS}{N} - \gamma I - \mu I, \\ \frac{dR}{dt} &= \gamma I, \\ \frac{dD}{dt} &= \mu I, \end{aligned} \tag{1}$$

where N represents total population, β, γ, μ are the infection, recovery and mortality rates, respectively. Furthermore, $N = S(t)+I(t)+R(t)+D(t)$ at each time t.

The aim of this work is the construction of a three-dimensional cellular automaton to represent the SIRD dynamics of the epidemic in connection with a Fuzzy Rule-Based System (FRBS) of two types: type-1 and interval type-2.

Cellular Automaton (CA) are mathematical models composed of a set of cells arranged in a n-dimensional space, in which the state of a cell at a given moment depends on local rules and its relationship with its surroundings. Introduced by von Neumann [5] and Ulman, an CA is composed of four elements: a set of initial states, a set of cells, a neighborhood type and transition rules. Each cell has a unique state in an i iteration. In turn, the transition rules determine what the cell state will be in the next iteration, $i+1$, according to its current state and the state of its neighboring cells. The values of all cells are updated simultaneously with each iteration. Two common types of neighborhoods are von Neumann and Moore [6]. The difference between the two in the three-dimensional case is that a cell has twenty-six adjacent neighbors in the Moore case and six in the von Neumann case, as shown in Figure 1. The environment where individuals artificially live, in CA, is represented by a cube. At the beginning of the dynamics, the occupied cells represent randomly distributed susceptible and infected individuals. The rest of the cells are free spaces, with the possibility of being occupied in the next iteration. This three-dimensional environment is a hypertorus, three-dimensional version of a torus. In fact, just as the two-dimensional torus can be seen as a complete square whose opposite sides are identified, the hypertorus is interpreted as a complete three-dimensional hypercube whose opposite faces are identified [7]. The cube is divided by cells in which the colors represent each type of group or free spaces.

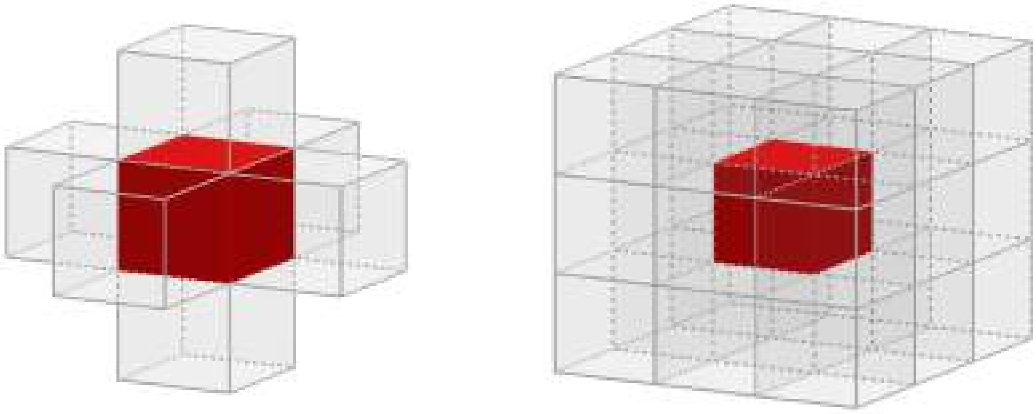

Fig. 1. On the left, the von Neumann neighborhood and, on the right, the Moore neighborhood. Source: image created by the authors in GeoGebra.

Fuzzy Logic was created by Lotfi Zadeh, professor emeritus of computer science at the University of California - Berkeley, where he proposed an alternative logic to the classical-Boolean one. Zadeh's first paper in this new area of knowledge, was released in 1965 [8]. Only ten years after, Zadeh introduced type-2 fuzzy sets in 1975 [9], which extended Zadeh's original fuzzy set theory, currently known as type-1 fuzzy sets. Jerry Mendel, pioneering in the rescue of the Zadeh's type-2 fuzzy sets theory, introduced his first paper on this topic in 1998 [10], twenty three years after Zadeh first paper on the subject [9]. The crucial difference between type-1 and type-2 fuzzy set is that the former has as membership degree, a single value, and the type-2 fuzzy set reveals the membership degree in two stages, the primary and secondary membership functions. When the secondary is constant, the result is called an interval type-2 fuzzy set; otherwise, it is called a general type-2 fuzzy set [11,12]. In 1998 the work of Nilesh Karnik and Jerry Mendel [10,11] appeared, with the presentation of the complete theory of the interval type-2 FRBS, including the type reducer as a previous stage of the defuzzification method. Another important contribution for the area was the Interval Type-2 Fuzzy Logic Toolbox [13] developed in 2007 by J. R. Castro, O. Castillo, L. G. Martínez. This toolbox is a set of functions for building interval type-2 FRBS, utilized in the applications of this work, and being kindly provided to the authors by Oscar Castillo for its use.

FRBS of type-1 and interval type-2 are added in order to establish particular characteristics of each individual and the place where they live. These characteristics chosen for the modeling are inspired by the epidemic that has generated a global crisis in 2019, that of the virus of the coronavirus family, qualified as SARS-CoV-2 and known as COVID-19 [14]. In fact, characteristics such as adherence to protection measures and social isolation, belonging to the so-called risk group, the quality of treatment, and the percentage of vacancies in hospitals have been determining factors for the dissemination or control of the COVID-19 epidemic. Thus, based on this current information, two FRBS of each type have been constructed whose outputs are the infection and recovery rates at which the individual is infected, remains infected, recovers or dies.

Related to this work, applications of the two-dimensional cellular automaton [15,16,17] can be found in the literature. Furthermore, again related with this study, is the one presented in [18] where a three-dimensional CA develops a SIRD dynamics with only a recovered rate as output for type-1 FRBS with the same inputs of the fuzzy system detailed in this manuscript. The novelty incorporated for this research from the former, has been an interval type-2 FRBS for the recovered rate and others fuzzy systems that model an infection rate in both types: type-1 and interval type-2. On the other hand, in both works, the CA's code built using Matlab software, has been developed by the authors. A type of SIRD dynamics through a three-dimensional automaton in which the rules are inferred through a FRBS of type-1 or interval type-2, there is no published literature on these combined topics, as far as the authors are aware.

The presentation of this research is organized as follows: in Section 2 a succinct review of the theory of fuzzy sets of type-1 and interval type-2 is developed with detailed bibliography. In Section 3 the dynamics of the cellular automaton are explained. Section 4 is related to the fuzzy rule-based systems content in the modeling, divided in two subsections: Subsection 4.1 confirming the features of the type-1 fuzzy system, and Subsection 4.2 the interval type-2 counterpart. The results of this work are presented in Section 5 to, finally, compose conclusions and future research in Section 6.

2 Type-1 and Interval Type-2 Fuzzy Sets

Through this section, a review of interval type-2 fuzzy sets is made. For more details refer to [11].

Firstly, recall that a type-1 fuzzy set, A, is characterized by its membership function μ_A defined in the universe of discourse X and $\mu_A(x) \in [0,1]$ [8]. In other words, it could be interpreted by the graph of its membership function:

$$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0,1]\}.$$

The support of A is the topological closure of the set $\{\mu_A(x) > 0, x \in X\}$.

The tool used in this work that the main component are type-1 fuzzy sets is the FRBS, described in the following. The architecture of a type-1 FRBS is composed by four components described as follows:

1. Input processor: For each coordinate of the crisp input vector is determined the membership degree corresponding to each linguistic variable.

2. Rule Base: Let n number of linguistic variables, M the number of rules, and F_i^m, $i = 1, 2, \ldots, n$, G^m, $m = 1, 2, \ldots, N$, type-1 fuzzy sets, then the rule R^m of type-1 FRBS is defined by

$$R^m : \text{If } x_1 \text{ is } F_1^m \text{ and } x_2 \text{ is } F_2^m \text{ and } \ldots \text{ and } x_n \text{ is } F_n^m \text{ then } y \text{ is } G^m, \quad (2)$$

3. Fuzzy Inference: The inference block performs the logical operations for type-1 FRBS. The fuzzy inference method used in this work is Mamdani [11]
4. Defuzzification: this component determines the real output of the modeled system from the inference method. The Center of Gravity or Centroid has been chosen for this study and is calculated for a fuzzy set, B, obtained by the inference method, an average like the weighted average for data distribution is performed, with the difference that the weights are the values of μ_B. For a discrete domain this value given by:

$$y = \frac{\sum_{i=0}^{n} z_i \mu_B(z_i)}{\sum_{i=0}^{n} \mu_B(z_i)}, \ z_i \text{ in the domain of } \mu_B. \quad (3)$$

Next is a mathematical concept that extends the possibilities to deal with uncertainties: the type-2 fuzzy sets [12,11].

A type-2 fuzzy set, $\tilde{A}$, is the graph of the function $\mu_{\tilde{A}} : X \times [0,1] \to [0,1]$, called the membership function of $\tilde{A}$. In mathematical symbols:

$$\widetilde{A} = \left\{ \left((x,u), \mu_{\widetilde{A}}(x,u) \right), (x,u) \in X \times [0,1], \mu_{\widetilde{A}}(x,u) \in [0,1] \right\}.$$

The secondary membership of $\widetilde{A}$, denoted by $\mu_{\widetilde{A}(x)}(u)$, is, by definition, the membership function of the type-1 fuzzy set, $\widetilde{A}(x)$, which is obtained by cutting a plane parallel to the u axis by x. The support of $\widetilde{A}(x)$ is denoted by I_x, that is:

$$I_x = \overline{\{u \in [0,1], \mu_{\widetilde{A}}(x,u) > 0\}}. \quad (4)$$

Let $x \in X$ be a fixed point, it is defined by the upper and lower membership functions of x, denoted respectively by $\overline{\mu}_{\widetilde{A}}(x)$, $\underline{\mu}_{\widetilde{A}}(x)$, by:

$$\overline{\mu}_{\widetilde{A}}(x) = \sup\{u \in [0,1], \ \mu_{\widetilde{A}}(x,u) > 0\} = \sup I_x, \quad (5)$$

$$\underline{\mu}_{\widetilde{A}}(x) = \inf\{u \in [0,1], \ \mu_{\widetilde{A}}(x,u) > 0\} = \inf I_x, \quad (6)$$

where $\sup I_x$ is the supremum of the set I_x and $\inf I_x$ is the infimum of the set I_x.

A type-1 fuzzy set, B, is said to be embedded in the interval fuzzy set of type–2, $\widetilde{A}$, if $\underline{\mu}_{\widetilde{A}}(x) \leq \mu_B(x) \leq \overline{\mu}_{\widetilde{A}}(x)$, for all $x \in X$.

If $I_x = [\underline{\mu}_{\widetilde{A}}(x), \overline{\mu}_{\widetilde{A}}(x)]$ then the called footprint of uncertainty of $\widetilde{A}$, denoted by $FOU(\widetilde{A})$, is the set defined by:

$$FOU(\widetilde{A}) = \{(x,u), x \in X, u \in I_x\}, \quad (7)$$

where $\overline{\mu}_{\widetilde{A}}(x)$ and $\underline{\mu}_{\widetilde{A}}(x)$ are defined in (5) and (6), respectively. Notice that the set of all embedded B, type–1 fuzzy set in $\widetilde{A}$ determined $FOU(\widetilde{A})$.

A type–2 fuzzy set is an interval type–2 fuzzy set if $\mu_{\widetilde{A}}(x,u) = 1$, for all $(x,u) \in X \times [0,1]$.

The construction of the interval type-2 FRBS follows similar type-1 counterpart steps with the difference that, in the first step, which fuzzifies the independent inputs and outputs of the model to be analyzed, the linguistic variables that compose them are elaborated, with the help of specialists or information from the literature. At least one of the input linguistic variables is transformed into an interval type-2 fuzzy set. The interval type-2 FRBS consists of five components: input processor, inference, rule base, type-reducer, and output processor, explained in the following.

1. Input Processor: this block calculates the upper and lower membership degree of each linguistic variable.
2. Rule Base: The rule-based of the interval type-2 FRBS has the same structure of the type-1 counterpart. The difference with rule described in (2) is the nature of the sets involved that are interval type-2 fuzzy sets.
3. Fuzzy Inference: the inference block performs the logical operations for interval type-2 FRBS. The Mamdani's method [11] is used as well for this section.
4. Type-Reducer: the type-reducer block aims to use the Karnik-Mendel (KM) [20] algorithm that determines the minimum, y_L, and the maximum, y_R of all the centroids of type-1 fuzzy sets contained in the interval type-2 fuzzy set. A brief outline of the Karnik-Mendel algorithm is described in the following: Let $X = \{x_1 < x_2 < \ldots < x_n\}$ be a universe of discourse. Given an interval type-2 fuzzy set, $\widetilde{A}$, defined in X, consider two specific embedded subsets [21] of $\widetilde{A}$, $A_e(l)$, defined as

$$\mu_{A_e(l)}(x_i) = \begin{cases} \overline{\mu}_{\widetilde{A}}(x_i) \text{ if } i \leq l \\ \underline{\mu}_{\widetilde{A}}(x_i) \text{ if } i > l \end{cases}$$

where $i = 1, 2, \ldots, n$ and l is the so called switch point of $A_e(l)$.
Likewise, consider fuzzy set $A_e(r)$ defined by

$$\mu_{A_e(r)}(x_i) = \begin{cases} \underline{\mu}_{\widetilde{A}}(x_i) \text{ if } i \leq r \\ \overline{\mu}_{\widetilde{A}}(x_i) \text{ if } i > r, \end{cases}$$

where $i = 1, 2, \ldots, n$ and r is the so called switch point of $A_e(r)$.
Then, it has been demonstrated [22] that exist L and R such that

$$y_L(\widetilde{A}) = \min_{l \in [1,n-1]} \frac{\sum_{i=1}^{l} x_i \overline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=l+1}^{n} x_i \underline{\mu}_{\widetilde{A}}(x_i)}{\sum_{i=1}^{l} \overline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=l+1}^{n} \underline{\mu}_{\widetilde{A}}(x_i)} = \frac{\sum_{i=1}^{L} x_i \overline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=L+1}^{n} x_i \underline{\mu}_{\widetilde{A}}(x_i)}{\sum_{i=1}^{L} \overline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=L+1}^{n} \underline{\mu}_{\widetilde{A}}(x_i)},$$

$$y_R(\widetilde{A}) = \max_{r\in[1,n-1]} \frac{\sum_{i=1}^{r} x_i \underline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=r+1}^{n} x_i \overline{\mu}_{\widetilde{A}}(x_i)}{\sum_{i=1}^{r} \underline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=r+1}^{n} \overline{\mu}_{\widetilde{A}}(x_i)} = \frac{\sum_{i=1}^{R} x_i \underline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=R+1}^{n} x_i \overline{\mu}_{\widetilde{A}}(x_i)}{\sum_{i=1}^{R} \underline{\mu}_{\widetilde{A}}(x_i) + \sum_{i=R+1}^{n} \overline{\mu}_{\widetilde{A}}(x_i)}.$$

The KM algorithm [22] locate the switch points L, R and, as a consequence, the generalized centroid of $\widetilde{A}$ defined in (8), as explained in the following.

KM algorithm to calculate y_L:
(a) Calculate the initial point:

$$y' = \frac{\sum_{i=1}^{n} x_i \theta_i}{\sum_{i=1}^{n} \theta_i}, \quad \text{with} \quad \theta_i = \frac{\underline{\mu}_{\widetilde{A}}(x_i) + \overline{\mu}_{\widetilde{A}}(x_i)}{2}, \quad i = 1, 2, \ldots, n.$$

(b) Find $1 \le k \le n-1$ such that $x_k \le y' \le x_{k+1}$.
(c) Define

$$\theta_i = \begin{cases} \overline{\mu}_{\widetilde{A}}(x_i) \text{ if } i \le k, \\ \underline{\mu}_{\widetilde{A}}(x_i) \text{ if } i > k, \end{cases}$$

and calculate,

$$y_l(k) = \frac{\sum_{i=1}^{n} x_i \theta_i}{\sum_{i=1}^{n} \theta_i}.$$

(d) If $y_l(k) = y'$, then stop and define $y_L = y_l(k), L = k$. If not, go to step 5.
(e) Define $y' = y_l(k)$ and go to step 2.

KM algorithm to calculate y_R:
(a) Calculate the initial point:

$$y' = \frac{\sum_{i=1}^{n} x_i \theta_i}{\sum_{i=1}^{n} \theta_i}, \quad \text{with} \quad \theta_i = \frac{\underline{\mu}_{\widetilde{A}}(x_i) + \overline{\mu}_{\widetilde{A}}(x_i)}{2}, \quad i = 1, 2, \ldots, n.$$

(b) Find $1 \le k \le N-1$ such that $x_k \le y' \le x_{k+1}$.
(c) Define

$$\theta_i = \begin{cases} \underline{\mu}_{\widetilde{A}}(x_i) \text{ if } i \le k, \\ \overline{\mu}_{\widetilde{A}}(x_i) \text{ if } i > k, \end{cases}$$

and calculate,

$$y_r(k) = \frac{\sum_{i=1}^{n} x_i \theta_i}{\sum_{i=1}^{n} \theta_i}.$$

(d) If $y_r(k) = y'$, then stop and define $y_R = y_r(k), R = k$. If not, go to step 5.
(e) Define $y' = y_r(k)$ and go to step 2.

5. Defuzzification: the defuzzified output of the interval type-2 FRBS is given by the average of y_L and y_R, that is,

$$y_C = \frac{y_L + y_R}{2}, \tag{8}$$

noticing that y_L and y_R depend on each input crisp value.

The dynamics is developed with a combination of two computational tools: a three-dimensional (3D) cellular automaton and two FRBS of type-1 and interval type-2. This methodology is detailed in next sections.

3 3D Cellular Automaton

As explained in the introduction, in the CA, individuals of the different groups are represented by colored cells in the following way: blue cells represent susceptible individuals, red cells are infected, green cells are recovered, black cells are dead and white are the free spaces.

The following items, 3, 4 and 5, include numbers that have been chosen as a result of empirical tests that are best adapted to the deterministic SIRD model.

The rules of 3D cellular automaton are as follows:

1. Cell movement:
   - Blue, red, and green cells move towards a white neighbor cell, chosen randomly. Black cells do not move.
   - Infected cells look for susceptible cells; the search for a susceptible neighbor is carried out in a pre-set order, analyzing all 26 neighbors. All susceptible cells are registered and in the next iteration they will possibly be infected. The representation of this movement is shown in Figure 2, Figure 3 and Figure 4.

2. Characteristics of susceptible and infected populations:
   - One of the characteristics of the individuals of each population is the quality of the protection measures. A random number between 0 and 1 is determined for each cell, which represents adherence to the protection measures recommended by the World Health Organization (WHO) [23] (mask, alcohol gel, and washing hands).
   - The second characteristic is adherence to social isolation, which is randomly determined between 0 and 1 for each cell.
3. Infection of cells representing susceptible individuals:

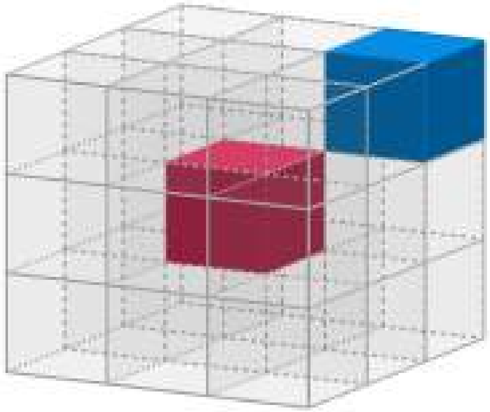

Fig. 2. The visualization of the blue cell by the red one.

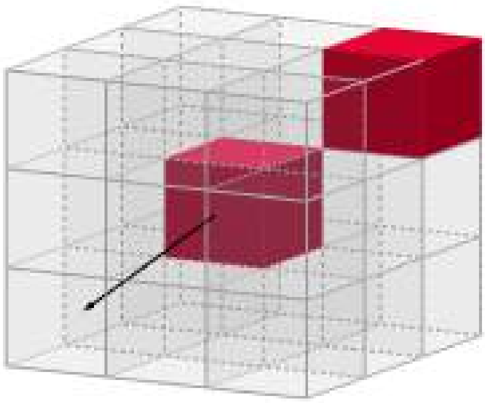

Fig. 3. Having a red cell as a neighbor, the blue cell of Figure 2 turns red.

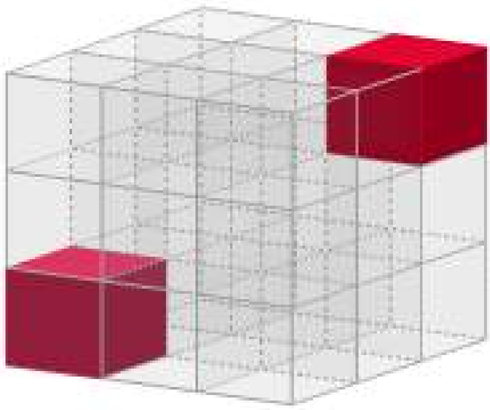

Fig. 4. In the same iteration the infected cell moves to the first free space found.

– Type-1 Fuzzy Model:

  The rule for a blue cell will be infected by a red one if the infection rate from the type-1 FRBS, (see Subsection 4.1) is greater than or equal to 0.8. Else, a new random number is generated to determine if the cell is infected. If the result is greater than 0.8, then the cell is infected. Otherwise, keep the blue color.

- Interval Type-2 Fuzzy Model:
  The decision upon the next condition of the cell is divided in three cases, corresponding to the three outputs originated by the interval type-2 FRBS with following rules:
  - For the interval type-2 FRBS left output, a susceptible cell becomes an infected cell if the infection rate from the interval type-2 FRBS, (see Subsection 4.2) is greater than 0.9, meanwhile if the cell value is less or than or equal to 0.9, a random number is generated to decide the cell's infected condition. Otherwise, it continues as susceptible.
  - For the center output, a susceptible cell becomes infected if the infection rate is greater than 0.8. If the value is less than or equal to 0.8, a random number greater than 0.8 decides the cell's infected condition. Otherwise, keep the blue color.
  - For the right output, the blue cell becomes red if the infection rate is greater than 0.6, otherwise a random number greater than 0.4 turns the blue color in red. Otherwise, keep the blue color.

4. Transformation of a cell representing an infected individual to continue in the infected group, recovers or dies. The rules are detailed in the following, divided by type of model:
   - Type-1 Fuzzy Model:
     An infected cell becomes recovered if the rate of recovery from the FRBS, explained in Subsection 4.1, is greater than 0.8. If the FRBS output is in the range $(0.3, 0.8]$, the next cell state is randomly determined: if the number drawn is greater than 0.5, then the cell turns green (recovered), otherwise it remains red (infected). If the recovery rate obtained by the FRBS is in the range $(0, 0.3]$, the next cell state is drawn again and if this number is greater than or equal to 0.4, then the cell remains red, otherwise, the cell changes to black (dead). If the FRBS output is zero then the cell represents a dead individual.
   - Interval Type-2 Fuzzy Model:
     - Left output: a red cell becomes green if the rate of recovery from the interval type-2 FRBS of the recuperation rate (see Subsection 4.2), is greater than 0.2. If this output is in the range $[0, 0.2]$ the cell turns black (dead).
     - Center output: An infected cell becomes recovered if the rate of recovery is greater than 0.8. If the FRBS output is in the range $(0.3, 0.8]$ a random number lesser than 0.5, then the cell turns green (recovered), otherwise it remains red. If the recovery rate obtained by the FRBS is in the range $(0, 0.3]$, a random number greater than or equal to 0.5 determines that the cell remains red, otherwise, the cell changes to black. If the FRBS output is zero then the cell becomes black.
     - Right output: a red cell becomes green if the rate of recovery from the interval type-2 FRBS of the recuperation rate (see Subsection 4.2), is greater than 0.53. If this output is in the range $[0, 0.53]$ the cell turns black.

5. Characteristics of infected individuals:
   – One of the characteristics of infected individuals is that they belong to a risk group. For each cell, a random number between 0 and 1 is determined, which if it is greater than 0.8 the individual is in the risk group. This decision is made at the beginning of the dynamic and whenever an individual ceases to be susceptible and becomes infected.
   – The selection of the treatment quality for each infected cell is performed in the first iteration in a random way, with values between 0 and 1.
6. Characteristics of the place where the infected individual resides:
   – Hospital vacancies at the location of infected individuals are randomly determined with values between 0 and 100%, in the first iteration and whenever an individual ceases to be susceptible and becomes infected.

The implemented SIRD type dynamics is completed with a FRBS that is described in Section 4.

4 Fuzzy Rule-Based System

In this section two FRBS included in the dynamics organized are described in two subsections corresponding to type-1 and interval type-2.

4.1 FRBS of Type-1

In the following the features of the two type-1 FRBS are detailed.

Infection Rate FRBS (FRBS1): The input variables are: adherence to protection measures and social isolation with values in the interval [0,1]. The linguistic terms for input variable Protection Measures (PM) are: Inappropriate (I) Medium (M), and Efficient (E). For the input variable Social Isolation (SI) are: Small (S), Medium (M), and Large (L). The output variable is the Infection Rate (IR), with the linguistic terms: Low (Lw), Medium (M), High (H). The membership functions of the input and output variables are trapezoidal. The fuzzy inference method used is the Mamdani method and the defuzzification method is the Center of Gravity [24]. The nine inference rules are giving by:

1. If PM is I and SI is S then IR is L;
2. If PM is I and SI is M then IR is L;
3. If PM is I and SI is L then IR is M;
4. If PM is M and SI is S then IR is L;
5. If PM is M and SI is M then IR is M;
6. If PM is M and SI is L then IR is S;
7. If PM is E and SI is S then IR is M;

8. If PM is E and SI is M then IR is S;
9. If PM is E and SI is L then IR is S.

Recovery Rate (FRBS2): The input variables are: the infected individual rather is or is not in the risk group; the quality of treatment; and the percentage of vacancies in hospitals. The linguistic terms for input variable Risk Group (RG) are True (T) or False (F); for the input variable Quality of Treatment (QT) are Inappropriate (I) or Efficient (E); for the percentage of Hospital Vacancies (HV) are No vacancy (N), Small (S) and Large (L). The output variable is the Recovery Rate (RR), with the linguistic terms: Dead (D), Low (Lw), Medium (M) and High (H). The membership functions of the input and output variables are trapezoidal and singleton. The fuzzy inference and defuzzification methods are the same as before. The twelve rules are giving by:

1. If RG is F and QT is I and VH is N then RR is M;
2. If RG is F and QT is I and VH is S then RR is M;
3. If RG is F and QT is I and VH is L then RR is M;
4. If RG is F and QT is E and VH is N then RR is H;
5. If RG is F and QT is E and VH is S then RR is H;
6. If RG is F and QT is E and VH is L then RR is H;
7. If RG is T and QT is I and VH is S then RR is D;
8. If RG is T and QT is I and VH is P then RR is D;
9. If RG is T and QT is I and VH is L then RR is D;
10. If RG is T and QT is E and VH is S then RR is Lw;
11. If RG is T and QT is E and VH is P then RR is Lw;
12. If RG is T and QT is E and VH is L then RR is D.

In Figure 5 and Figure 6 the complete type-1 FRBS corresponding to the infection and recovery rate are shown .

4.2 FRBS of interval Type-2

The interval type-2 fuzzy rule base is the same as used in the type-1 fuzzy rule base substituting the name of the type-1 fuzzy set that represents the linguistic term, for the corresponding name of the interval type-2 fuzzy set. For instance, the set L is transformed in $\widetilde{L}$. Shown in Figure 7 is the complete interval type-2 FRBS corresponding to the infection rate and, in Figure 8, the corresponding to the recovery rate.

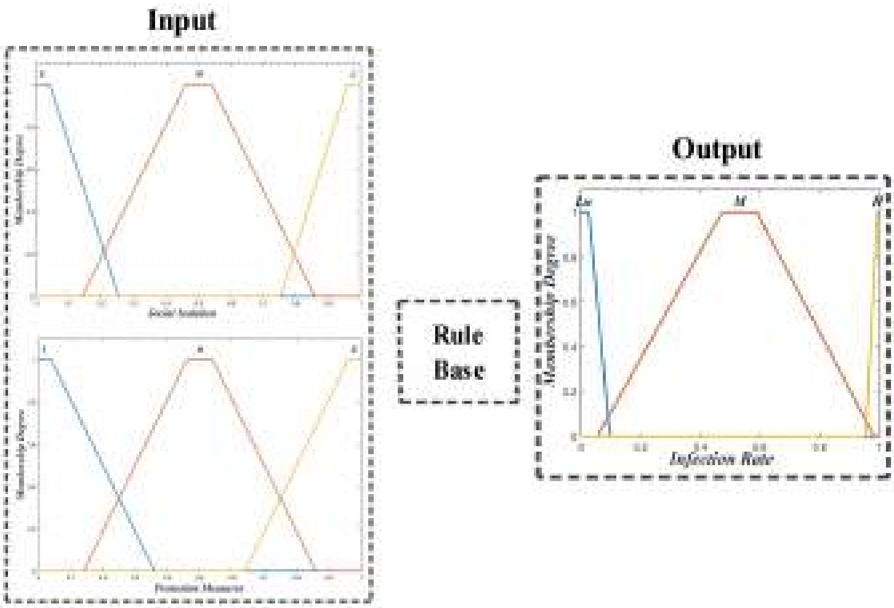

Fig. 5. The FRBS of the CA individual's infection rate.

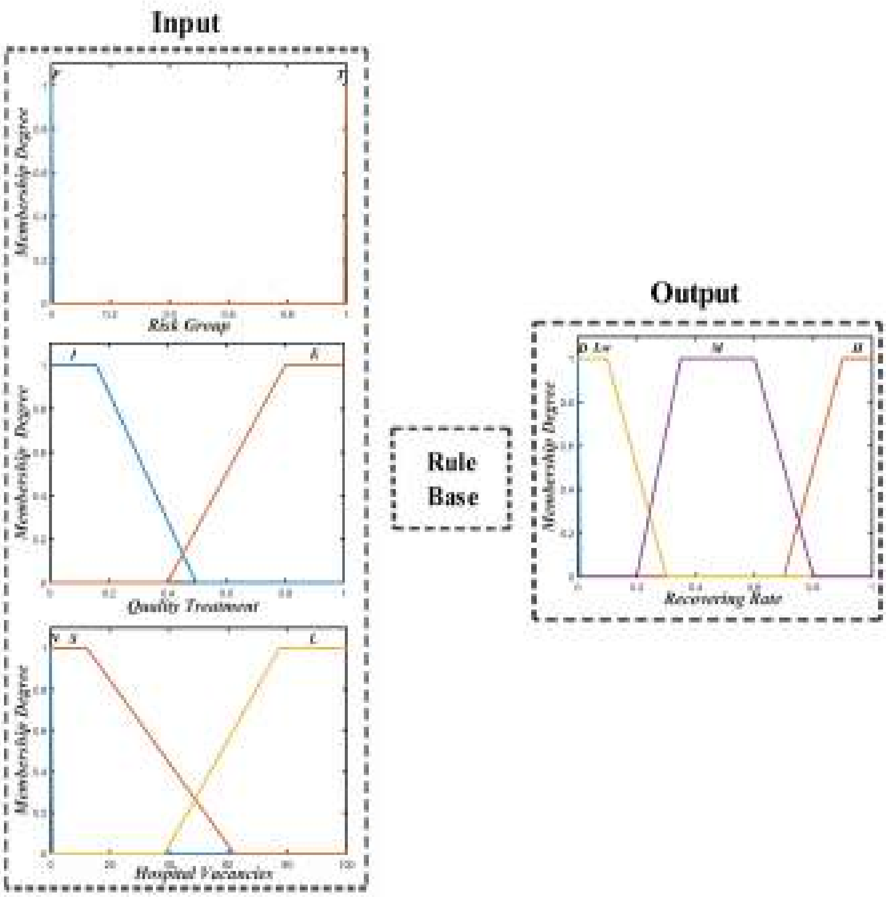

Fig. 6. The FRBS of the CA infected individual's recovery rate.

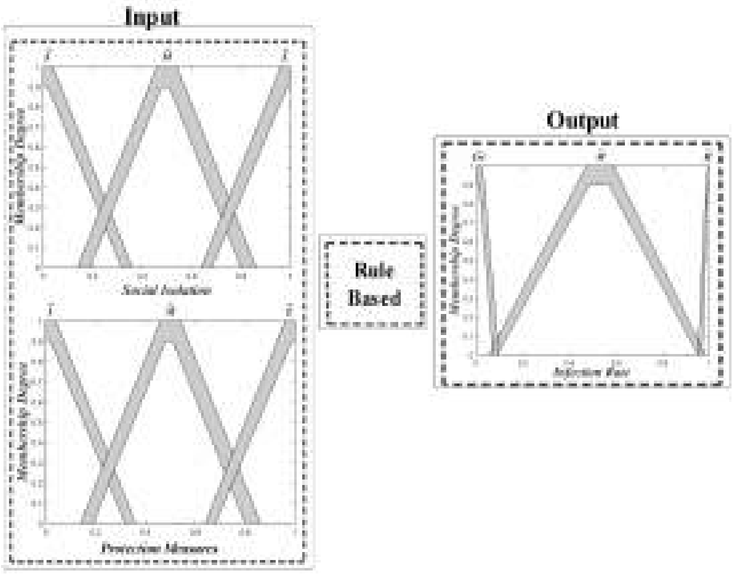

Fig. 7. Interval type-2 FRBS corresponding to the infection rate.

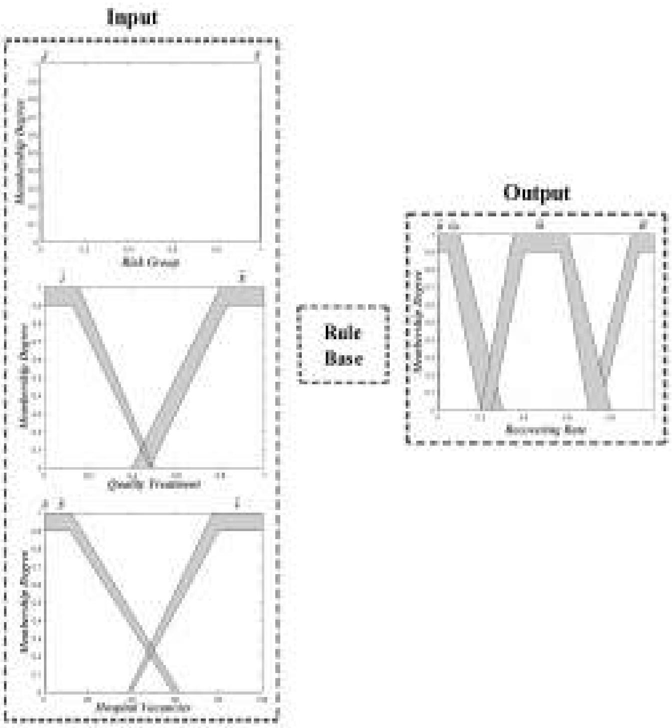

Fig. 8. The interval type-2 FRBS corresponding to the recovery rate.

The complete structure with the FRBS1 and FRBS2 described in the previous subsection along with the three-dimensional CA rules is detailed in the flowchart illustrated in Figure 9.

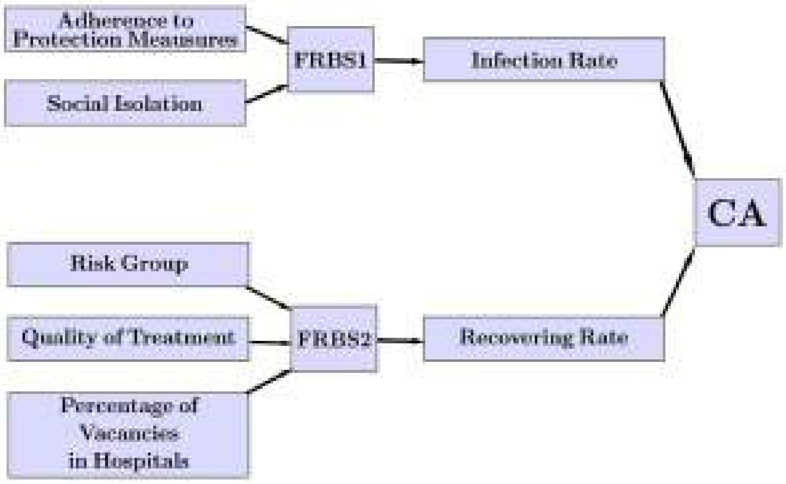

Fig. 9. Flowchart complete procedure for the modeling.

5 Results

For the purpose of validating the modeling through Fuzzy–CA combination, numerical simulations are performed to obtain an approximation of the solution of the ODE system (1). The numerical method used is the fourth-order Runge–Kutta [25], using as parameters $\beta = 1.52$, $\gamma = 0.37$ and $\mu = 0.06$ and the same initial conditions imposed for the automaton, that is, $S(0) = 1900$, $I(0) = 70$, $R(0) = D(0) = 0$. The result of the numerical simulation is shown in Figure 10.

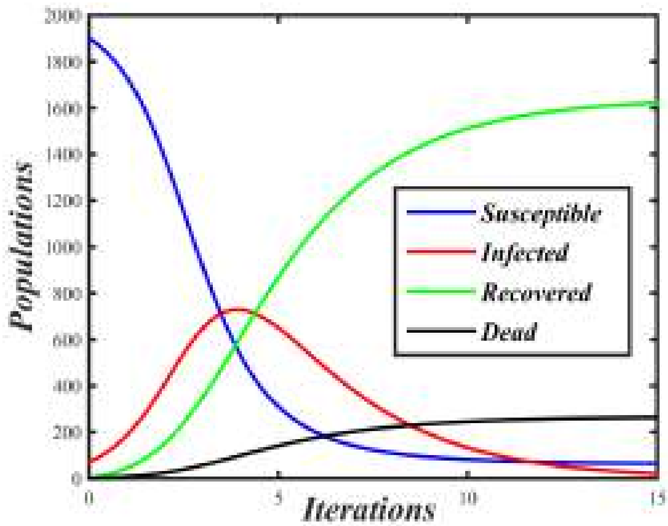

Fig. 10. Approximation of the deterministic solution for the SIRD model corresponding to the ODE system (1).

The modeling of the SIRD dynamics through the 3D cellular automaton has been generated from a cube of 19^3 cells and with initial populations: 1900 cells representing susceptible individuals, 70 cells representing infected individuals, null recovered and dead individuals. The total number of iteration is 15. These numbers have been chosen as result of empirical tests that best fit the deterministic SIRD model.The pseudo code corresponding to the dynamics of interval type-2 FRBS, is depicted in Algorithm 1. In details, there are more numbers, including random ($rand$), that complete the 15 experimental parameters (EP) used in the dynamics. A similar algorithm is the corresponding to the type-1 FRBS interacting with the 3D CA. The criteria for the output of type-1 FRBS is the same as the center output of the interval type-2 FRBS.

Thus, via the automaton rules combined with the FRBS output, a dynamic is obtained at each iteration as shown in Figure 11.

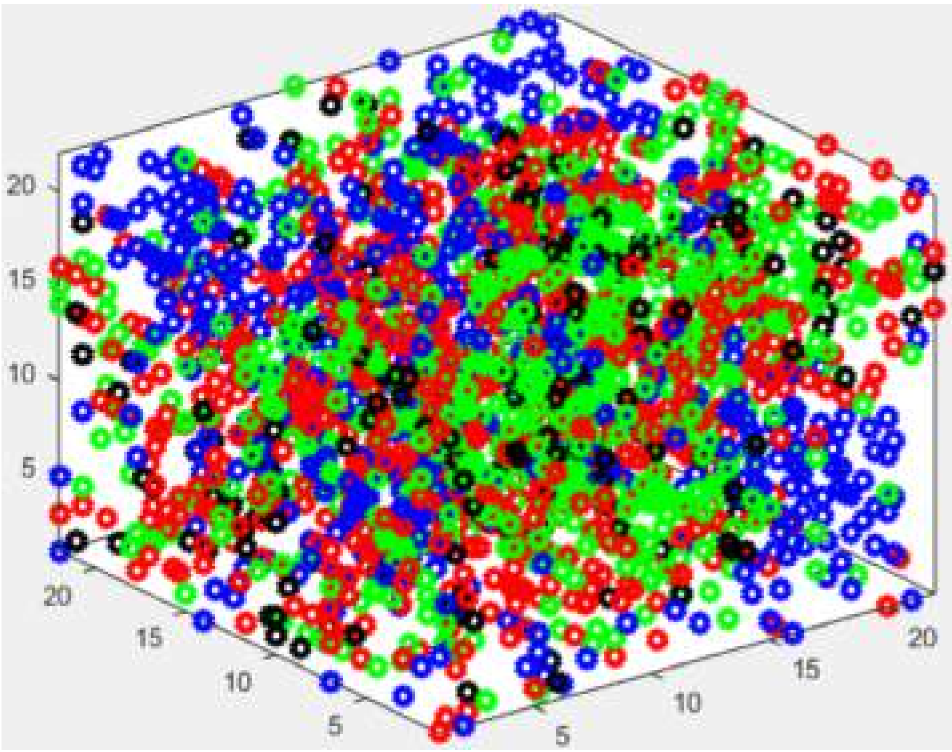

Fig. 11. An iteration of the cellular automaton dynamics in which all communities of individuals appear displayed in circular rings.

As a result of counting individuals at each iteration, for each population of individuals, three output curves originated by the FRBS type-2 and combined with the CA rules, are obtained outputs: left, center, and right. Along with these curves, the deterministic solution and the output of the type-1 FRBS combined with the CA are shown in the Figures 12, 13, 14, and 15, corresponding to the population of susceptible, infected, recovered, and dead, respectively.

The outputs from the CA provide the evolution of the populations at each iteration for the four classes given by: susceptible, infected, recovered, and dead.

Algorithm 1 : Interaction CA and Interval Type-2 FRBS

```
Require: IniNumSuscept = InitialCondition1;
Require: IniNumInfec = InitialCondition2;
Require: IniNumRec = 0;
Require: IniNumDead = 0;
Require: ExperimentalParameterNumber(EP#)
Require: CellNum = EP1;
Require: GenerationsNum = EP2;

  procedure funcCreateSuscepInfect
    Function to create susceptible and infected cells.
  end procedure

  for iteration = 1, 2, ..., EP2 do

    procedure funcFuzzyInfec
      Fuzzy block (FRBS1) determines the cell state, infected or not, through three outputs:
        LeftOutput1, CenterOutput1, RightOutput1.
    end procedure.

    if LeftSide then
      if LeftOutput1 > EP3 then
        The Cell is infected.
      else if rand > EP4 then
        The Cell is infected.
      end if
    end if
    if Center then
      if CenterOutput1 > EP5 then
        The Cell is infected.
      else if rand > EP6 then
        The Cell is infected.
      end if
    end if
    if RightSide then
      if RightOutput1 > EP7 then
        The Cell is infected.
      else if rand > EP8 then
        The Cell is infected.
      end if
    end if

    procedure funcFuzzyRec
      Fuzzy block (FRBS2) determines the cell state, recovered, remains infected or die, through three
        outputs: LeftOutput2, CenterOutput2, RightOutput2.
    end procedure.

    if LeftSide then
      if (LeftOutput2 ≥ 0)and(LeftOutput ≤ EP9) then
        The cell dies.
      else if (LeftOutput2 ≥ EP9)and(LeftOutput2 ≤ EP10) then
        The cell recovers.
      end if
    end if
    if Center then
      if (CenterOutput2 ≥ 0)and(CenterOutput2 ≤ EP11) then
        if rand < EP12 then
          The cell dies.
        end if
        if (CenterOutput2 ≥ EP11)and(CenterOutput2 ≤ EP13) then
          The cell recovers.
        end if
        if CenterOutput2 > EP14 then
          if rand < EP15 then
            The cell recovers.
          end if
        end if
      end if
    end if
    procedure funcMoveAll
      All cells move.
    end procedure
    procedure funcDataStore
      Function that stores the data of the current generation.
    end procedure

  end for
```

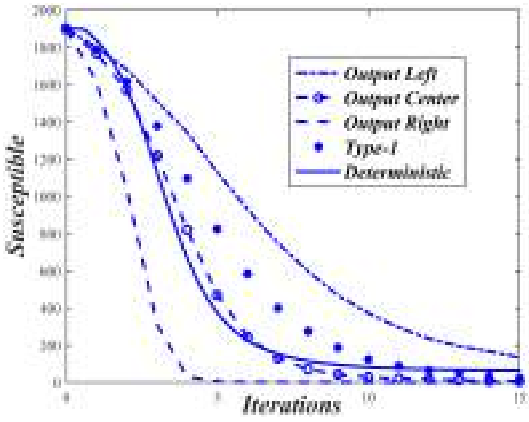

Fig. 12. The model's outputs for susceptible population.

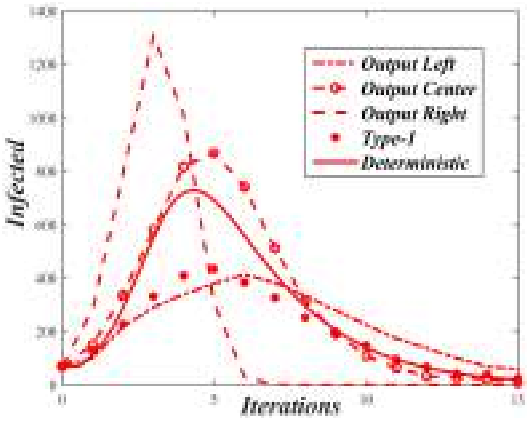

Fig. 13. The model's outputs for infected population.

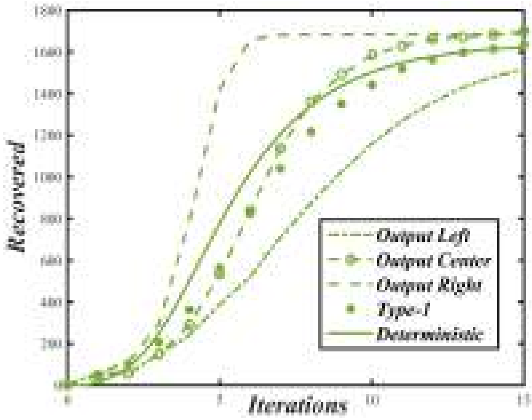

Fig. 14. The model's outputs for recovered population.

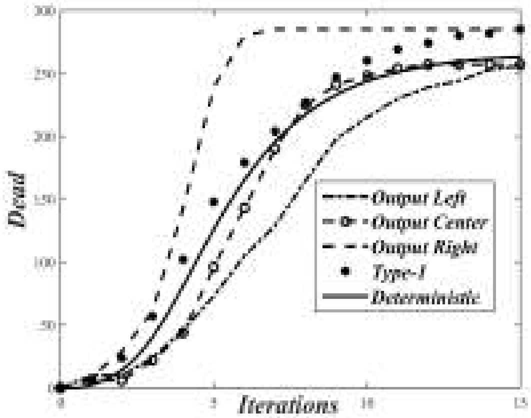

Fig. 15. The model's outputs for dead population.

The output of each class is acquired using the three values, y_L, y_R and y_C (see equation (8)), obtained through the interval type-2 $FRBS1$ and $FRBS2$. The graphics of Figures 16, 17, 18, and 19 are composed by the populations at each iteration, where the maximum and the minimum of values, y_L, y_R and y_C are calculated. Notice that a range of values is obtained for the CA output, and that output corresponding the type-1 modeling is inside the range.

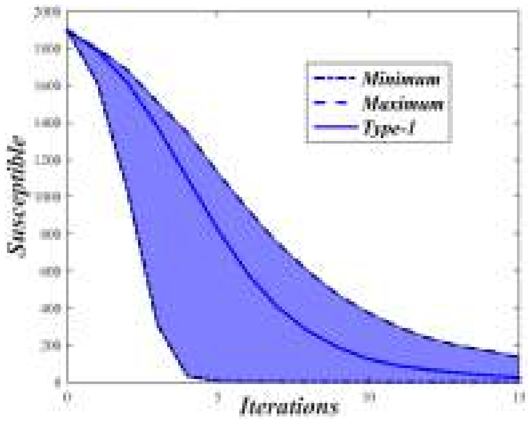

Fig. 16. The range determined by output model for susceptible population.

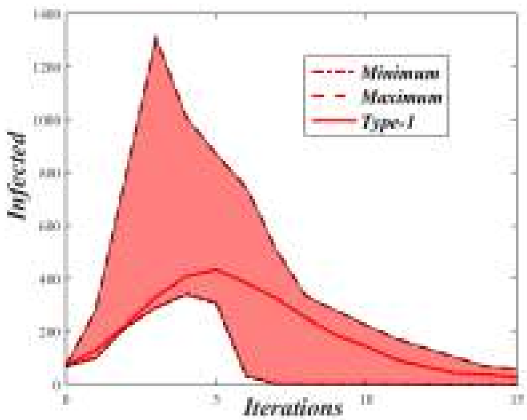

Fig. 17. The range determined by output model for infected population.

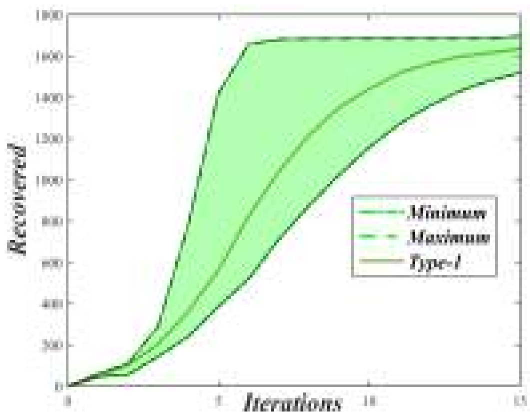

Fig. 18. The range determined by output model for recovered population.

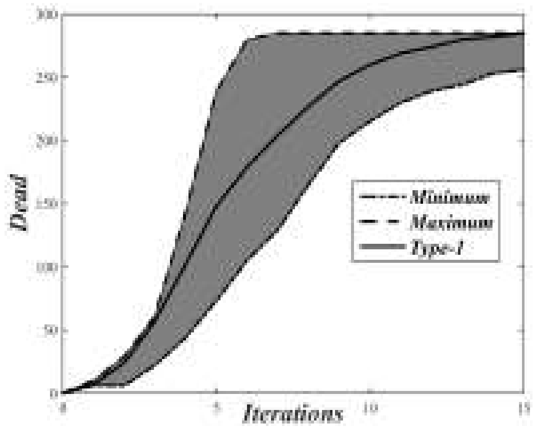

Fig. 19. The range determined by output model for dead population.

6 Conclusion

The evolution of an epidemic is simulated in the computational environment of a three-dimensional cellular automaton combined with two fuzzy rule-based systems that interpret the infection and recovery rate, being each one of type-1 and interval type-2. Its inputs, outputs and fuzzy rules are inspired by the crucial characteristics of the epidemiological phenomenon that the contemporary world is going through, the one generated by COVID-19. The modeling result is validated by means of a SIRD dynamics containing the same initial conditions imposed on the automaton: it starts with a group of susceptible and infected non-null and the other two groups, recovered and dead, as null. The rules that determine the next state of the cell of the cellular automaton interact with the of type-1 and interval type-2 fuzzy rule-based systems in order to establish the following situation of an individual represented by a cell: is infected, remains infected, recovers or is dead.

Three innovative elements are highlighted in this study: the dynamics of the evolution of an epidemic inspired by the classic SIRD model; the development of rule base of two types that governs the infected and recovery rates, and the fuzzy approach that integrates in two manners the uncertainty of decision making. The validation is made through the span obtained for each population of individuals, originated by the cellular automaton outputs related to the interval type-2 fuzzy rule-based system. In the range can be found the deterministic solution and the modeling output corresponding to the type-1 fuzzy rule-based system. Equivalent qualitative behaviors are highlighted. Thus, the follow-up of the research in future works, intends to compare results with data from the real dynamics of those infected by the disease.

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